CS 341: Algorithms

Lecture 6: Greedy algorithms, continued

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based on lecture notes by many other CS341 instructors

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Minimizing lateness

The problem

Input:

- jobs $J_1, \ldots, J_n$ with processing times $t(1), \ldots, t(n)$ and deadlines $d(1), \ldots, d(n)$
- can only do one thing at a time

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Output:

- a **scheduling** of the jobs which **minimizes maximal lateness**
  - job J_i starts at time $s(i)$ (TBD) and finishes at $f(i) = s(i) + t(i)$
  - if $f(i) \geq d(i)$, lateness $\ell(i) = f(i) - d(i)$, otherwise $\mathbf{0}$
- maximal lateness $= \max_i \ell(i)$

Example: 3 jobs

- **prepare my slides:** need $\boldsymbol{t(1) = 4}$ hours, deadline $\boldsymbol{d(1) = 2}$ hours
- **write solutions to assignments:** need $\boldsymbol{t(2) = 6}$ hours, deadline $\boldsymbol{d(2) = 1}$ hour
- **bake a panettone:** need $\boldsymbol{t(3) = 10}$ hours, deadline $\boldsymbol{d(3) = 24}$ hours

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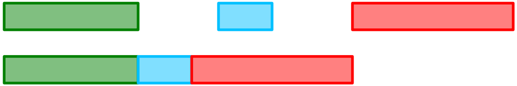
- **1, then 2, then 3:** latenesses $[2, 9, 0]$
- **2, then 1, then 3:** latenesses $[8, 5, 0]$ (optimal)

No breaks

Observation:

- if a scheduling has **idle time**, we can improve it by removing the breaks



- so the optimal has no idle time, and is given by a **permutation** of $[1, \ldots, n]$

A few attempts

Attempt 1:

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- do jobs with little slack first $\qquad$ slack $= d(i) - t(i)$
- **no**

  take $\boldsymbol{t(1) = 8}$, $\boldsymbol{d(1) = 10}$ so $\boldsymbol{s(1) = 2}$ and $\boldsymbol{t(1) = 2}$, $\boldsymbol{d(2) = 5}$ so $\boldsymbol{s(2) = 3}$

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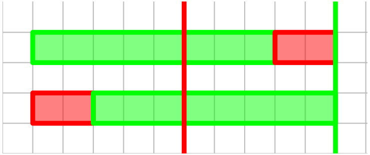

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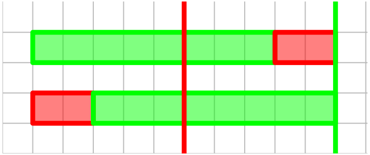


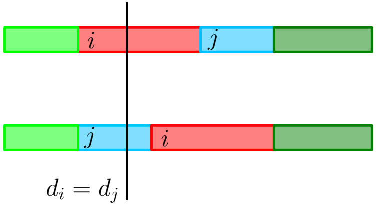
Attempt 3:

- do jobs in non-decreasing deadline order

Non-uniqueness

Observation:

- if $d(i) = d(j)$, the orderings $[\ldots, \boldsymbol{i}, \boldsymbol{j}, \ldots]$ and $[\ldots, \boldsymbol{j}, \boldsymbol{i}, \ldots]$ have the same max-lateness (because the second job is the latest)



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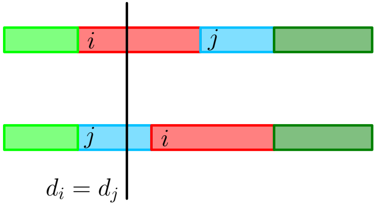

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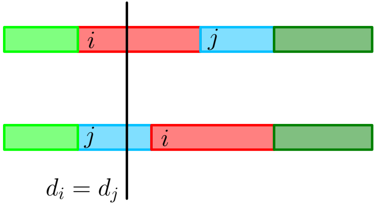

- so **all** orderings in non-decreasing deadline order have the same max-lateness

Definition:

- take a permutation $L = [e_1, \ldots, e_n]$ of $[1, \ldots, n]$
- **inversion**: a pair (i, j) with $i < j$ and $d(e_i) > d(e_j)$ (= an inversion in $[d(e_1), \ldots, d(e_n)]$ in the sense of lecture 3)
- no inversion $\iff$ L in non-decreasing deadline order

Correctness: exchange argument

- let $L = [e_1, \ldots, e_n]$ be **any** permutation of $[1, \ldots, n]$
- suppose that L is **not** in non-decreasing order of deadlines
- want: $\mathbf{max\_lateness}(\boldsymbol{L}) \geq \mathbf{max\_lateness}(\boldsymbol{L}_{\mathbf{greedy}})$

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- want: $\mathbf{max\_lateness}(\boldsymbol{L}) \geq \mathbf{max\_lateness}(\boldsymbol{L}_{\mathbf{greedy}})$
- there exists i such that $\boldsymbol{d(e_i) > d(e_{i+1})}$.
- now, **swap $\boldsymbol{e_i}$ and $\boldsymbol{e_{i+1}}$ to get a permutation $\boldsymbol{L'}$.** What about $\mathrm{max\_lateness}(L')$?

