CS 341: Algorithms

Lecture 10: Graphs, breadth first search

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based on lecture notes by many other CS341 instructors

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Definitions

Undirected graphs

Definition, notation: a graph $\boldsymbol{G}$ is pair $\boldsymbol{(V, E)}$:

- V is a finite set, whose elements are called **vertices**
  (we often take $V = \{1, \ldots, n\}$)
- E is a finite set, whose elements are **sets of two (distinct) vertices**, and are called **edges**.

Convention: $\boldsymbol{n}$ is the number of vertices, $\boldsymbol{m}$ is the number of edges.

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Data structures:

- **adjacency lists:** an array $A[1..n]$ s.t. $A[v]$ is the **linked list** of all edges connected to v.
  $\boldsymbol{2m}$ list cells, total size $\boldsymbol{\Theta(n + m)}$, but testing if an edge exists is not $O(1)$

- **adjacency matrix:** a $(0, 1)$ matrix M of size $n \times n$, with $M[v, w] = 1$ iff $\{v, w\}$ is an edge.
  size $\boldsymbol{\Theta(n^2)}$, but testing if an edge exists is $O(1)$

Connected graphs, path, cycles, trees

Definition:

- **walk:** a sequence $\boldsymbol{v_0, \ldots, v_k}$ of vertices, with $\{v_i, v_{i+1}\}$ in E for $i = 0, \ldots, k-1$.
  length = number of steps = k ($k = 0$ is OK)

- **path:** a walk with distinct vertices

- **connected graph:** $G = (V, E)$ such that for all v, w in V, there is a path/walk $v \rightsquigarrow w$

- **cycle:** a walk $\boldsymbol{v_0, \ldots, v_k, v_0}$ with $\boldsymbol{k \geq 2}$ and v_i's distinct
  ($k = 1$ would be v_0, v_1, v_0, this is not a cycle)

- **tree:** a connected graph with no cycle
  (equiv: a connected graph with $\boldsymbol{m = n - 1}$)
  (equiv: a graph with $m = n - 1$ and no cycle)

- **rooted tree:** a tree with a special vertex called **root**

Breadth-first search

Breadth-first exploration of a graph

BFS(G, s)

G: a graph with n vertices, given by adjacency lists

s: a vertex from G

```
let Q be an empty queue
let visited be an array of size n, with all entries set to false
enqueue(s, Q)
visited[s] ← true
while Q not empty do
    v ← dequeue(Q)
    for all w neighbours of v do
        if visited[w] is false
            enqueue(w, Q)
            visited[w] ← true
```

Runtime

Anaysis:

- each vertex is enqueued at most once
- so each vertex is dequeued at most once $\quad$ $\boldsymbol{O(n)}$ for steps 5-6
- so each adjacency list is read at most once

For all v, write d_v = number of neighbours of v = length of $A[v]$ = **degree** of v.

Then total cost at step 7 is

$$O\left(\sum_v d_v\right) = O(m)$$

cf. the adjacency array A has $2m$ cells

Total: $\boldsymbol{O(n+m)}$

Correctness 1

Claim

For all vertices v, if visited$[v]$ is true at the end, there is a walk $s \rightsquigarrow v$ in G

Proof. Let $s = v_0, \ldots, v_K$ be the vertices for which visited is set to true, in this order. We prove: **for all $\boldsymbol{i}$, there is a walk $\boldsymbol{s \rightsquigarrow v_i}$** by induction.

- OK for $i = 0$
- suppose true for $v_0, \ldots, v_{i-1}$.

  when visited$[v_i]$ is set to true, we are examining the neighbours of a certain v_j, $\boldsymbol{j < i}$.

  by assumption, there is a walk $s \rightsquigarrow v_j$

  because $\{v_j, v_i\}$ is in E, there is a walk $s \rightsquigarrow v_i$

Correctness 2

Claim

For all vertices v, if there is a walk $s \rightsquigarrow v$ in G, $\mathsf{visited}[v]$ is true at the end

Proof. Let $v_0 = s, \ldots, v_k = v$ be a walk $s \rightsquigarrow v$. We prove $\mathsf{visited}[v_i]$ is true for all i by induction.

- $\mathsf{visited}[v_0]$ is true
- if $\mathsf{visited}[v_i]$ is true, we will examine all neighbours v of v_i

  so at the end of Step 7, all $\mathsf{visited}[v]$ will be true, for v neighbour of v_i

  in particular, $\mathsf{visited}[v_{i+1}]$ will be true

Correctness

Summary

For all vertices v, there is a walk $s \rightsquigarrow v$ in G **if and only if** visited$[v]$ is true at the end

Applications

- testing if there is a walk $s \rightsquigarrow v$
- testing if G is connected
- a spanning tree, if G is connected (tree that covers all vertices)

in $\boldsymbol{O(n+m)}$.

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Exercise

For a connected graph, $n - 1 \leq m$, so $O(n+m) = O(m)$.

