CS 341: ALGORITHMS

Lecture 7: finishing greedy

Readings: see website

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LAST TIME: EXCHANGE ARGUMENT FOR INTERVAL SELECTION

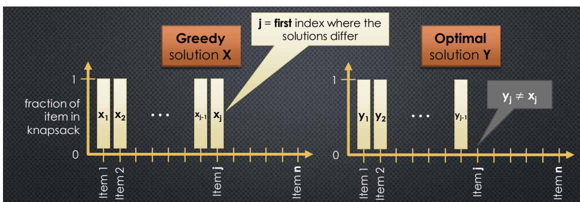
ASSUMED: PROFIT / WEIGHT RATIOS ARE **DISTINCT**

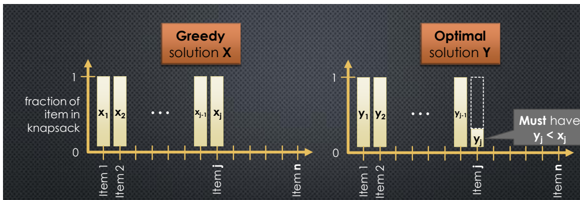
WHAT IF PROFIT/WEIGHT RATIOS ARE **NOT DISTINCT**?

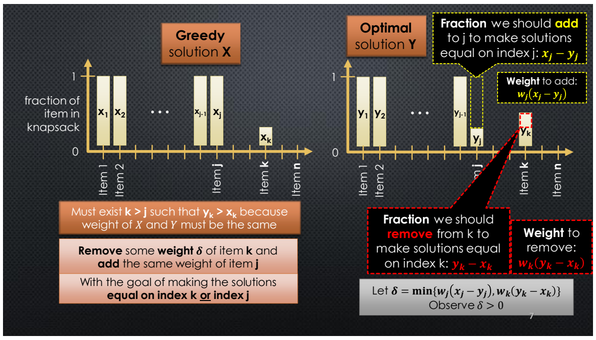
OR, MORE GENERALLY, **WHAT IF THERE ARE MANY OPT SOLUTIONS?**

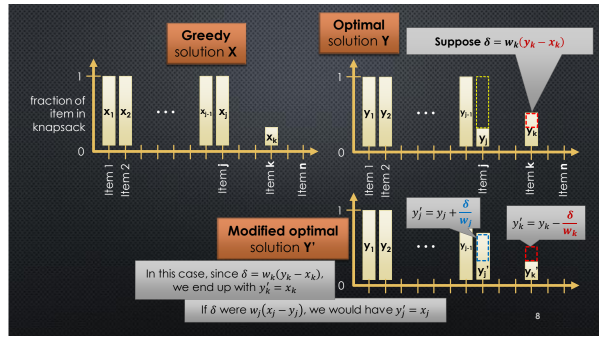
WHAT IF THERE ARE **MANY** OPTIMAL SOLUTIONS

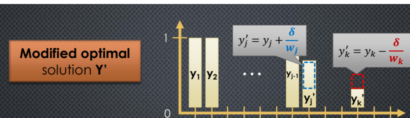
- Can't just assume X != Y and obtain a contradiction!
- **Key idea:** focus on **one particular optimal solution**
  - Let Y be an optimal solution that **matches X on a maximal number of indices**
  - **Observe**: if X is really optimal, then $Y = X$
- Suppose X != Y for contra
  - We will modify Y, preserving its optimality, but making it match X on **one more index** (a contradiction!)











To show Y' is feasible, we show $weight(Y') \le M$ and $y_k' \ge 0, y_j' \le 1$

Weight	We move δ weight from item k to item j This does not change the total weight! So $weight(Y') = weight(Y) = M$

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FEASIBILITY OF Y'

- Showing $y_k' \ge 0$
  - By definition, $y_k' = y_k - \frac{\delta}{w_k} \ge 0$ **iff** $\boldsymbol{\delta \le y_k w_k}$
  - But δ is the **minimum** of $w_j(x_j - y_j)$ and $w_k(y_k - x_k)$
  - And $w_k(y_k - x_k) \le w_k y_k$ **so** $\boldsymbol{\delta \le y_k w_k}$
- Showing $y_j' \le 1$
  - $y_j' = y_j + \frac{\delta}{w_j} \le 1$ **iff** $\boldsymbol{\delta \le w_j(1 - y_j)}$ (rearranging)
  - $\delta \le \boldsymbol{w_j(x_j - y_j)}$ (definition of δ)
  - and $w_j(x_j - y_j) \le \boldsymbol{w_j(1 - y_j)}$ (by feasibility of X, i.e., $x_j \le 1$)

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PROFIT OF Y'

(Fraction of item j **added**) × (profit for entire item)

- $profit(Y') = profit(Y) + \frac{\delta}{w_j} p_j - \frac{\delta}{w_k} p_k = profit(Y) + \delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right)$
- Since j is before k, and we consider items with more profit per unit weight first, we have $\frac{p_j}{w_j} \ge \frac{p_k}{w_k}$.
- Since $\delta > 0$ and $\frac{p_j}{w_j} \ge \frac{p_k}{w_k}$, we have $\delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right) \ge 0$
- Since Y is optimal, this **cannot be positive**
- So Y' is a new optimal solution that **matches X on one more index than Y**
- Contradiction: Y matched X on a **maximal** number of indices!

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SUMMARIZING EXCHANGE ARGUMENTS

- If there is a **unique optimal solution**
  - Let O != G be an optimal solution that beats greedy
  - Show how to change O to obtain a better solution
- If there is **more than one optimal solution**
  - Let O != G be an optimal solution that matches greedy on as many choices as possible
  - Show how to change O to obtain an optimal solution O' that matches greedy for even more choice(s)

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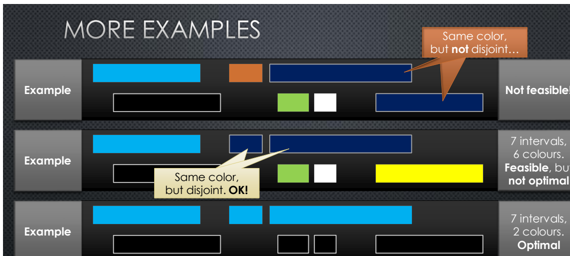
FINISHING UP GREEDY

INTERVAL **COLOURING**

PROBLEM: INTERVAL COLOURING

Instance: *A set $\mathcal{A} = \{A_1, \ldots, A_n\}$ of* **intervals**.
For $1 \leq i \leq n$, $A_i = [s_i, f_i)$, where s_i is the **start time** *of interval A_i and f_i is the* **finish time** *of A_i.*
Feasible solution: *A c-***colouring** *is a mapping $col : \mathcal{A} \to \{1, \ldots, c\}$ that assigns each interval a* **colour** *such that two intervals receiving the same colour are always disjoint.*
Find: *A c-colouring of $\mathcal{A}$ with the minimum number of colours.*

MORE EXAMPLES



Greedy Strategies for Interval Colouring

As usual, we consider the intervals one at a time.

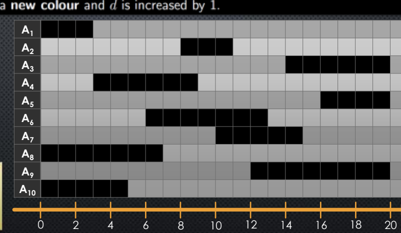
At a given point in time, suppose we have coloured the first $i < n$ intervals using d colours.

We will colour the $(i+1)$st interval with **any permissible colour**. If it cannot be coloured using any of the existing d colours, then we introduce a **new colour** and d is increased by 1.

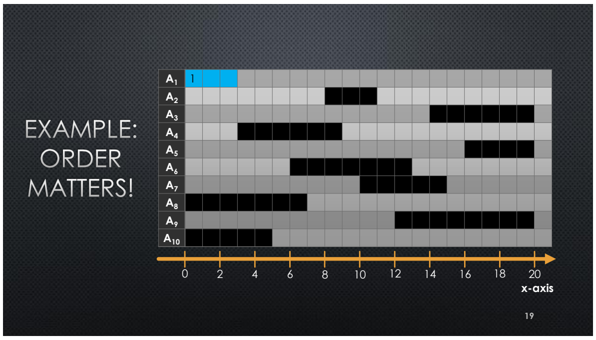
Question: In **what order** should we consider the intervals?

