CS 341: Algorithms

Lecture 13: Minimum spanning trees

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based on lecture notes by many other CS341 instructors

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Spanning trees

Input and output:

- $G = (V, E)$ is a **weighted, connected undirected graph**
- edges have **weights** $w(e_i)$
- a **spanning tree** is a tree with edges from E that covers all vertices
- examples: BFS tree, DFS tree

Remark: will assume $w(e_i)$ distinct, using $W(e_i) = [w(e_i), i]$ to break ties if needed

Goal:

- a spanning tree with **minimal weight**
- notation: $w(T) = \sum_{e \text{ edge in } T} w(e)$
- all weights fit in a word, as usual

Exercise

what about maximal weight spanning trees?

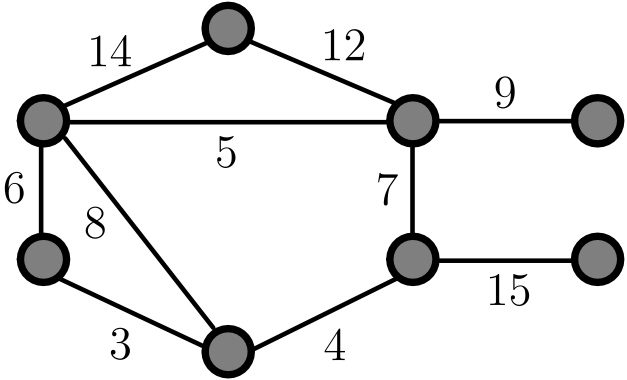
Kruskal's algorithm

Kruskal's algorithm

GreedyMST(G)

1. $F \leftarrow [\ ]$
2. sort edges by non-decreasing weight
3. **for** $k = 1, \ldots, m$ **do**
4.   **if** e_k does not create a cycle in (V, F) **then**
5.   append e_k to F
6. **return** $A = (V, F)$

Example



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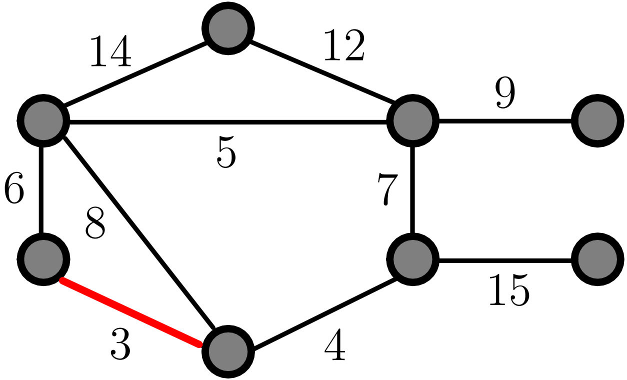

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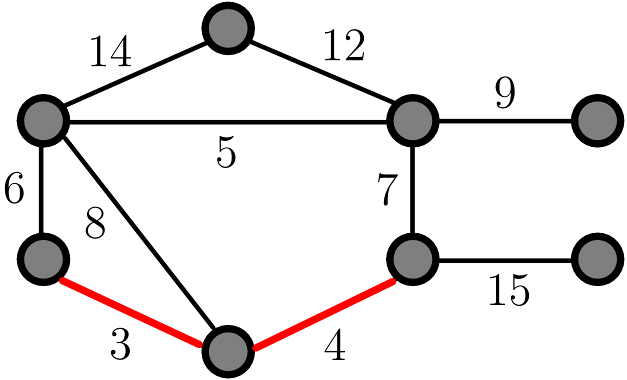

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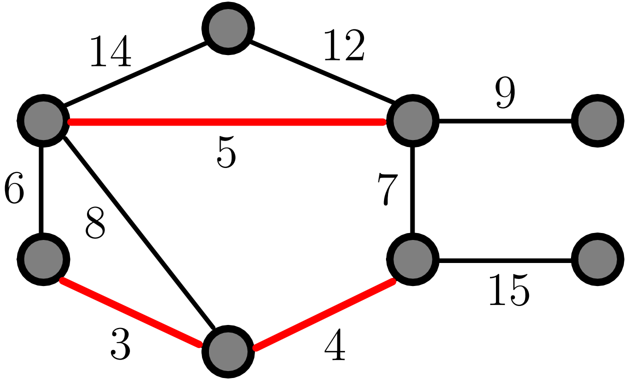

