CS 341: Algorithms

Lecture 12: Directed graphs

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based on lecture notes by many other CS341 instructors

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Directed graphs

Directed graphs basics

Definition:

- $G = (V, E)$ as in the undirected case, with the difference that edges are **(directed)** pairs (v, w)
  - edges also called **arcs**
  - we allow **loops**, with $v = w$
- walks, paths and cycles as before; here, cycles have at least one edge
- a **directed acyclic graph** (DAG) is a directed graph with no cycle

BFS and DFS for directed graphs

The algorithms work **without any modification**.

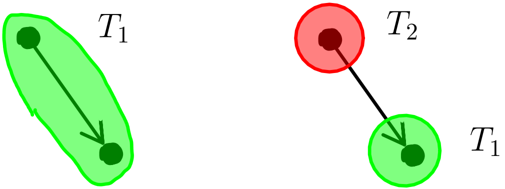
BFS: still get shortest paths

DFS: still have

- a partition of V into **vertex-disjoint trees** $T_1, \ldots, T_k$
- white path lemma (when we start exploring a vertex v, any w with an **unvisited path** $v \rightsquigarrow w$ becomes a descendant of v)
- properties of start and finish times

New for DFS:

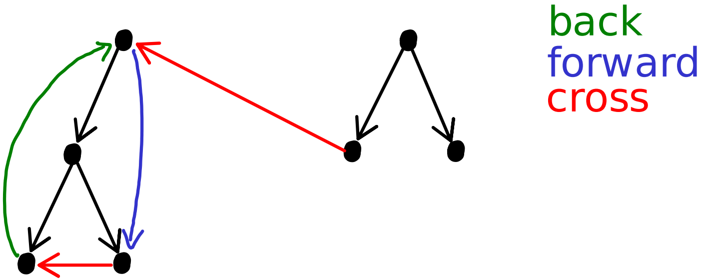
- there can exist edges connecting the trees T_i



Classification of edges

Suppose we have a DFS forest. Edges of G are one of the following:

- **tree edges**
- **back edges:** from descendant to ancestor
- **forward edges:** from ancestor to descendant (but not tree edge)
- **cross edges:** all others



(depends on the order of vertices we chose in the main DFS loop)

Classification of edges

```
explore(v)
    visited[v] = true
    start[v] = t, t++
    for all w neighbour of v do
        if visited[w] = false
            explore(w)                    (v, w) tree edge
    finish[v] = t, t++
```

If w **was visited**:

- if w not finished, (v, w) **back edge**
- else if **start**$[v] <$ **start**$[w] <$ **finish**$[w]$, (v, w) **forward edge**
- else, **start**$[w] <$ **finish**$[w] <$ **start**$[v]$, (v, w) **cross edge**

Testing acyclicity

Claim

G has a cycle if and only if there is a back edge in the DFS forest

Proof

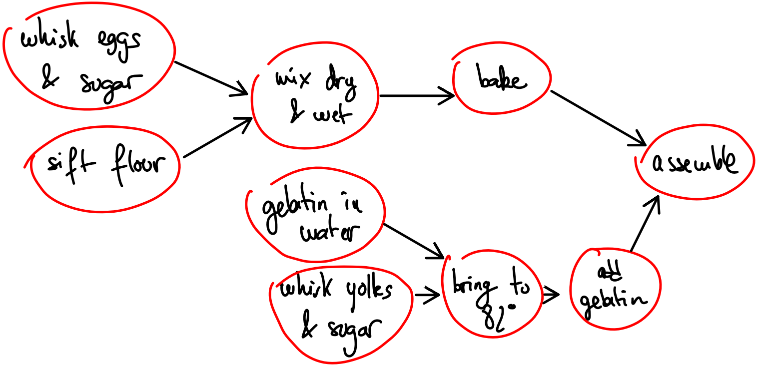
- Suppose there is a back edge (v, w). Then v is a descendant of w, so there is a path $w \rightsquigarrow v$, and a cycle $w \rightsquigarrow v \to w$
- Suppose there is a cycle $\boldsymbol{v_1, \dots, v_k, v_1}$. Up to renumbering, assume we find v_1 first in the DFS.

  Starting from v_1, we will reach v_k (white path lemma). We check the edge (v_k, v_1), and v_1 is not finished. So back edge.

Consequence: acyclicity test in $\boldsymbol{O(n+m)}$

Topological ordering

Definition: Suppose $G = (V, E)$ is a DAG. A **topological order** is an ordering $<$ of V such that for any edge (v, w), we have $v < w$.



Remark: exists a topological order **iff** G is a DAG.