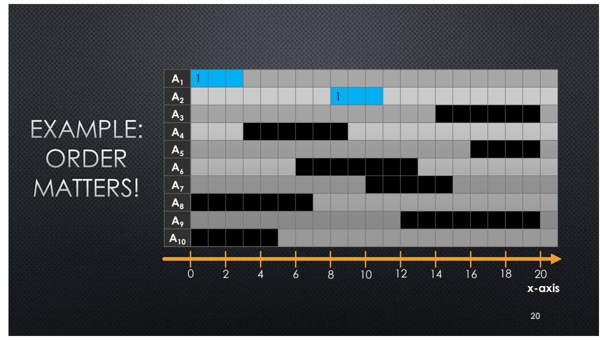
EXAMPLE: ORDER MATTERS!

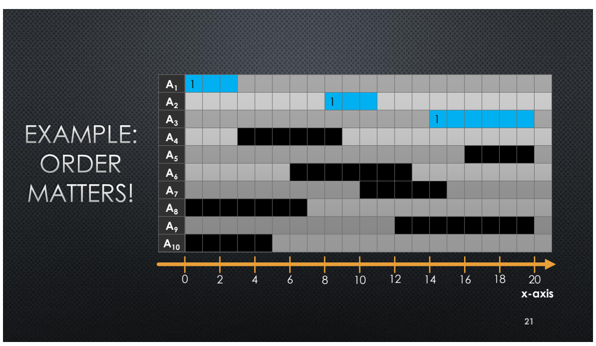
We will colour the $(i+1)$st interval with **any permissible colour**. If it cannot be coloured using any of the existing d colours, then we introduce a **new colour** and d is increased by 1.

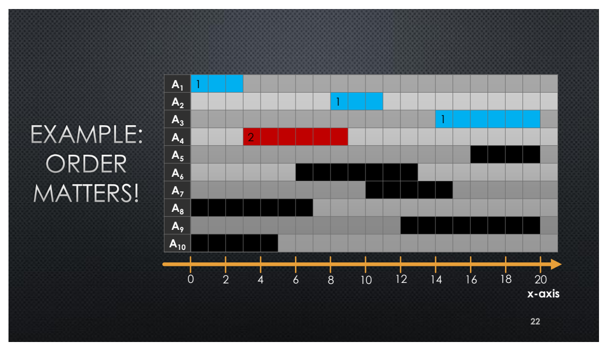


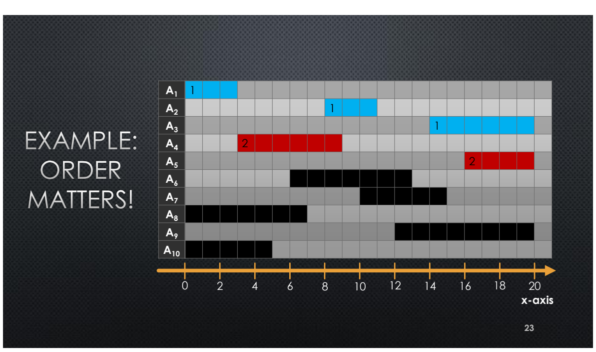
Consider intervals in the order they are given in the input: $A_1 \ldots A_{10}$

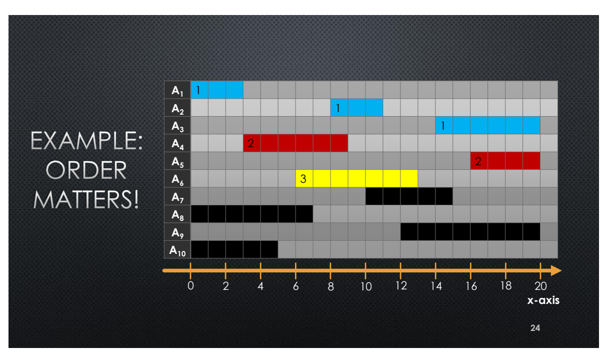

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
0 2 4 6 8 10 12 14 16 18 20
x-axis
19

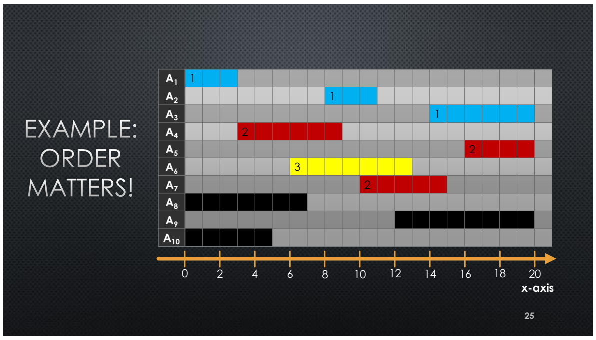

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
1
0 2 4 6 8 10 12 14 16 18 20
x-axis
20

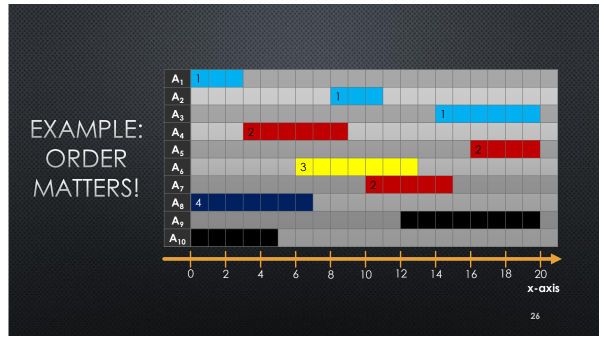

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
1
1
0 2 4 6 8 10 12 14 16 18 20
x-axis
21

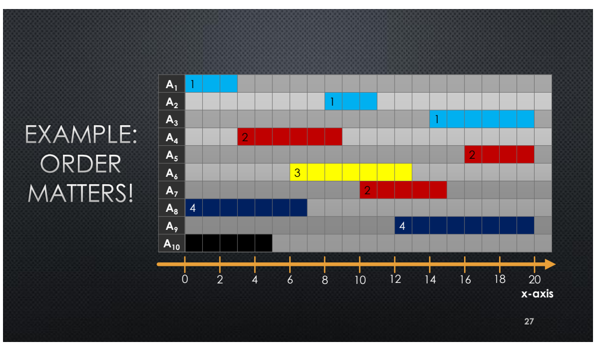

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
1
1
2
0 2 4 6 8 10 12 14 16 18 20
x-axis
22

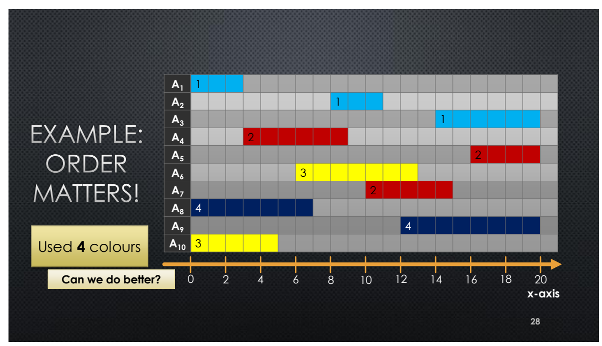

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
1
1
2
2
0 2 4 6 8 10 12 14 16 18 20
x-axis
23

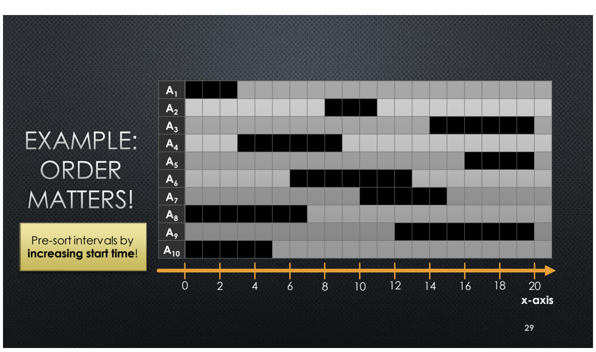

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
1
1
2
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
24