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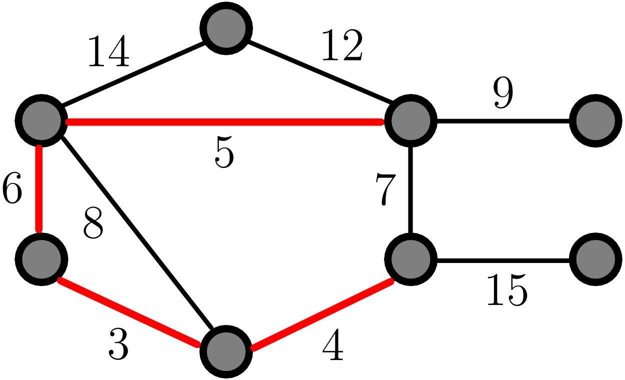

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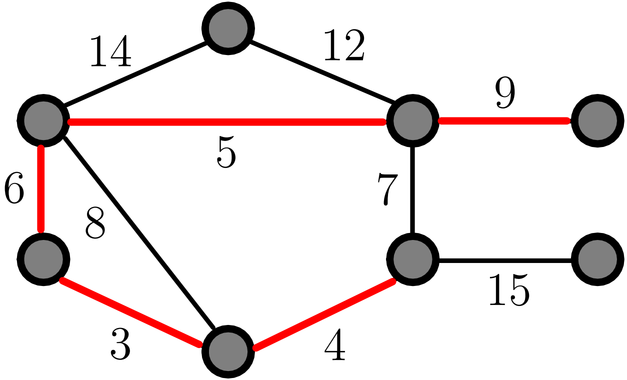

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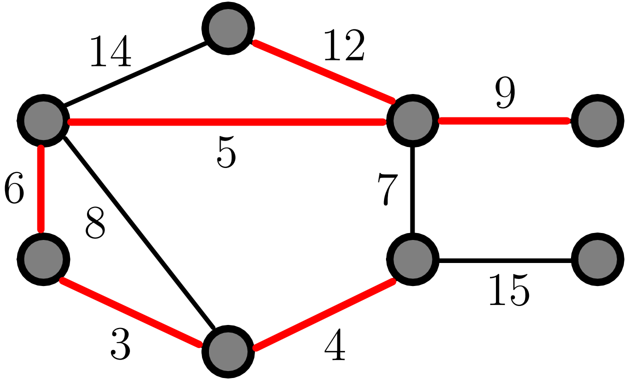

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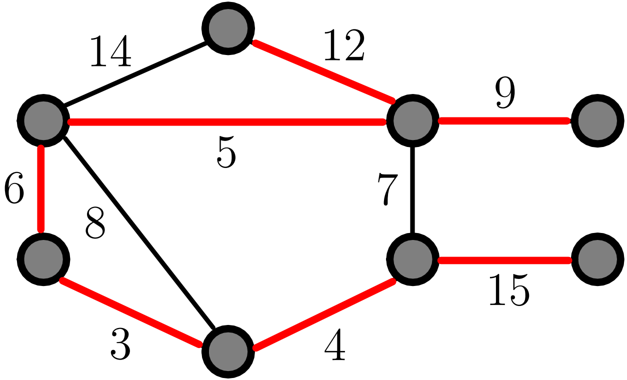


Properties of the output

Claim

The output $\boldsymbol{A = (V, F)}$ **is a spanning tree**

Proof:

- of course, A has no cycle: it is a **forest**
- suppose A is **not connected**. Then, there exists an edge e not in F, such that $(V, F \cup \{e\})$ still has no cycle (join two connected components)
- when we checked e, we did not include it
- that's because that it created a cycle with some edges already in F: **impossible**.