From a DFS forest

Observation:

- start times do not help
- finish times do, but we have to reverse their order

From a DFS forest

Claim

Assume G is a DAG. Suppose that V is ordered using the reverse of the finishing times: $v < w \iff \textsf{finish}[w] < \textsf{finish}[v]$.

This is a topological order.

Proof. Have to prove: for any edge (v, w), $\textsf{finish}[w] < \textsf{finish}[v]$.

- if we discover v **before** w, w will become a descendant of v (white path lemma), and we finish exploring it before we finish v.
- if we discover w **before** v, because there is no path $w \rightsquigarrow v$ (G is a DAG), we will finish w before we start v.

Consequence: topological order in $O(n + m)$.

Testing strong connectivity

Definition. A directed graph G is **strongly connected** if for all v, w in G, there is a path $\boldsymbol{v \rightsquigarrow w}$ (and thus a path $\boldsymbol{w \rightsquigarrow v}$).

Algorithm:

- call **explore twice**, starting from a same vertex s
- edges reversed the second time

Correctness:

- first run tells whether for all v, there is a path $s \rightsquigarrow v$
- second one tells whether for all v, there is a path $s \rightsquigarrow v$ in the reverse graph (which is a path $v \rightsquigarrow s$ in G)

Consequence: test in $\boldsymbol{O(n+m)}$

Structure of directed graphs

Definition: a **strongly connected component** of G is

- a subgraph of G
- which is strongly connected
- but not contained in a larger strongly connected subgraph of G.

Exercise

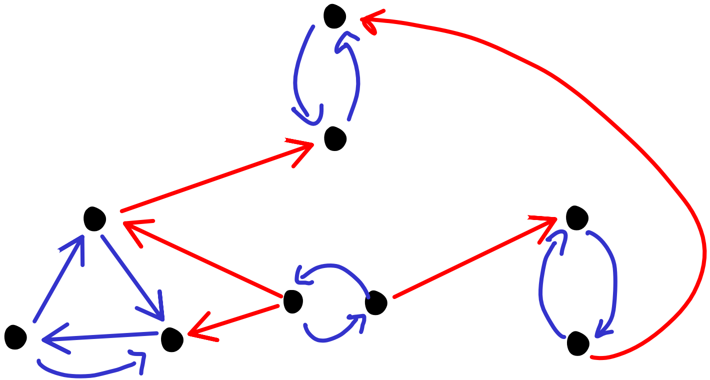
v and w are in the same strongly connected component if and only if there are paths $v \leadsto w$ and $w \leadsto v$.

Exercise

The vertices of strongly connected components form a partition of V.

Structure of directed graphs

A directed graph G can be seen as a **DAG** of disjoint **strongly connected components**.



Structure of directed graphs

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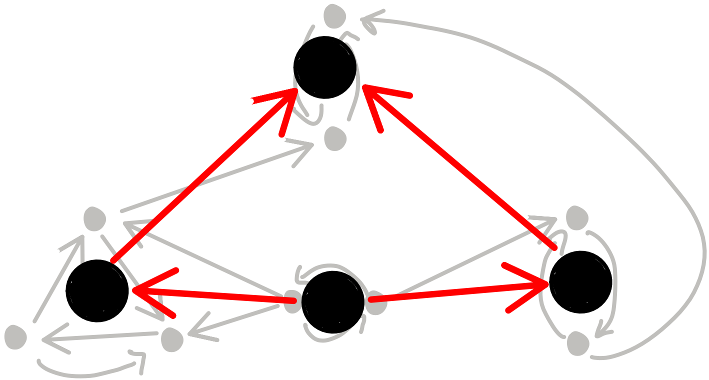

Kosaraju's algorithm for strongly connected components

Definition: for a directed graph $G = (V, E)$, the **reverse** (or **transpose**) graph $G^T = (V, E^T)$ is the graph with same vertices, and reversed edges.

SCC(G)
1. run a DFS on G and record finish times
2. run a DFS on G^T, with vertices ordered in **decreasing finish time**
3. return the trees in the DFS forest of G^T

Complexity: $\boldsymbol{O(n+m)}$ (don't forget the time to reverse G)

Exercise

check that the strongly connected components of G and G^T are the same

The idea behind the algorithm

Claim

If S and T are two strongly connected components of G and there is an edge $S \to T$,
latest finish time in S > latest finish time in T

Proof:

- if we visit a vertex in S first, all vertices in T will be its descendants
- if we visit a vertex in T first, we won't reach S before T is finished.

Consequence:

- start second run from the last-finished vertex s
- in G^T, every vertex reachable from s is in the same strongly connected component
- continue