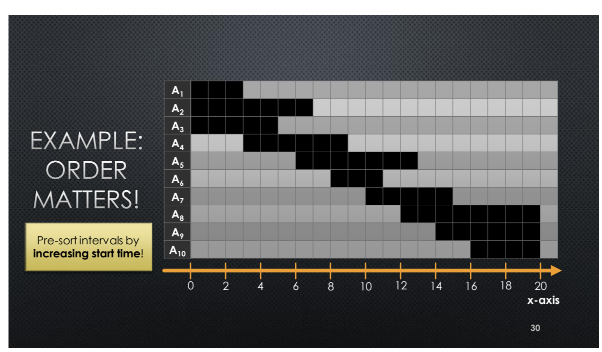

EXAMPLE: ORDER MATTERS!
A1
A2
A3
A4
A5
A6
A7
A8
A9
A10
1
1
1
2
2
3
2
0 2 4 6 8 10 12 14 16 18 20
x-axis
25

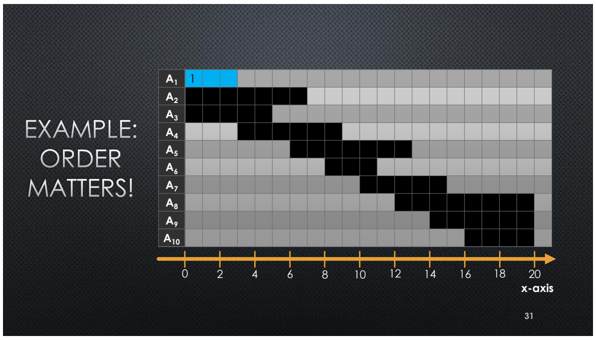

EXAMPLE: ORDER MATTERS!
A1
A2
A3
A4
A5
A6
A7
A8
A9
A10
1
1
1
2
2
3
2
4
0 2 4 6 8 10 12 14 16 18 20
x-axis
26

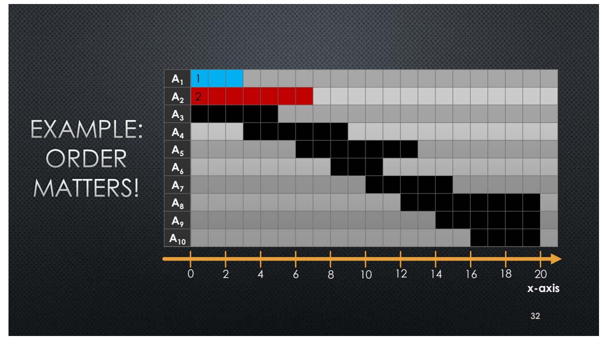

EXAMPLE: ORDER MATTERS!
A1
A2
A3
A4
A5
A6
A7
A8
A9
A10
1
1
1
2
2
3
2
4
4
0 2 4 6 8 10 12 14 16 18 20
x-axis
27

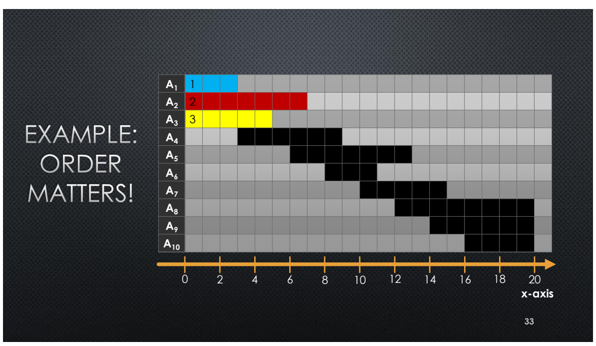

EXAMPLE: ORDER MATTERS!
Used 4 colours
Can we do better?
A1
A2
A3
A4
A5
A6
A7
A8
A9
A10
1
1
1
2
2
3
2
4
4
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
28

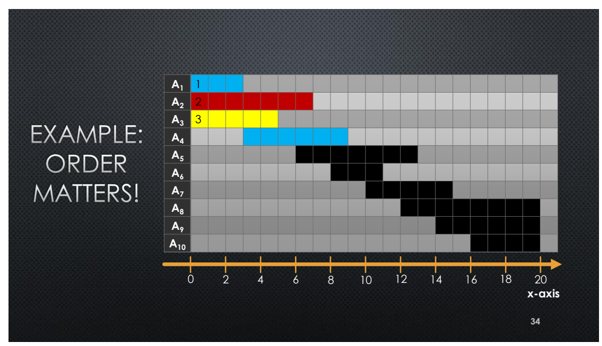

EXAMPLE: ORDER MATTERS!
Pre-sort intervals by increasing start time!
A1
A2
A3
A4
A5
A6
A7
A8
A9
A10
0 2 4 6 8 10 12 14 16 18 20
x-axis
29

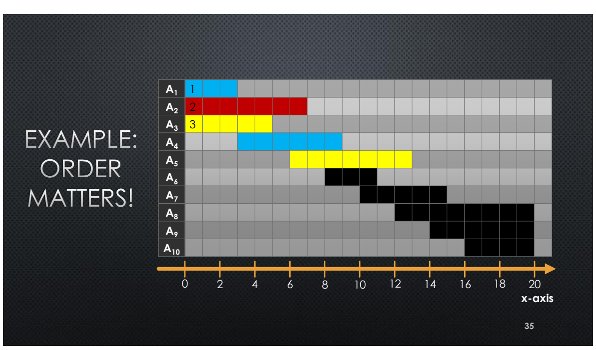

EXAMPLE: ORDER MATTERS!
Pre-sort intervals by increasing start time!
A1
A2
A3
A4
A5
A6
A7
A8
A9
A10
0 2 4 6 8 10 12 14 16 18 20
x-axis
30

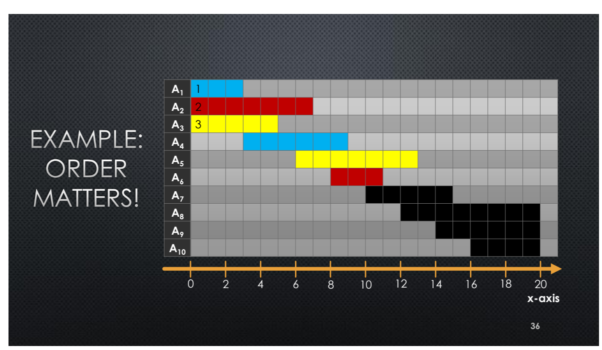

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
0 2 4 6 8 10 12 14 16 18 20
x-axis
31

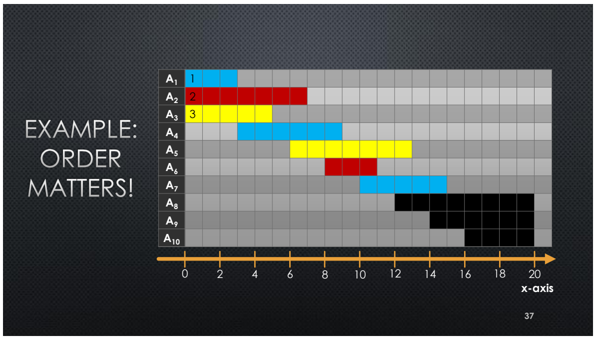

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
0 2 4 6 8 10 12 14 16 18 20
x-axis
32