The cut property

Definition

cut: a partition of the vertices into sets $\boldsymbol{S}$ and $\boldsymbol{V} - \boldsymbol{S}$
cutset: the edges between $\boldsymbol{S}$ and $\boldsymbol{V} - \boldsymbol{S}$

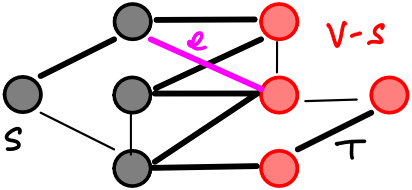
Claim

For **any** cut, the minimal weight edge in the cutset is in **any** minimum spanning tree.

Proof

For any cut, the minimal weight edge e in the cutset is in any minimum spanning tree.

- let T be a minimum spanning tree **that does not contain e**
- adding e to T creates a cycle C, and there must be an edge $e' \neq e$ in C connecting S and $V - S$



consider $\boldsymbol{T' = T - \{e'\} \cup \{e\}}$

- $w(T') < w(T)$
- but T' is still a spanning tree
  - $n - 1$ edges
  - connected: can replace edge e' by $C - \{e'\}$ to connect vertices
- contradiction

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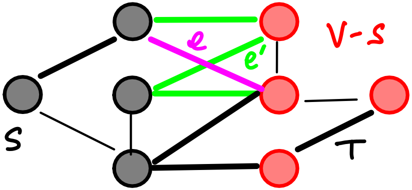

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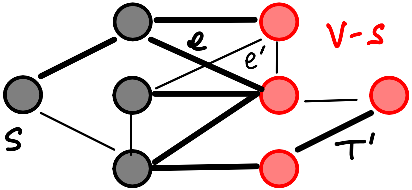

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Kruskal is optimal

Claim: every edge we add to the output is in every minimal spanning tree

Proof: consider $A = (V, F)$ the forest just before inserting $e = \{v, w\}$, let S be the vertices in the tree containing v

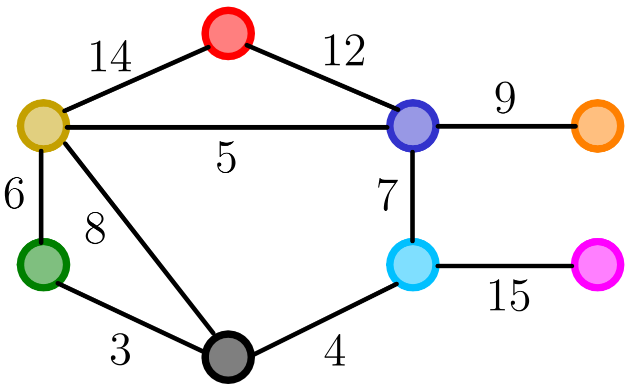
fact 1: w is in $V - S$ (otherwise, cycle)
fact 2: the other edges in the cutset have not been considered yet
(they do not create cycles, so they would have been put in F)

so e is has minimal weight in the cutset, and it is in every minimal spanning tree

Remark 1: this proves that the minimum spanning tree is unique

Remark 2: proof by exchange argument doable as well

Merging connected sets of vertices



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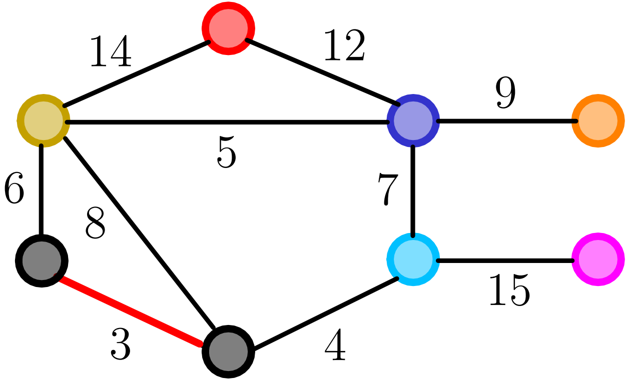

