CS 341: Algorithms

Lecture 20: Reductions, P, NP, co-NP

Éric Schost

based on lecture notes by many other CS341 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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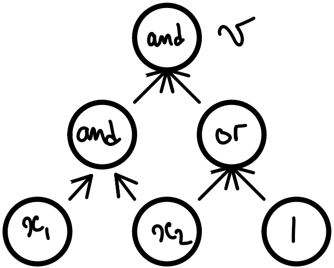
More examples of Karp reductions

Circuit satisfiability

CircuitSAT.

- instance: a **circuit** = DAG with labels on the vertices
- inputs labelled by boolean variables $x_1, \ldots, x_n$ or $0, 1$
- internal vertices labelled by **and**, **or**, **not**
- there is a marked vertex v for the output
- **problem:** is there a choice of boolean x_i that makes v **true**?



k-terms conjonctive formula satisfiability

kSAT.

- instance: a **boolean formula** in n variables $x_1, \dots, x_n$ in **CNF**

$$(y_{\mathbf{1,1}} \vee \cdots \vee y_{\mathbf{1,k_1}}) \quad \wedge \quad \cdots \quad \wedge \quad (y_{\boldsymbol{\ell},\mathbf{1}} \vee \cdots \vee y_{\boldsymbol{\ell},\boldsymbol{k_\ell}})$$

  with literals $y_{i,j}$ of the form x_m, $\overline{x_m}$, 1 or 0 and $k_i \leq k$
- **problem:** is there a choice of the variables that makes it true?

Remark 1: in clause i, can have repeated $y_{i,j}$ (then we only write them once)

$$(\overline{z} \vee x) \wedge (\overline{z} \vee y) \wedge (z \vee \overline{x} \vee \overline{y}) \qquad\qquad k = 3$$

Remark 2: can assume there are no constants 1 or 0

- if $y_{i,j} = 0$, remove the literal, if $y_{i,j} = 1$ remove the clause

Remark 3: key cases are $\boldsymbol{k = 2}$ and $\boldsymbol{k = 3}$

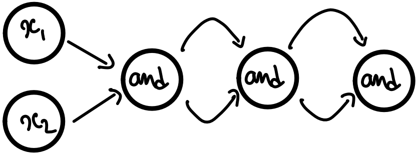
CircuitSAT $\leq_P$ 3SAT

Reduction:

- **given:** circuit $\boldsymbol{C}$ with $\boldsymbol{s}$ gates, variables $x_1, \ldots, x_n$, output v
- **build:** 3-CNF formula $\boldsymbol{F}$ with $\boldsymbol{O(s)}$ clauses
- **ensure:** C satisfiable $\iff$ F satisfiable

Remark:

- easy to build a formula: do it for all vertices bottom-up
- not polynomial, not 3CNF



$$((x_1 \wedge x_2) \wedge (x_1 \wedge x_2)) \wedge ((x_1 \wedge x_2) \wedge (x_1 \wedge x_2))$$

CircuitSAT $\leq_P$ 3SAT

Reduction:

- **given:** circuit $\boldsymbol{C}$ with $\boldsymbol{s}$ gates, variables $x_1, \ldots, x_n$, output v
- **build:** 3-CNF formula $\boldsymbol{F}$ with $\boldsymbol{O(s)}$ clauses
- **ensure:** C satisfiable $\iff$ F satisfiable

Key idea: introduce one new variable y_i per non-input gate and use

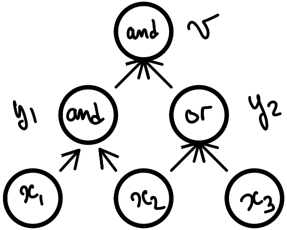
$$\boldsymbol{y_i = z \iff (z \implies y_i) \wedge (y_i \implies z) \iff (y_i \vee \overline{z}) \wedge (\overline{y_i} \vee z)}$$

- **and gate:** $z = t \wedge u$, and so $\overline{z} = \overline{t} \vee \overline{u}$

$$(y_i \vee \overline{t} \vee \overline{u}) \wedge (\overline{y_i} \vee (t \wedge u)) = \boldsymbol{(y_i \vee \overline{t} \vee \overline{u}) \wedge (\overline{y_i} \vee t) \wedge (\overline{y_i} \vee u)}$$

- **or gate:** $z = t \vee u$ gives $\boldsymbol{(y_i \vee \overline{t}) \wedge (y_i \vee \overline{u}) \wedge (\overline{y_i} \vee t \vee u)}$
- **not gate:** $z = \overline{t}$ gives $\boldsymbol{(y_i \vee t) \wedge (\overline{y_i} \vee \overline{t})}$