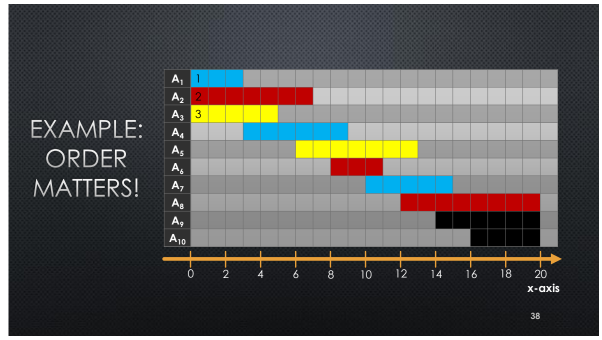

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
33

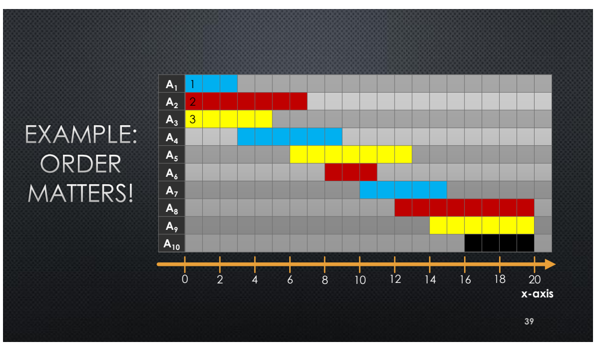

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
34

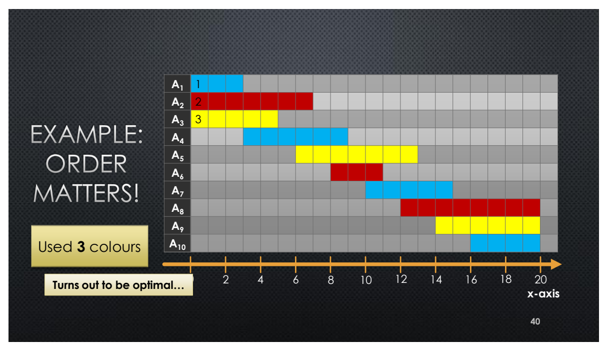

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
35

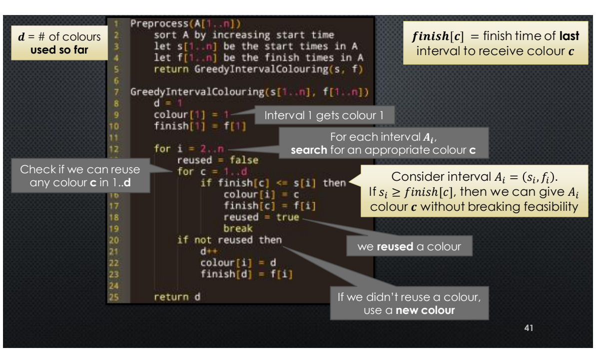

EXAMPLE: ORDER MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
36

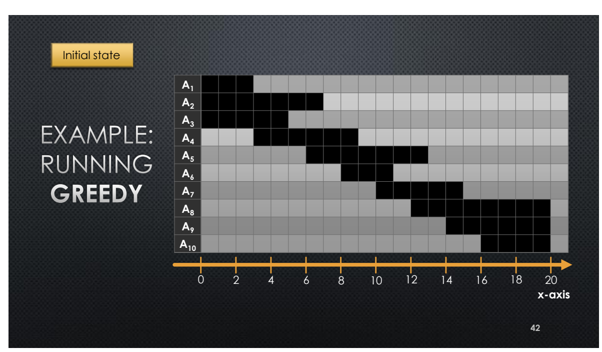

EXAMPLE:
ORDER
MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
37

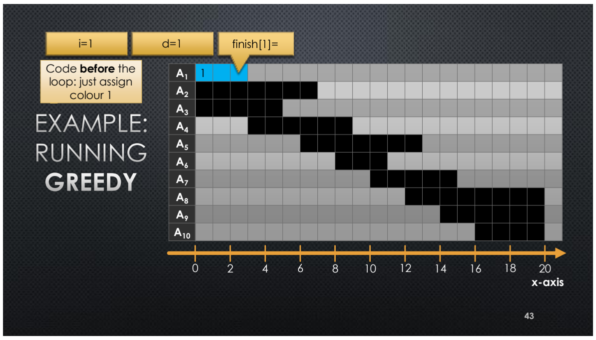

EXAMPLE:
ORDER
MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
38

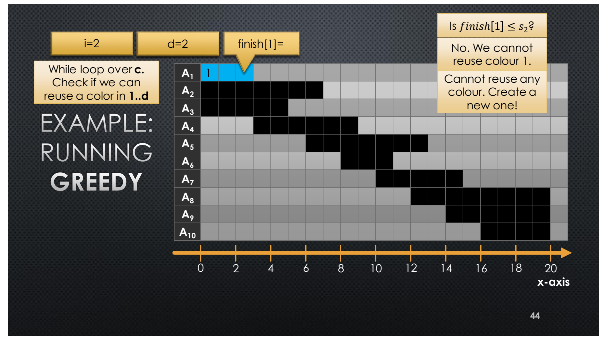

EXAMPLE:
ORDER
MATTERS!
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
39

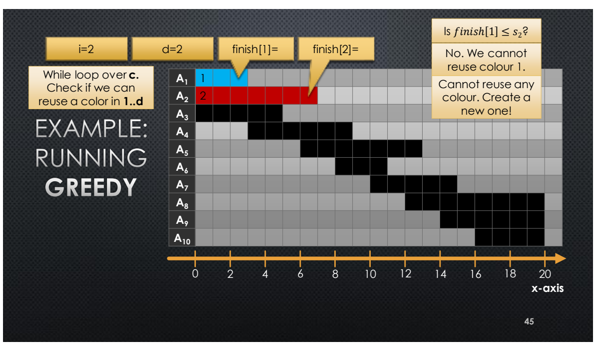

EXAMPLE:
ORDER
MATTERS!
Used 3 colours
Turns out to be optimal…
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
2 4 6 8 10 12 14 16 18 20
x-axis
40

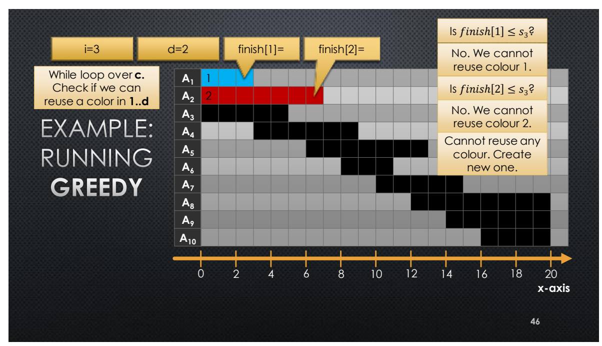

d = # of colours used so far
Preprocess(A[1..n])
sort A by increasing start time
let s[1..n] be the start times in A
let f[1..n] be the finish times in A
return GreedyIntervalColouring(s, f)
GreedyIntervalColouring(s[1..n], f[1..n])
d = 1
colour[1] = 1
finish[1] = f[1]
for i = 2..n
reused = false
for c = 1..d
if finish[c] <= s[i] then
colour[i] = c
finish[c] = f[i]
reused = true
break
if not reused then
d++
colour[i] = d
finish[d] = f[i]
return d
finish[c] = finish time of last interval to receive colour c
Interval 1 gets colour 1
For each interval Ai, search for an appropriate colour c
Check if we can reuse any colour c in 1..d
Consider interval Ai = (si, fi). If si ≥ finish[c], then we can give Ai colour c without breaking feasibility
we reused a colour
If we didn't reuse a colour, use a new colour
41

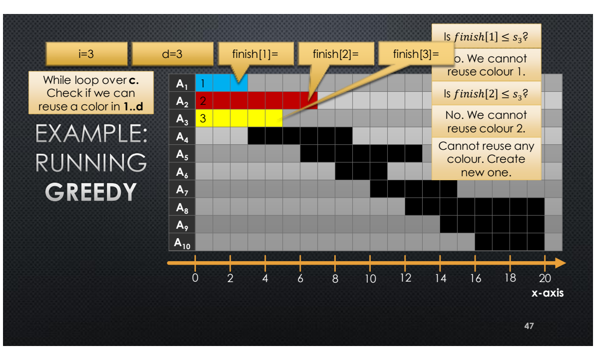

Initial state
EXAMPLE:
RUNNING
GREEDY
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
0 2 4 6 8 10 12 14 16 18 20
x-axis
42

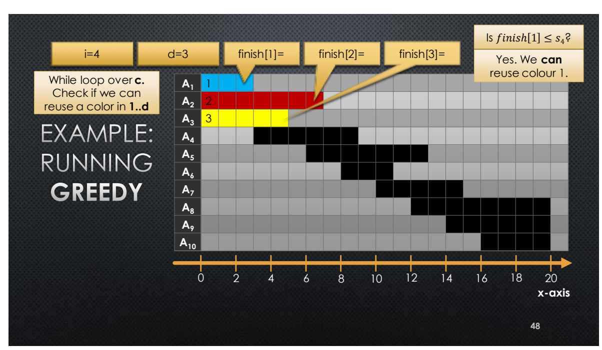

i=1
d=1
finish[1]=
Code **before** the loop: just assign colour 1
EXAMPLE: RUNNING GREEDY
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
0 2 4 6 8 10 12 14 16 18 20
x-axis
43