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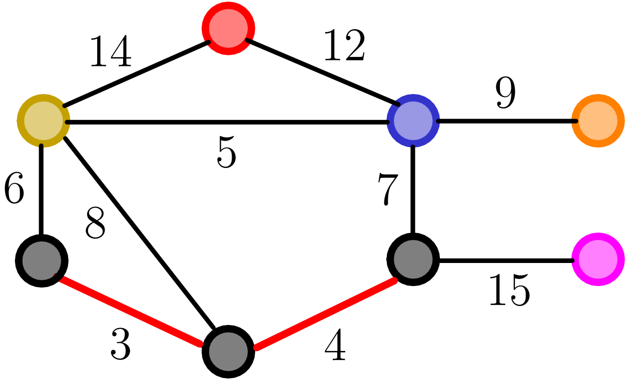

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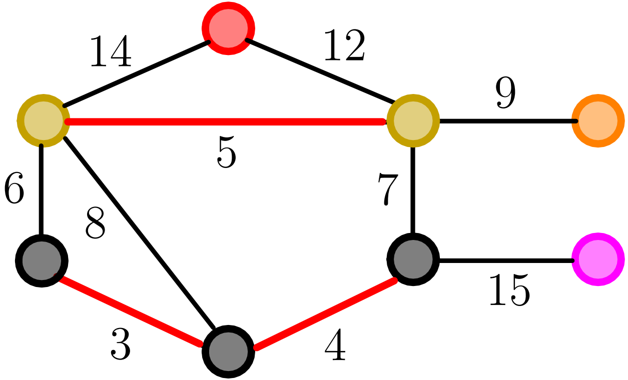

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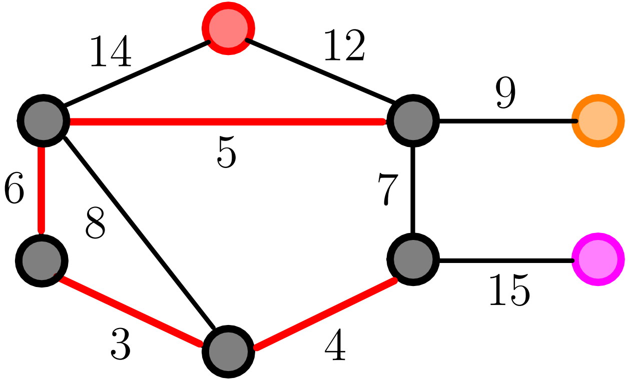

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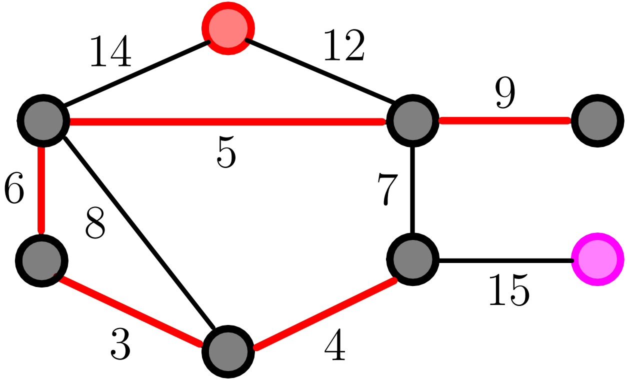

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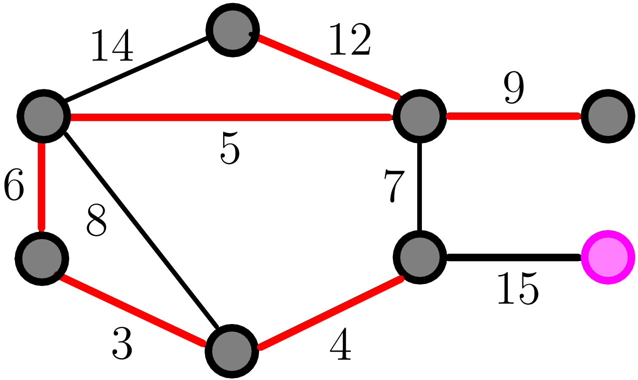