CircuitSAT $\leq_P$ 3SAT



gives

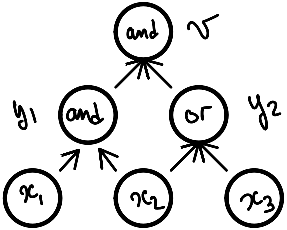
$$(y_1 = (x_1 \wedge x_2)) \ \wedge \ (y_2 = (x_2 \vee x_3)) \ \wedge \ (v = (y_1 \wedge y_2)) \ \wedge \ v$$

and

$$\begin{aligned}(y_1 \vee \overline{x_1 \wedge x_2}) \ \wedge \ (\overline{y_1} \vee (x_1 \wedge x_2)) \wedge \\ (y_2 \vee \overline{x_2 \vee x_3}) \ \wedge \ (\overline{y_2} \vee (x_2 \vee x_3)) \wedge \\ (v \vee \overline{y_1 \wedge y_2}) \ \wedge \ (\overline{v} \vee (y_1 \wedge y_2)) \ \wedge \ v\end{aligned}$$

given C, F can be constructed in polynomial time

CircuitSAT $\leq_P$ 3SAT



gives

$$(y_1 = (x_1 \wedge x_2)) \;\wedge\; (y_2 = (x_2 \vee x_3)) \;\wedge\; (v = (y_1 \wedge y_2)) \;\wedge\; v$$

and

$$\begin{aligned} F = &(y_1 \vee \overline{x_1} \vee \overline{x_2}) \;\wedge\; (\overline{y_1} \vee x_1) \;\wedge\; (\overline{y_1} \vee x_2) \;\wedge \\ &(y_2 \vee \overline{x_2}) \;\wedge\; (y_2 \vee \overline{x_3}) \;\wedge\; (\overline{y_2} \vee x_2 \vee x_3) \;\wedge \\ &(v \vee \overline{y_1} \vee \overline{y_2}) \;\wedge\; (\overline{v} \vee y_1) \;\wedge\; (\overline{v} \vee y_2) \;\wedge\; v \end{aligned}$$

given C, F can be constructed in polynomial time

Aside: polynomial-time Turing reductions

A stronger form of reduction

Consider two problems PROB1, PROB2, **not necessarily decision problems**

Definition

PROB1 is **polynomial-time Turing reducible** to PROB2 if there is an algorithm that solves PROB1 using

- a polynomial number of operations
- a polynomial number of calls to a solver (oracle) for PROB2

Remark:

- inputs/output transfers to/from the oracle count as "operations"
- so all inputs to the oracle have polynomial size

Notation:

- PROB1 $\leq_P^T$ PROB2

Examples and key property

Example 1

- reducing an optimization problem to its decision version (if optimal is an integer of polynomial size)

Example 2

- Karp reductions for decision problems (only one oracle call, at the end)

Claim

if Prob1 $\leq_P^T$ Prob2 and Prob2 can be solved in polynomial time, then it's also the case for Prob1

Proof: same as for Karp reductions

Example: factoring

Effective version: FACTOR

- **input:** integer M — input size $\Theta(\log M)$
- **output:** the prime factors of M

Decision version: HASFACTOR

- **input:** integers M and $0 \leq k \leq M$ — input size $\Theta(\log M)$
- **output:** **yes** iff M has a prime factor $\leq k$

Remark: polynomial time = $\boldsymbol{\log(M)^{O(1)}}$

Claim 1:

HASFACTOR $\leq_P^T$ FACTOR

Proof: factor M and check

Example: factoring

Claim 2:

$\textsc{Factor} \leq_P^T \textsc{HasFactor}$

1. Find the first ℓ such that M has a prime factor between 2^ℓ and $2^{\ell+1} - 1$
   - test all $\ell = 1, 2, 3, \ldots, \log(M)$ — $O(\log M)$ calls to $\textsc{HasFactor}$ with inputs $\leq M$
   - if all **no**, M is prime, done
2. Find the smallest factor between 2^ℓ and $2^{\ell+1} - 1$
   - binary search — $O(\log M)$ calls to $\textsc{HasFactor}$ with inputs $\leq M$
3. We found one prime factor P. Repeat on M/P
   - $\log M$ prime factors at most

Conclusion: if $\textsc{HasFactor}$ can be solved in polynomial time, we can factor integers in polynomial time.

P, NP, co-NP

The classes P and NP

Definition

P is the set of decision problems that can be solved in polynomial time
NP is the set of decision problems where **yes**-instances can be **certified** in polynomial time.