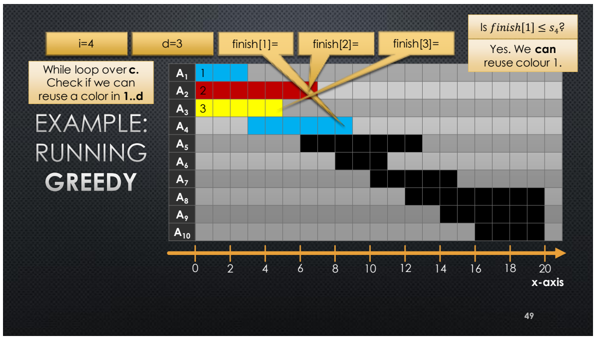

i=2
d=2
finish[1]=
While loop over **c.** Check if we can reuse a color in **1..d**
Is $finish[1] \le s_2$?
No. We cannot reuse colour 1.
Cannot reuse any colour. Create a new one!
EXAMPLE: RUNNING GREEDY
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
0 2 4 6 8 10 12 14 16 18 20
x-axis
44

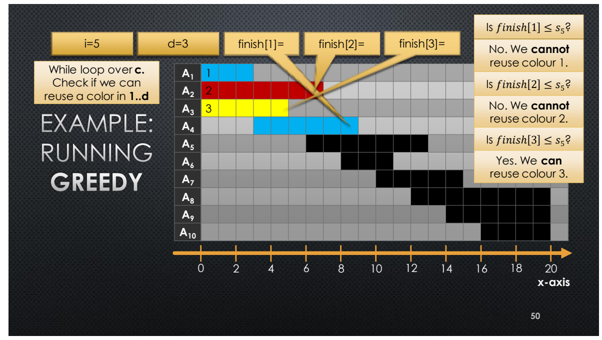

i=2
d=2
finish[1]=
finish[2]=
While loop over **c.** Check if we can reuse a color in **1..d**
Is $finish[1] \le s_2$?
No. We cannot reuse colour 1.
Cannot reuse any colour. Create a new one!
EXAMPLE: RUNNING GREEDY
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
0 2 4 6 8 10 12 14 16 18 20
x-axis
45

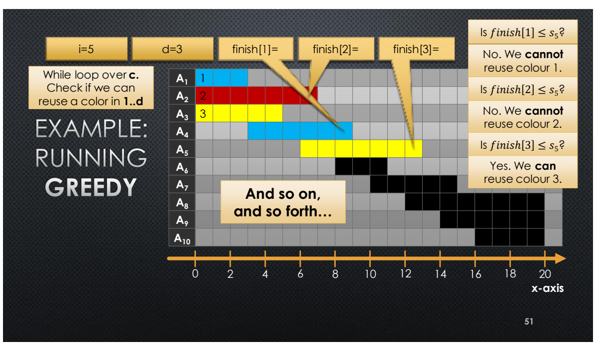

i=3
d=2
finish[1]=
finish[2]=
While loop over **c.** Check if we can reuse a color in **1..d**
Is $finish[1] \le s_3$?
No. We cannot reuse colour 1.
Is $finish[2] \le s_3$?
No. We cannot reuse colour 2.
Cannot reuse any colour. Create new one.
EXAMPLE: RUNNING GREEDY
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
0 2 4 6 8 10 12 14 16 18 20
x-axis
46


i=3
d=3
finish[1]=
finish[2]=
finish[3]=
While loop over **c.** Check if we can reuse a color in **1..d**
Is $finish[1] \le s_3$?
o. We cannot reuse colour 1.
Is $finish[2] \le s_3$?
No. We cannot reuse colour 2.
Cannot reuse any colour. Create new one.
EXAMPLE: RUNNING GREEDY
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
47


i=4
d=3
finish[1]=
finish[2]=
finish[3]=
While loop over **c.** Check if we can reuse a color in **1..d**
Is $finish[1] \le s_4$?
Yes. We **can** reuse colour 1.
EXAMPLE: RUNNING GREEDY
A1 A2 A3 A4 A5 A6 A7 A8 A9 A10
1
2
3
0 2 4 6 8 10 12 14 16 18 20
x-axis
48





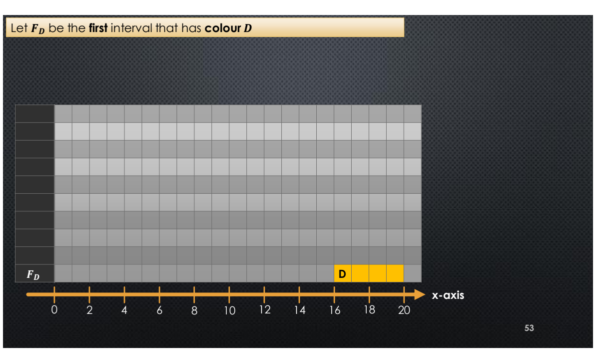


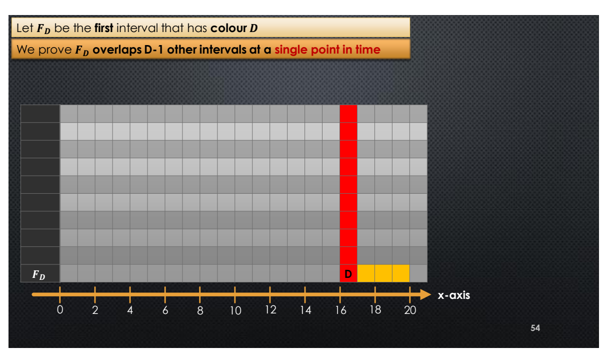
Correctness of the Algorithm

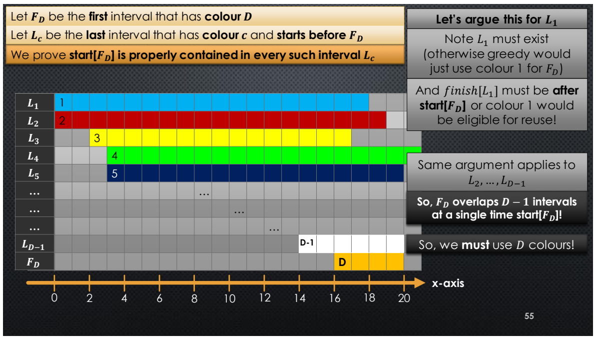
The correctness of this greedy algorithm can be proven inductively as well as by a "slick" method—we give the "slick" proof:

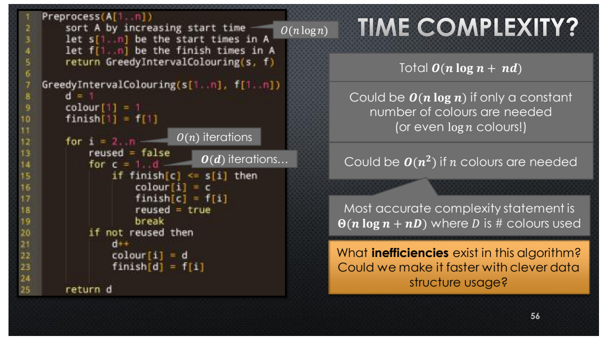
Let D denote the number of colours used by the algorithm.