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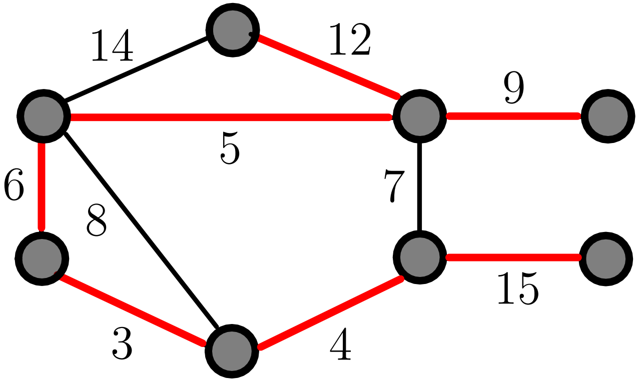


Data structures

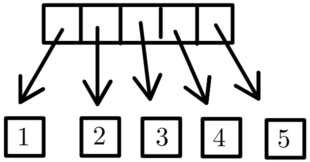
Operations on **disjoint sets of vertices:**

- **Find:** identify which set contains a given vertex
- **Union:** replace two sets by their union

GreedyMST_UnionFind(G)
1. $T \leftarrow [\,]$
2. $S \leftarrow \{\{v_1\}, \ldots, \{v_n\}\}$
3. sort edges by non-decreasing weight
4. **for** $k = 1, \ldots, m$ **do**
5. **if find**$(S, e_k.1) \neq$ **find**$(S, e_k.2)$ **then**
6. **union**$(S,$ **find**$(S, e_k.1),$ **find**$(S, e_k.2))$
7. append e_k to T

An OK solution

a data structure for union: an array U of **linked lists**



union_v1(U, s, t)
1. **while** $U[s]$ not **NULL do**
2. $U[t] \leftarrow$ new list$(U[s]$.value$, U[t])$
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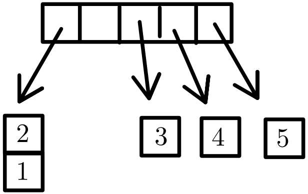

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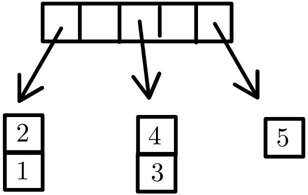

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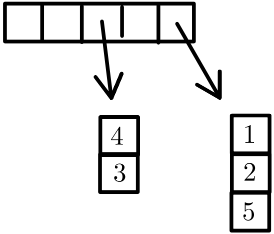

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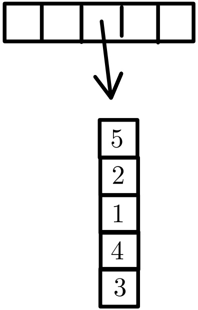

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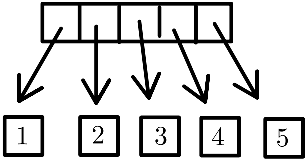
for find, use an **array of indices**, $X[i]$ = index of the set that contains i (find returns $X[i]$)



$$X = [1, 2, 3, 4, 5]$$

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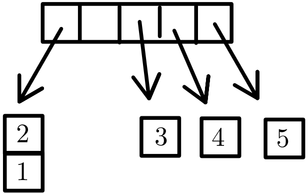

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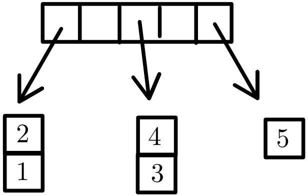

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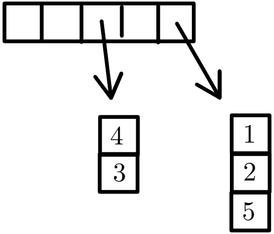

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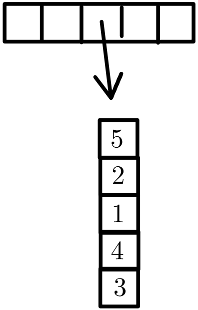

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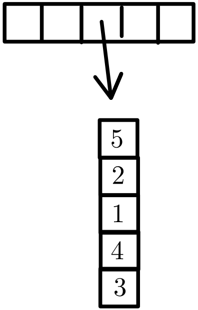

union_v2(X, U, s, t)

while $U[s]$ not **NULL do**

  $U[t] \leftarrow$ new list$(U[s]$.value$, U[t])$

  $X[U[s]$.value$] \leftarrow t$

  $U[s] \leftarrow U[s]$.next

Analysis

Worst case:

- **Find** is $O(1)$
- **Union** **traverses** one of the linked lists and **updates** corresponding entries of X.
  Worst case $\Theta(n)$

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Kruskal's algorithm:

- sorting edges $\boldsymbol{O(m\log(m))}$
- $O(m)$ **Find**
- $O(n)$ **Union**

Worst case $\boldsymbol{O(m\log(m)+n^2)}$

A simple heuristics for Union

Modified Union

- each list in U keeps track of its size
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Key observation: worst case for **one** union **still** $\Theta(n)$, but the amortized cost is better.

- for any vertex v, the size of the set containing v **at least doubles** when we update $X[v]$
- so $X[v]$ updated at most $\log(n)$ times
- so the **total** cost of union **per vertex** is $O(\log(n))$

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Conclusion: $\boldsymbol{O(n\log(n))}$ for all unions and $\boldsymbol{O(m\log(m))}$ total

Prim's algorithm

The idea

Goal

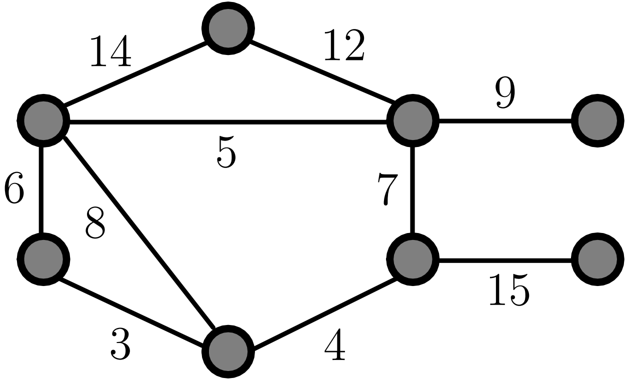
- G is an undirected graph
- $w : E \to R$ a weight function
- as before, want a **minimum weight spanning tree**

The idea:

- start from an **arbitrary source**
- **grow a tree** (connected, no cycle) edge-by-edge
- new edges chosen in a greedy manner

Growing a tree

we grow $\boldsymbol{A} = (\boldsymbol{S}, \boldsymbol{F})$ by adding the **minimal weight edge** $\boldsymbol{S} \leftrightarrow (\boldsymbol{V} - \boldsymbol{S})$



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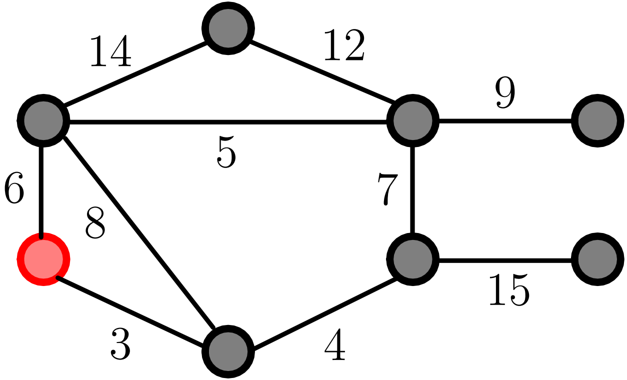

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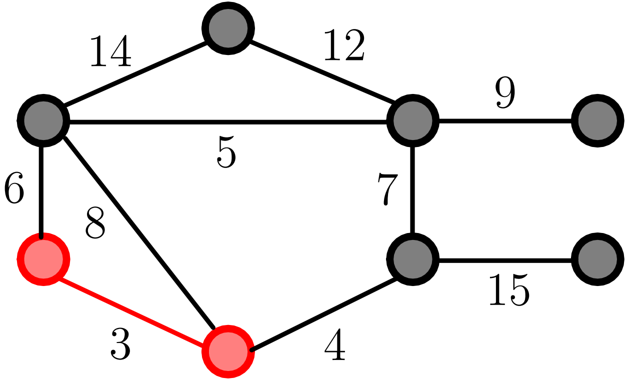