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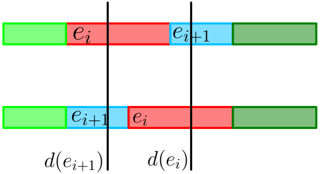
- now, **swap e_i and e_{i+1} to get a permutation L'.** What about max_lateness(L')?
- the lateness of e_{i+1} cannot increase (because we do e_{i+1} earlier than before), so at most max_lateness(L)
- the **new** lateness of e_i is **at most** the **old** lateness of e_{i+1}, so at most max_lateness(L)



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- nothing else changes, so max_lateness(L') $\leq$ max_lateness(L)
- we removed an inversion
- keep going: after at most $n(n-1)/2$ iterations, we have L_{ord} with **no inversion** and such that $\mathbf{max\_lateness}(\boldsymbol{L}_{\mathbf{ord}}) \leq \mathbf{max\_lateness}(\boldsymbol{L})$
- we saw that $\mathbf{max\_lateness}(\boldsymbol{L}_{\mathbf{ord}}) = \mathbf{max\_lateness}(\boldsymbol{L}_{\mathbf{greedy}})$

Fractional knapsack

The problem

Input:

- items $I_1, \ldots, I_n$ with weights $\boldsymbol{w_1, \ldots, w_n}$ and positive values $\boldsymbol{v_1, \ldots, v_n}$
- a capacity W

Output:

- **fractions** $E = e_1, \ldots, e_n$ such that
  - $0 \leq e_j \leq 1$ for all j
  - $e_1 w_1 + \cdots + e_n w_n \leq W$
  - $e_1 v_1 + \cdots + e_n v_n$ maximal

Example:

- $\boldsymbol{w_1 = 10, v_1 = 60}$, $\boldsymbol{w_2 = 30, v_2 = 90}$, $\boldsymbol{w_3 = 20, v_3 = 100}$
- $\boldsymbol{W = 50}$

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Example:

- $\boldsymbol{w_1 = 10, v_1 = 60}$, $\boldsymbol{w_2 = 30, v_2 = 90}$, $\boldsymbol{w_3 = 20, v_3 = 100}$
- $\boldsymbol{W = 50}$
- optimal is $\boldsymbol{e_1 = 1}$, $\boldsymbol{e_2 = 2/3}$, $\boldsymbol{e_3 = 1}$, total value **220**

The problem

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Remark:

- 0/1-**version:** $e_j \in \{0, 1\}$ for all j
- dynamic programming

The knapsack should be full

Remark:

- if $\sum_i w_i < W$, just take all $e_i = 1$
- so assume $\boldsymbol{\sum_i w_i \geq W}$

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Observation:

- suppose we have an assignment with $\sum_i e_i w_i < W$
- then some e_i must be **less than** 1
- so we can increase the value by increasing this e_i

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- if $\sum_i w_i < W$, just take all $e_i = 1$
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Observation:

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- then some e_i must be **less than** 1
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Consequence:

- it is enough to consider assignments with $\boldsymbol{\sum_i e_i w_i = W}$

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Attempt 1:

- pack with items in **decreasing value** v_i

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- **no**, previous example (we get $[0, 1, 1]$ with total value **190**)

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Attempt 2:

- pack with items in **increasing weight** w_i

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Attempt 2:

- pack with items in **increasing weight** w_i
- **no**: $\boldsymbol{W = 10}$, $\boldsymbol{w_1 = 10, v_1 = 100}$, $\boldsymbol{w_2 = 5, v_2 = 1}$

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Attempt 1:

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- pack with items in **increasing weight** w_i
- **no**: $\boldsymbol{W = 10}$, $\boldsymbol{w_1 = 10, v_1 = 100}$, $\boldsymbol{w_2 = 5, v_2 = 1}$

Attempt 3:

- pack with items in **non-increasing "value per kilo"** $\boldsymbol{v_i/w_i}$
- first example $\mathbf{[6, 3, 5]}$, second example $\mathbf{[10, 1/5]}$

Pseudo-code

GreedyKnapsack$(\boldsymbol{v}, \boldsymbol{w}, W)$

$E \leftarrow [0, \ldots, 0]$
sort items by non-increasing order of v_i/w_i
for $k = 1, \ldots, n$ **do**
    if $w_k < W$ **then**
        $E[k] \leftarrow 1$
        $W \leftarrow W - w_k$
    else
        $E[k] \leftarrow W/w_k$
        return

Remark: output is $S = [1, \ldots, 1, e_k, 0, \ldots, 0]$

Runtime: $O(n \log(n))$

Correctness: exchange argument

- let $E = [e_1, \ldots, e_n]$ be the **output**, with $\boldsymbol{\sum e_i w_i = W}$
- let $S = [s_1, \ldots, s_n]$ be **any assignment**, with $\boldsymbol{\sum s_i w_i = W}$
- assume that $\boldsymbol{S \neq E}$, want $\mathbf{value}(\boldsymbol{E}) \geq \mathbf{value}(\boldsymbol{S})$

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- let i be the **first** index with $\boldsymbol{e_i \neq s_i}$
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- set $\boldsymbol{s'_i = s_i + \alpha / w_i}$ and $\boldsymbol{s'_j = s_j - \alpha / w_j}$, for α TBD > 0, all other $\boldsymbol{s'_k = s_k}$
- in any case, $\boldsymbol{\sum s'_i w_i = W}$ and $\textbf{value}(\boldsymbol{S'}) \geq \textbf{value}(\boldsymbol{S})$

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- because their weights are the same, there is $j > i$ with $s_j > e_j$
- set $s'_i = s_i + \alpha / w_i$ and $s'_j = s_j - \alpha / w_j$, for α TBD > 0, all other $s'_k = s_k$
- in any case, $\sum s'_i w_i = W$ and $\textbf{value}(S') \geq \textbf{value}(S)$
- choose **the first** α such that **either** $s'_i = e_i$ **or** $s'_j = e_j$

$$\alpha = \min(w_i(e_i - s_i), w_j(s_j - e_j))$$

- we found S' with **one more common entry** with E, and $\textbf{value}(S') \geq \textbf{value}(S)$
- if $S' \neq E$, repeat, $\dots$, until $S''''^{\cdots} = E$