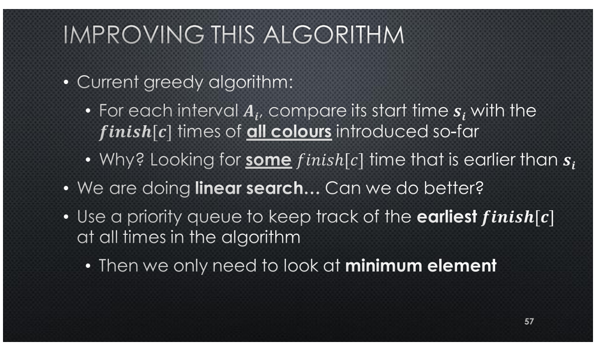
52






Let F_D be the **first** interval that has **colour** D
Let L_c be the **last** interval that has **colour** c and **starts before** F_D
We prove **start[F_D] is properly contained in every such interval L_c**
Let's argue this for L_1
Note L_1 must exist (otherwise greedy would just use colour 1 for F_D)
And $finish[L_1]$ must be **after start[F_D]** or colour 1 would be eligible for reuse!
Same argument applies to $L_2, \dots, L_{D-1}$
So, F_D overlaps $D-1$ intervals at a single time start[F_D]!
So, we **must** use D colours!
L_1 L_2 L_3 L_4 L_5 … … … L_{D-1} F_D
1 2 3 4 5 … … … D-1 D
0 2 4 6 8 10 12 14 16 18 20
x-axis
55


Preprocess(A[1..n])
sort A by increasing start time
let s[1..n] be the start times in A
let f[1..n] be the finish times in A
return GreedyIntervalColouring(s, f)
GreedyIntervalColouring(s[1..n], f[1..n])
d = 1
colour[1] = 1
finish[1] = f[1]
for i = 2..n
reused = false
for c = 1..d
if finish[c] <= s[i] then
colour[i] = c
finish[c] = f[i]
reused = true
break
if not reused then
d++
colour[i] = d
finish[d] = f[i]
return d
$O(n \log n)$
$O(n)$ iterations
$O(d)$ iterations…
TIME COMPLEXITY?
Total $O(n \log n + nd)$
Could be $O(n \log n)$ if only a constant number of colours are needed (or even $\log n$ colours!)
Could be $O(n^2)$ if n colours are needed
Most accurate complexity statement is $\Theta(n \log n + nD)$ where D is # colours used
What **inefficiencies** exist in this algorithm? Could we make it faster with clever data structure usage?
56


IMPROVING THIS ALGORITHM
• Current greedy algorithm:
• For each interval A_i, compare its start time s_i with the $finish[c]$ times of **all colours** introduced so-far
• Why? Looking for **some** $finish[c]$ time that is earlier than s_i
• We are doing **linear search…** Can we do better?
• Use a priority queue to keep track of the **earliest** $finish[c]$ at all times in the algorithm
• Then we only need to look at **minimum element**
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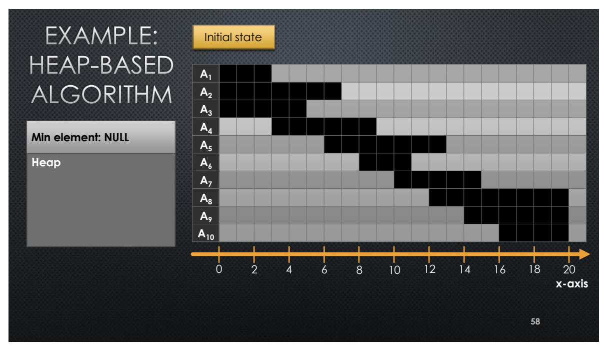
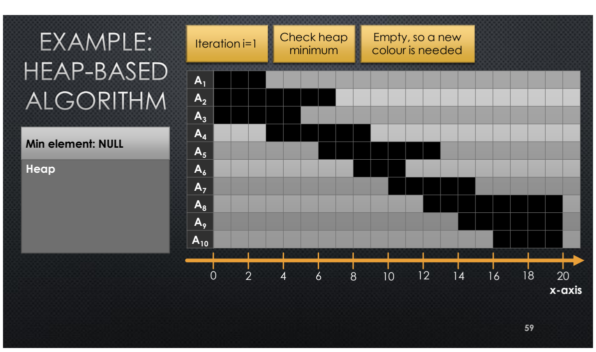
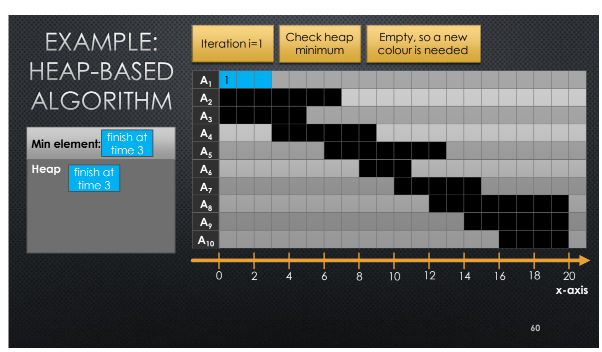
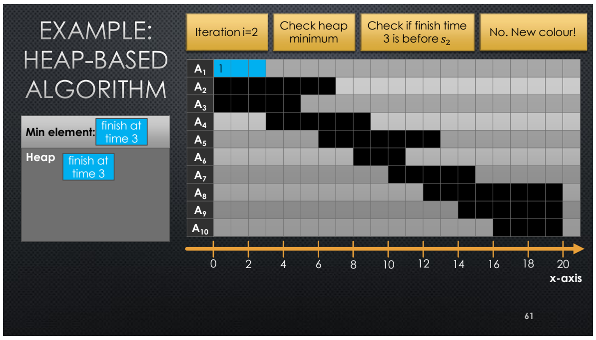
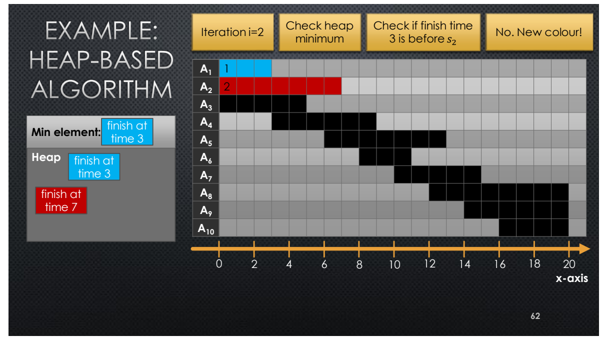
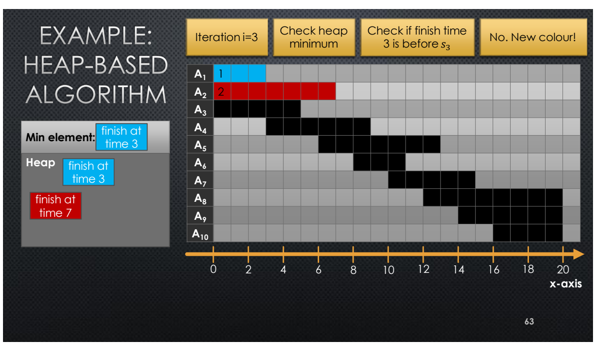

EXAMPLE: HEAP-BASED ALGORITHM
Initial state
Min element: NULL
Heap
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
0 2 4 6 8 10 12 14 16 18 20
x-axis
58

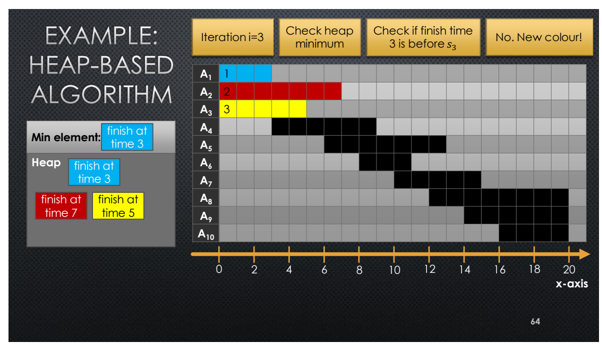

EXAMPLE: HEAP-BASED ALGORITHM
Iteration i=1
Check heap minimum
Empty, so a new colour is needed
Min element: NULL
Heap
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
0 2 4 6 8 10 12 14 16 18 20
x-axis
59

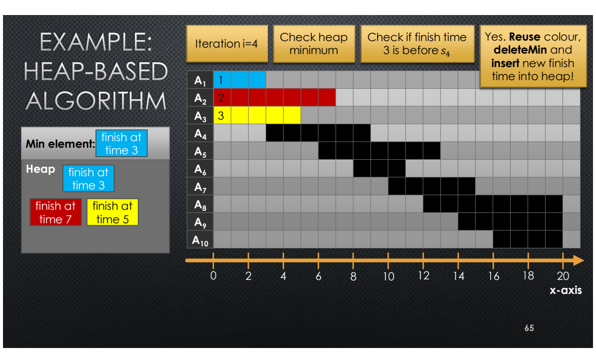

EXAMPLE: HEAP-BASED ALGORITHM
Iteration i=1
Check heap minimum
Empty, so a new colour is needed
Min element: finish at time 3
Heap
finish at time 3
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
1
0 2 4 6 8 10 12 14 16 18 20
x-axis
60