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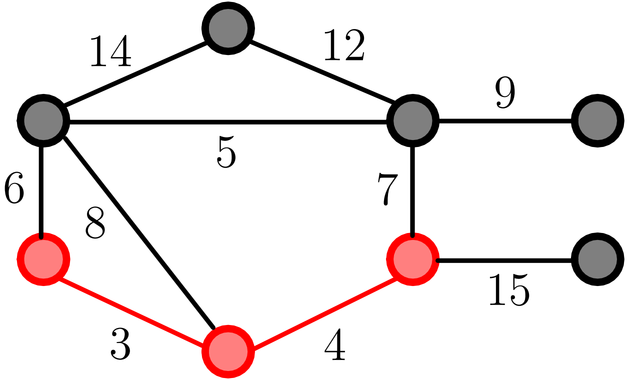

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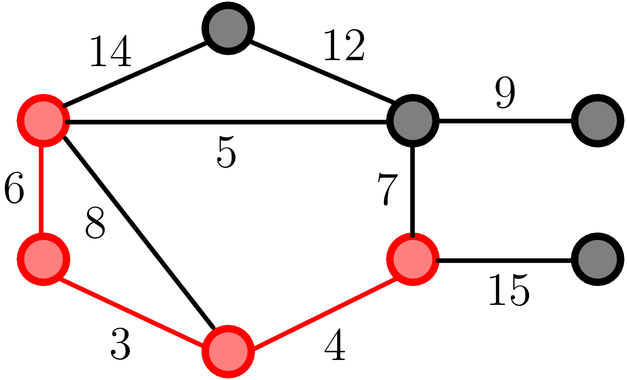

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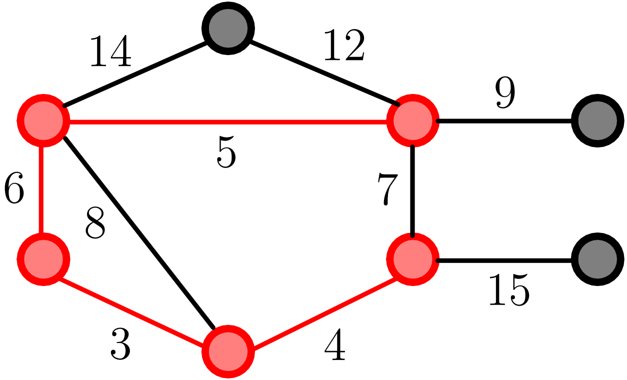

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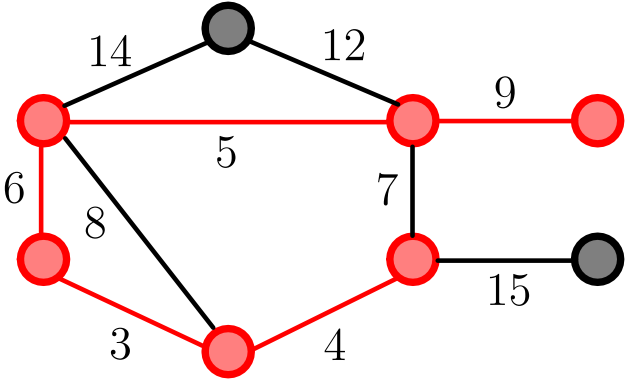

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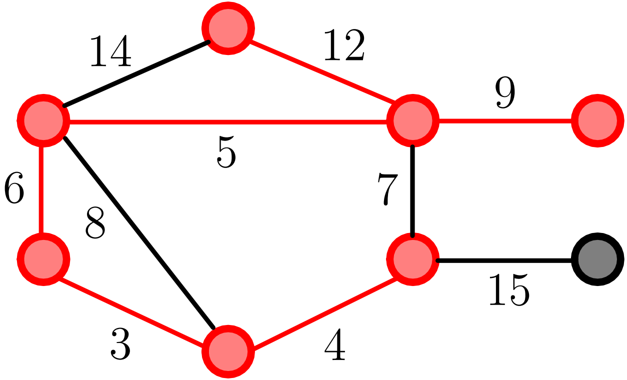

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