Keeping track of parents and levels

```
BFS(G, s)
    let Q be an empty queue
    let parent be an array of size n, with all entries set to NIL
    let level be an array of size n, with all entries set to ∞
    enqueue(s, Q)
    parent[s] ← s
    level[s] ← 0
    while Q not empty do
        v ← dequeue(Q)
        for all w neighbours of v do
            if parent[w] is NIL
                enqueue(w, Q)
                parent[w] ← v
                level[w] ← level[v] + 1
```

BFS tree

Definition: the **BFS tree** T is the subgraph made of:

- all w such that $\mathsf{parent}[w] \neq \mathbf{NIL}$.
- all edges $\{w, \mathsf{parent}[w]\}$, for w as above (except $w = s$)

Claim

The BFS tree T is a tree

Proof: T connected, n_s vertices, $n_s - 1$ edges
(n_s is the number of vertices reachable from s)

Remark: we make it a **rooted** tree by choosing s as root

Shortest paths from the BFS tree

Claim

If there is a walk $s \rightsquigarrow v$ in G then $\mathsf{level}[v] = \mathrm{dist}(s, v)$

Observation 1:

- $\mathrm{dist}(s, v) =$ length of the shortest path $s \rightsquigarrow v$
- so $\mathrm{dist}(s, v) \leq \mathsf{level}[v]$
- want: $\boldsymbol{\mathsf{level}[v] \leq \mathrm{dist}(s, v)}$

Observation 2: the levels in the queue are always of the form

$$[\boldsymbol{\ell}, \ldots, \boldsymbol{\ell}] \text{ or } [\boldsymbol{\ell}, \ldots, \boldsymbol{\ell}, \boldsymbol{\ell} + \mathbf{1}, \ldots, \boldsymbol{\ell} + \mathbf{1}]$$

so **if we dequeue v before w, $\mathsf{level}[v] \leq \mathsf{level}[w]$**

Shortest paths from the BFS tree

Claim

If there is a walk $s \rightsquigarrow v$ in G then $\mathsf{level}[v] = \mathrm{dist}(s, v)$

Proof

- take a shortest path $\boldsymbol{s = v_0 \to v_1 \to v_2 \to \cdots \to v_k = v}$ $\qquad k = \mathrm{dist}(s, v)$
- prove $\mathsf{level}[v_i] \leq i$ for all i by induction $\qquad i = k$ gives $\mathsf{level}[v] \leq \mathrm{dist}(s, v)$
- OK for $i = 0$

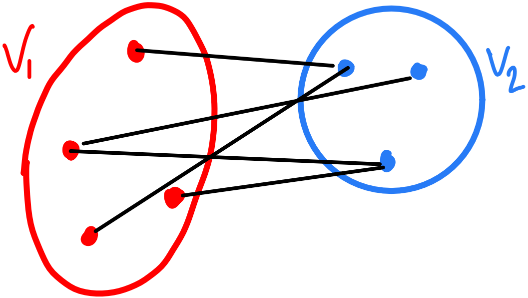
Induction step: suppose OK for $i - 1$

- the parent of v_i is either v_{i-1}, or a vertex we processed before v_{i-1}
- so in any case, $\mathsf{level}[\mathsf{parent}(v_i)] \leq \mathsf{level}(v_{i-1})$ $\qquad$ previous slide
- lhs is $\mathsf{level}(v_i) - 1$, rhs is at most $i - 1$ (induction assumption)
- done

Bipartite graphs

Definition

- a graph $G = (V, E)$ is **bipartite** if there is a partition $V = \boldsymbol{V_1} \cup \boldsymbol{V_2}$ such that all edges have **one end in** V_1 and **one end in** V_2.



Using BFS to test bipartite-ness

Claim.

Suppose G connected, run BFS from any s, and set

- V_1 = vertices with odd level
- V_2 = vertices with even level.

Then G is bipartite if and only all edges have one end in V_1 and one end in V_2 (testable in $\boldsymbol{O(n+m)}$)

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Proof. $\Longleftarrow$ obvious.

For $\Longrightarrow$, let W_1, W_2 be a bipartition. Because paths alternate between W_1, W_2:

- V_1 (= vertices with odd level) is included in W_1 (say)
- V_2 (= vertices with even level) is included in W_2

So $\boldsymbol{V_1 = W_1}$ and $\boldsymbol{V_2 = W_2}$.

Subgraphs, connected components

Definition:

- **subgraph** of $G = (V, E)$: a graph $G' = (V', E')$, where
  - $V' \subset V$
  - $E' \subset E$, with all edges E' joining vertices from V'
- **connected component** of $G = (V, E)$
  - a connected subgraph of G
  - that is not contained in a larger connected subgraph of G

Let $G_i = (V_i, E_i)$, $i = 1, \ldots, s$ be the connected components of $G = (V, E)$.

- the V_i's are a partition of V, with $\sum_i n_i = n$ $\quad\quad$ $\boldsymbol{n_i = |V_i|}$
- the E_i's are a partition of E, with $\sum_i m_i = m$ $\quad\quad$ $\boldsymbol{m_i = |E_i|}$

Computing the connected components

Idea: add an outer loop that runs BFS on successive vertices

Exercise

Fill in the details.

Complexity:

- each pass of BFS $\boldsymbol{O(n_i + m_i)}$, if $G_i(V_i, E_i)$ is the ith connected component
- total $\boldsymbol{O(n + m)}$