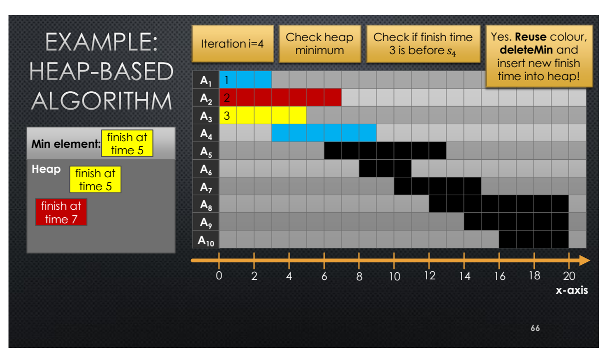

EXAMPLE: HEAP-BASED ALGORITHM
Iteration i=2
Check heap minimum
Check if finish time 3 is before s_2
No. New colour!
Min element: finish at time 3
Heap
finish at time 3
1
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
0 2 4 6 8 10 12 14 16 18 20
x-axis
61

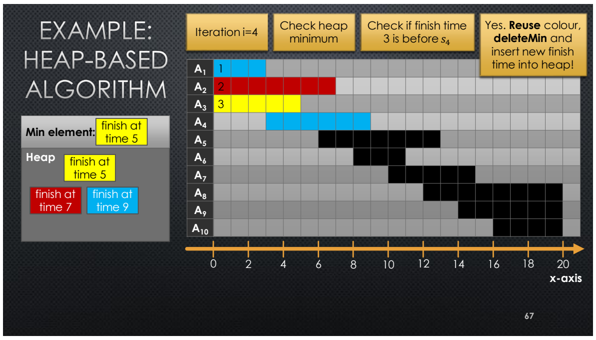

EXAMPLE: HEAP-BASED ALGORITHM
Iteration i=2
Check heap minimum
Check if finish time 3 is before s_2
No. New colour!
Min element: finish at time 3
Heap
finish at time 3
finish at time 7
1
2
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
0 2 4 6 8 10 12 14 16 18 20
x-axis
62

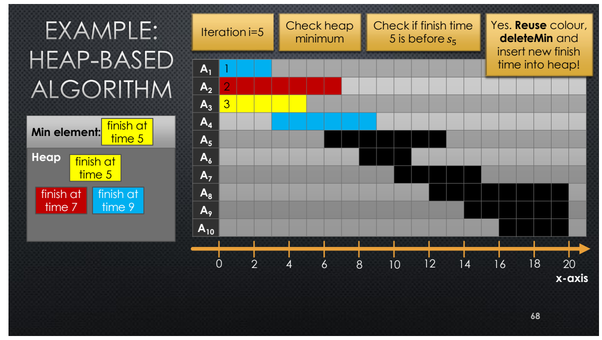

EXAMPLE: HEAP-BASED ALGORITHM
Iteration i=3
Check heap minimum
Check if finish time 3 is before s_3
No. New colour!
Min element: finish at time 3
Heap
finish at time 3
finish at time 7
1
2
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
0 2 4 6 8 10 12 14 16 18 20
x-axis
63

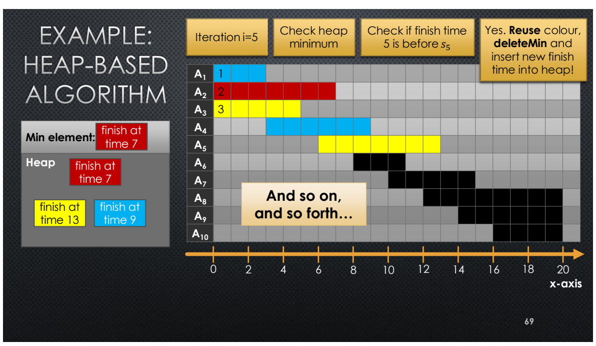

EXAMPLE: HEAP-BASED ALGORITHM
Iteration i=3
Check heap minimum
Check if finish time 3 is before s_3
No. New colour!
Min element: finish at time 3
Heap
finish at time 3
finish at time 7
finish at time 5
1
2
3
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
0 2 4 6 8 10 12 14 16 18 20
x-axis
64


EXAMPLE: HEAP-BASED ALGORITHM
Iteration i=4
Check heap minimum
Check if finish time 3 is before s_4
Yes. **Reuse** colour, **deleteMin** and **insert** new finish time into heap!
Min element: finish at time 3
Heap
finish at time 3
finish at time 7
finish at time 5
1
2
3
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
0 2 4 6 8 10 12 14 16 18 20
x-axis
65


EXAMPLE: HEAP-BASED ALGORITHM
Iteration i=4
Check heap minimum
Check if finish time 3 is before s_4
Yes. **Reuse** colour, **deleteMin** and insert new finish time into heap!
Min element: finish at time 5
Heap
finish at time 5
finish at time 7
1
2
3
A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 A_9 A_{10}
0 2 4 6 8 10 12 14 16 18 20
x-axis
66








```
Preprocess(A[1..n])
    sort A by increasing start time
    let s[1..n] be the start times in A
    let f[1..n] be the finish times in A
    return GreedyIntervalColouring(s, f)

GreedyIntervalColouring(s[1..n], f[1..n])
    d = 1
    colour[1] = 1
    h = new minPQ
    h.insert([f[1],colour[1]])

    for i = 2..n
        (fc, c) = h.min()
        if fc <= s[i] then
            h.deleteMin()
            colour[i] = c
        else
            d++
            colour[i] = d
        h.insert([f[i], colour[i]])

    return d
```

$O(\log S)$ where $S = \text{size(priority queue)}$

$O(1)$

$O(1)$

$O(\log D)$

$O(\log D)$

Total $\Theta(n \log n) + \Theta(n \log D)$

Since $n \geq D$, $\Theta(n \log n)$

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DYNAMIC PROGRAMMING

What?

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—Richard Bellman, Eye of the Hurricane: An Autobiography (1984, excerpts from page 159)

Where did the name, dynamic programming, come from? The 1950s were not good years for mathematical research.

We had a very interesting gentleman in Washington named Wilson. He was Secretary of Defense, and he actually had a pathological fear and hatred of the word "research"… He would turn red, and he would get violent if people used the term research in his presence. You can imagine how he felt, then, about the term mathematical.

I felt I had to do something to shield Wilson … from the fact that I was really doing mathematics… What title, what name, could I choose? In the first place I was interested in planning, in decision making, in thinking. But planning, is not a good word for various reasons. I decided therefore to use the word "programming." I wanted to get across the idea that this was "dynamic," this was multistage, this was time-varying. I thought, let's kill two birds with one stone.

I thought dynamic programming was a good name.
It was something not even a Congressman could object to.

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COMPUTING FIBONACCI NUMBERS **INEFFICIENTLY**

A **TOY** EXAMPLE TO COMPARE D&C TO DYNAMIC PROGRAMMING

```
BadFib(n)
   if n == 0 or n == 1 then return n
   return BadFib(n-1) + BadFib(n-2)
```

RUNTIME

- **In unit cost model**
  - **(UNREALISTIC!)**

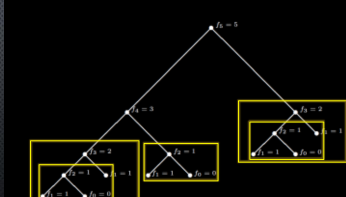
```
BadFib(n)
   if n == 0 or n == 1 then return n
   return BadFib(n-1) + BadFib(n-2)
```

- $T(n) = T(n-1) + T(n-2) + O(1)$ — This O(1) would change in the bit complexity model
  - $T(n) \geq 2T(n-2) + O(1)$
  - $T(n) \leq 2T(n-1) + O(1)$
- n/2 levels of recursion for the first expression
- n levels for the second expression
- Work doubles at each level
- $T(n)$ is certainly in $\Omega(2^{n/2})$ **and** $O(2^n)$

WHY IS THIS SO SLOW?

- Subproblems have LOTS of overlap!
- Every subtree on the right appears on the left
- … recursively …
- Each subtree is computed **exponentially** often in its depth

The Recursion Tree to Evaluate f_5:



This **overlap** suggests dynamic programming may be able to help!

