University of Waterloo

CS 341 Winter 2025

Written Assignment 3

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Due Date: Friday, March 7 at 11:59pm to Crowdmark
All work submitted must be the student's own.

- Make sure to read the Assignments section on the course webpage for instructions on submission and question expectations ("Instructions for Assignments"): `https://student.cs.uwaterloo.ca/~cs341/#Assignments`
- Space complexity is not required for this assignment.
- Assume all graphs are stored using adjacency list representation.

Question 1 [10 marks]

Prove or disprove each of the following statements, where in each case $G = (V, E)$ is a connected undirected weighted graph with n vertices and m edges.

a) If G has $m \geq n$ edges and a unique heaviest edge e, then e is not part of any minimum spanning tree of G.

b) If G has $m \geq n$ edges and a unique lightest edge e, then e is part of every minimum spanning tree of G.

c) If e is a maximal weight edge of a cycle of G, then there is a minimum spanning tree of G that does not include e.

Question 2 [10 marks]

A *vertex-and-edge-weighted graph* is a directed graph $G = (V, E)$ where each vertex $v \in V$ has a *cost* $c(v)$ and every edge $e \in E$ has a *weight* $w(e)$. The *length* of a path in G is the sum of the weights of the edges in the path *and* the cost of the vertices in the path. In the Single-Source Shortest Path (SSSP) problem on vertex-and-edge-weighted graphs, we are given G and a source vertex s and we want to determine the minimum length of a path from s to v for every $v \in V$. (Note that the minimum length of a path from s to s is $c(s)$.)

Solve the SSSP problem on vertex-and-edge-weighted graphs in the special case where all the weights and the costs are positive integers.

Question 3 [10 marks]

In the JobSelection problem, the input is a positive integer t and a sequence of n pairs of positive integers $(r_1, p_1), (r_2, p_2), \ldots, (r_n, p_n)$ that correspond to the *reward* r_i you earn if you complete job i and the *penalty* p_i that you must pay if you do not complete job i. You can only complete t jobs; a valid solution to the problem is a subset $S \subseteq \{1, 2, \ldots, n\}$ of $|S| = t$ jobs that maximizes the profit

$$\text{profit}(S) = \sum_{i \in S} r_i - \sum_{j \notin S} p_j$$

earned by completing the set S of jobs.

Design a greedy algorithm to solve this optimization problem. Prove that it always returns an optimal solution.

Question 4 [10 marks]

There are n houses to be connected to the internet in a very sparse forest neighbourhood. There are two options for every household i:
1) to put an ethernet link to a neighbour j, which costs c_{ij} dollars, or
2) to get a router, which costs r_i dollars (price for the routers can be different depending on the location).

A house will be connected to the internet if either (1) there is some path from it along ethernet links that leads to a house with the router or (2) there is a router in the house.

Design an efficient algorithm that, given the array $r[1..n]$ and the array $c[1..n, 1..n]$ as input, finds the minimum cost to connect every house to the internet. Note that not every two houses can be connected by a link, in which case the entry $c_{ij} = \infty$. You may not use dynamic programming.