Precisely, a **decision problem** PROB is in **NP** if

- there exists an algorithm B (**a certifier**) that takes as input an instance x and an extra input y (**a certificate**) and outputs "yes" or "no" in polynomial time in size(x)+size(y)
- x **yes**-instance for PROB if and only if there exists y of size polynomial in size(x), such that $B(x, y) =$"yes"

Remarks

1. if we can solve Prob in polynomial time, we can certify it as well (with an empty certificate) so

$$\mathbf{P} \subset \mathbf{NP}$$

\$1,000,000 question: $\mathbf{P} = \mathbf{NP}$?

2. **NP** means **N**on-deterministic **P**olynomial time

- nothing to do with randomized algorithms
- non-deterministic Turing machines have several transitions available each step
- existence of one accepting path $\simeq$ existence of a certificate

Examples

Independent set

- **instance**: graph G, integer K
- **certificate:** a set S of vertices
- **certification:** test if $|S| \geq K$ and S independent

Vertex cover

- **instance**: graph G, integer K
- **certificate:** a set S of vertices
- **certification:** test if $|S| \leq K$ and S covers all edges

Clique

- **instance**: graph G, integer K
- **certificate:** a set S of vertices
- **certification:** test if $|S| \geq K$ and S clique

Examples

Circuit sat

- **instance**: boolean circuit C
- **certificate:** a sequence x of bits
- **certification:** test if $C(x) = \mathsf{true}$

3SAT

- **instance**: a boolean formula F in 3CNF
- **certificate:** a sequence x of bits
- **certification:** test if $F(x)$ is true

SAT

- **instance**: a boolean formula F
- **certificate:** a sequence x of bits
- **certification:** test if $F(x)$ is true

Examples

Hamiltonian cycle

- **instance**: graph G
- **certificate:** a sequence S of vertices
- **certification:** test if S is a Hamiltonian cycle in G

Hamiltonian path

- **instance**: graph G
- **certificate:** a sequence S of vertices
- **certification:** test if S is a Hamiltonian path in G

Factors

- **instance**: integers M and $0 \leq k \leq M$
- **certificate:** integer P
- **certification:** test if P is prime, P divides M and $P \leq k$

co-NP

Definition

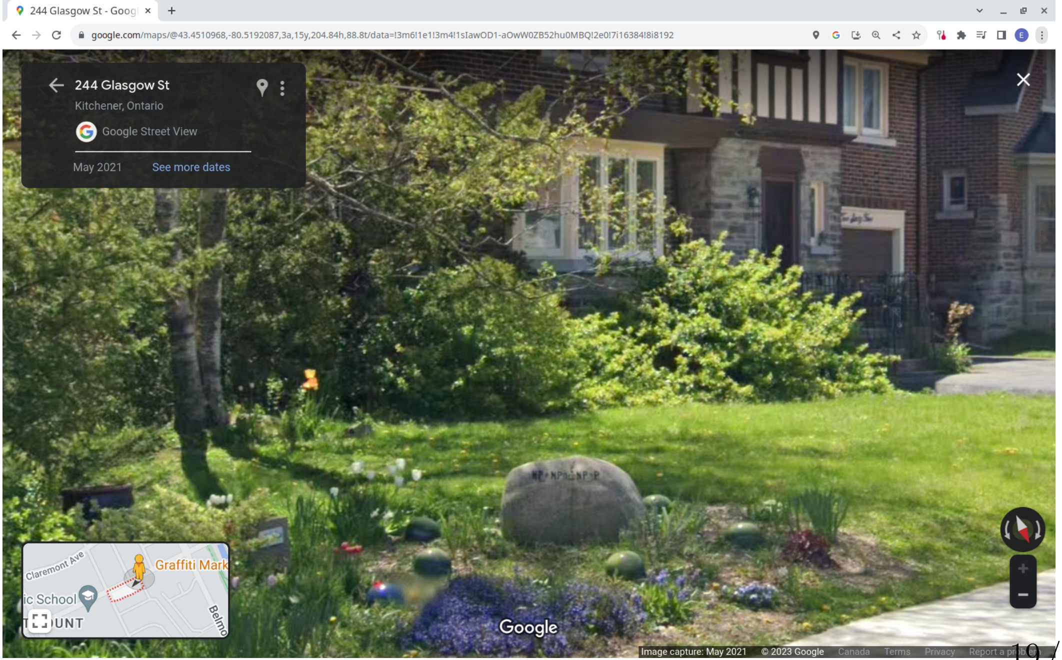
co-NP is the set of decision problems whose **no**-instances can be certified in polynomial time.

Remark: most problems so far are thought to not be in **co-NP**

- certify that a formula not satisfiable?
- certify that a graph has no Hamiltonian path?
- but HasFactor is in **co-NP** (certificate = all prime factors)

Exercise (after we see NP-completeness)

If a single NP-complete problem is in **co-NP**, **NP=co-NP**
(so doubtful that HasFactor is NP-complete)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