Designing Dynamic Programming Algorithms for Optimization Problems

(Optimal) Recursive Structure
Examine the structure of an optimal solution to a problem instance I, and determine if an optimal solution for I can be expressed in terms of optimal solutions to certain **subproblems** of I.

Define Subproblems
Define a set of subproblems $\mathcal{S}(I)$ of the instance I, the solution of which enables the optimal solution of I to be computed. I will be the last or largest instance in the set $\mathcal{S}(I)$.

Designing Dynamic Programming Algorithms (cont.)

Recurrence Relation
Derive a **recurrence relation** on the optimal solutions to the instances in $\mathcal{S}(I)$. This recurrence relation should be completely specified in terms of optimal solutions to (smaller) instances in $\mathcal{S}(I)$ and/or base cases.

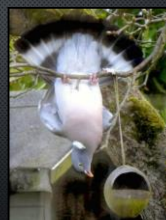
Compute Optimal Solutions
Compute the optimal solutions to all the instances in $\mathcal{S}(I)$. Compute these solutions using the recurrence relation in a **bottom-up** fashion, filling in a table of values containing these optimal solutions. Whenever a particular table entry is filled in using the recurrence relation, the optimal solutions of relevant subproblems can be looked up in the table (they have been computed already). The final table entry is the solution to I.

SOLVING FIB USING DYNAMIC PROGRAMMING

- (Optimal) Recursive Structure
  - Solution to n-th Fibonacci number $f(n)$ can be expressed as the addition of smaller Fibonacci numbers
  - No notion of **optimality** for this particular problem
- Define Subproblems
  - The set subproblems that will be combined to obtain $Fib(n)$ is $\{Fib(n-1), Fib(n-2)\}$
  - $S(I) = \{Fib(0), Fib(1), \ldots, Fib(n)\}$
- Recurrence Relation $f(n) = \begin{cases} f(n-1) + f(n-2) & : i \geq 2 \\ 1 & : i = 1 \\ 0 & : i = 0 \end{cases}$
- Computing (Optimal) Solutions
  - Create **table f[1..n]** and compute its entries **"bottom-up"**

FILLING THE TABLE "BOTTOM-UP"

- Key idea:
  - When computing a table entry
  - Must have **already computed** the **entries** it depends on!
- Dependencies
  - Extract directly from recurrence
  - Entry n depends on n-1 and n-2
- **Computing entries in order 1..n** guarantees n-1 and n-2 are already computed when we compute n



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DP SOLUTION

```
FibDP(n)
    f = new array of size n

    f[0] = 0
    f[1] = 1

    for i = 2..n
        f[i] = f[i-1] + f[i-2]

    return f[n]
```

```
FibDP(n)
    fi2 = 0
    fi1 = 1

    for i = 2..n
        temp = fi

        fi = fi1 + fi2

        fi2 = fi1
        fi1 = temp

    return fi
```

represents f[i-2]

represents f[i-1]

Save f[i] before overwriting it (so its value can be stored in f[i-1] later)

Contains f[n]

- **Space saving** optimization:
  - We never look at f[i-3] or earlier
  - Can make do with a few variables instead of a table

This is still considered to be dynamic programming… We've just optimized out the table.

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CORRECTNESS

- **Step 1**
  - Prove that when computing a table entry, dependent entries are **already computed**
- **Step 2** (similar to D&C)
  - Suppose subproblems are solved correctly (optimally)
  - Prove these (optimal) subsolutions are combined into a(n optimal) solution

Order 0..n means i-1 and i-2 are already computed when we compute i

Suppose f[i-1] and f[i-2] are the (i-1)th and (i-2)th Fib #s

Then prove f[i] = the n-th Fib #

```
FibDP(n)
    f = new array of size n

    f[0] = 0
    f[1] = 1

    for i = 2..n
        f[i] = f[i-1] + f[i-2]

    return f[n]
```

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MODEL OF COMPUTATION FOR RUNTIME

- Unit cost model is **not very realistic** for this problem, because Fibonacci numbers grow quickly
  - F[10]=55
  - F[100]=354224848179261915075
  - F[300]=222232244629420445529739893461909967206666939096499764990979600
  - Value of F[n] is exponential in n: $f_n \in \Theta(\phi^n)$ *where* $\phi \cong 1.6$
  - ϕ^n needs $\log(\phi^n)$ bits to store it
  - So F[n] needs $\Theta(n)$ bits to store!

But let's use unit cost anyway for simplicity

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RUNNING TIME (UNIT COST)

- $T(n) \in \Theta(n)$

• $T(n) \in \Theta(n)$

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A BRIEF ASIDE

- Is this **linear runtime**?
- NO! This is **"a linear function of n"**
- When we say **"linear runtime"** we mean **"a linear function of the input size"**
- What is the input size S?
  - The input is the number n.
  - How many bits does it take to store n? $O(\log n)$
  - **So** $S = \log n$ **bits**

Express T(n) as a function of the input size S (in bits)

$T(n) \in \Theta(n)$
$2^S = 2^{\log n} = n$
So $T(n) \in \Theta(2^S)$

This algorithm is **exponential in the input size!**

… but still exponentially faster than 2^n

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UNLIKELY THAT WE GET HERE

ROD CUTTING

A "REAL" DYNAMIC PROGRAMMING EXAMPLE

- Input:
  - n: length of rod
  - $p_1, \dots, p_n$: p_i = price of a rod of length i
- Output:
  - Max **income** possible by cutting the rod of length n into any number of **integer** pieces (maybe **no** cuts)

$n = 4$

length i	1	2	3	4
price p_i	1	5	8	9

All ways of cutting a rod of length 4

Example output: 10

DYNAMIC PROGRAMMING APPROACH

- High level idea **(can just think recursively to start)**
  - Given a rod of length n
  - Either make no cuts, or make a cut and **recurse** on the remaining parts

  - **Where** should we cut?

DYNAMIC PROGRAMMING APPROACH

- Try **all ways** of making that cut
  - I.e., try a cut at positions $1, 2, \dots, n-1$
  - In each case, recurse on two rods $[0, i]$ and $[i, n]$
- Take the max income over **all possibilities** (each i / no cut)

Optimal substructure: Max income from two rods w/sizes i and $n-i$

… is max income we can get from the rod size i

+ max income we can get from the rod size $n-i$

RECURRENCE RELATION

Critical step! Must define what M(k) means, semantically!

- Define $M(k)$ = maximum income for rod of length k
- If we do **not** cut the rod, max income is p_k
- If we **do** cut a rod **at** i

  - max income is $M(i) + M(k-i)$
  - Want to maximize this **over all** i
    - $max_i\{M(i) + M(k-i)\}$ (for $0 < i < k$)
- $M(k) = \max\{p_k, \max_{1 \le i \le k-1}\{M(i) + M(k-i)\}\}$

COMPUTING SOLUTIONS BOTTOM-UP

- **Recurrence:** $M(k) = max\{p_k, \max_{1 \le i \le k-1}\{M(i) + M(k-i)\}\}$
- Compute **table** of solutions: $\mathrm{M}[1..n]$

- Dependencies: **entry** k depends on
  - $M[i]$ → $M[1..(k-1)]$
  - $M[k-i]$ → $M[1..(k-1)]$
  - All of these dependencies are $< k$
- So we can fill in the table entries in order $1..n$

Recall, semantically, $M(k)$ = maximum income for rod of length k

Recurrence: $M(k) = max\{p_k, \max_{1 \le i \le k-1}\{M(i) + M(k-i)\}\}$

```
RodCutting(n, p[1..n])
    M = new array[1..n]

    // compute each entry M[k]
    for k = 1..n
        M[k] = p[k] // current best = no cuts

        // try each cut in 1..(k-1)
        for i = 1..(k-1)
            M[k] = max(M[k], M[i] + M[k-i])

    return M[n]
```

Time complexity (unit cost)? $\Theta(n^2)$

Aside: Is this a "quadratic time" algorithm?

Exercise: devise an even simpler DP solution (hint: try "recursing" only once)

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MISCELLANEOUS TIPS

- Building a table of results bottom-up is what makes an algorithm DP
- There is a similar concept called **memoization**
  - But, for the purposes of this course, we want to see bottom-up table filling!
- Base cases are **critical**
  - They often completely determine the answer
  - Try setting f[0]=f[1]=0 in FibDP…

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