CS 341: Algorithms
Lecture 2: Solving Recurrences

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Based on lecture notes by Éric Schost and many previous CS 341 instructors

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Design Idea for MergeSort

Input: Array A of n integers

- **Step 1:** We split A into two subarrays: A_L consists of the first $\lceil \frac{n}{2} \rceil$ elements in A and A_R consists of the last $\lfloor \frac{n}{2} \rfloor$ elements in A.
- **Step 2:** *Recursively* run *MergeSort* on A_L and A_R.
- **Step 3:** After A_L and A_R have been sorted, use a function *Merge* to merge them into a single sorted array.

Recurrence Relations

The mergesort recurrence is

$$T(n) = \begin{cases} T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \Theta(n) & \text{if } n > 1 \\ \Theta(1) & \text{if } n = 1. \end{cases}$$

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It is simpler to consider the following *exact* recurrence, with constant factors c and d replacing Θ's:

$$T(n) = \begin{cases} T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + cn & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

Recurrence Relations (cont.)

The following is the corresponding *sloppy* recurrence (it has floors and ceilings removed):

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

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We solve the sloppy recurrence when $n = 2^j$ using the *recursion tree method*.

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3. Repeat this process recursively, terminating when a node receives the value $T(1) = d$.
4. Sum the values on each level of the tree, and then compute the sum of all these sums; the result is $T(n)$.

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- If this solution is expressed as a function of n using Θ-notation, then we obtain the complexity of the solution of the exact recurrence for *all* n.
- This is not a proof, however. If a real mathematical proof is required, then it is necessary to use induction.

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Theorem (Master theorem)

Suppose that $a \geq 1$ *and* $b > 1$. *Consider the recurrence*

$$T(n) = a\,T\left(\frac{n}{b}\right) + \Theta(n^y)$$

in sloppy or exact form. Denote $x = \log_b a$. *Then*

$$T(n) \in \begin{cases} \Theta(n^x) & \text{if } y < x \\ \Theta(n^y \log n) & \text{if } y = x \\ \Theta(n^y) & \text{if } y > x. \end{cases}$$

The Master Method

Suppose that $n = b^j$, $a \geq 1$, $b \geq 2$ are integers and

$$T(n) = a\,T\left(\frac{n}{b}\right) + c\,n^y, \qquad T(1) = d.$$

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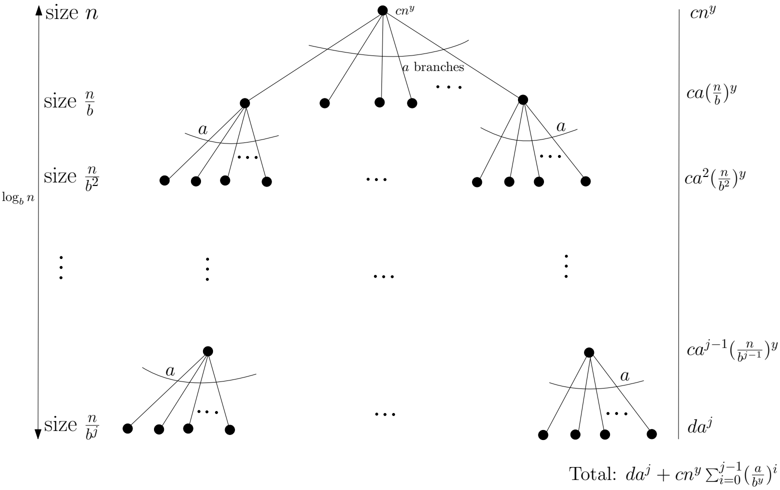

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size of subproblem	# nodes	cost/node	total cost
$n = b^j$	1	$c\,n^y$	$c\,n^y$
$n/b = b^{j-1}$	a	$c\,(n/b)^y$	$c\,a\,(n/b)^y$
$n/b^2 = b^{j-2}$	a^2	$c\,(n/b^2)^y$	$c\,a^2\,(n/b^2)^y$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n/b^{j-1} = b$	a^{j-1}	$c\,(n/b^{j-1})^y$	$c\,a^{j-1}\,(n/b^{j-1})^y$
$n/b^j = 1$	a^j	d	$d\,a^j$

Computing $T(n)$

Summing the costs of all levels of the recursion tree, we have that

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The formula for $T(n)$ is a geometric sequence with ratio $r = \frac{a}{b^y} = b^{x-y}$:

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$$T(n) = d\,n^x + c\,n^y \sum_{i=0}^{j-1} r^i.$$

There are three cases, depending on whether $r > 1$, $r = 1$ or $r < 1$.

Complexity of $T(n)$

case	r	y, x	complexity of $T(n)$
heavy leaves	$r > 1$	$y < x$	$T(n) \in \Theta(n^x)$
balanced	$r = 1$	$y = x$	$T(n) \in \Theta(n^y \log n)$
heavy top	$r < 1$	$y > x$	$T(n) \in \Theta(n^y)$

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"heavy top" means that cost of the recursion tree is dominated by the cost of the root node.

The substitution method

To solve a recurrence do the following:

- Guess the solution (or the form of the solution)
- Use induction to prove it (and if needed, find a constant)

Example: $T(n) = 2T(\frac{n}{2}) + n$