CS 341: Algorithms

Lecture 23: Misc

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based on lecture notes by many other CS341 instructors

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EXP and beyond

Exponential time

Definition

EXP is the set of decision problems that can be solved in exponential time $2^{O(\mathsf{size}(x)^k)}$ for some k.

Observation: **NP** $\subset$ **EXP** so problems in **NP** cannot be extraordinarily bad

Idea: brute-force, try all possible certificates

- for a given x, we look for a certificate of size $\mathsf{size}(x)^k$, for some constant k
- if we work with binary symbols, there are $2^{\mathsf{size}(x)^k}$ certificates
- each of them takes polynomial time

Bounded halting

Definition

- **instance:** program / Turing maching P, input x to P, integer t
- **output:** does $P(x)$ stop on input x within t steps?
- **remark:** input size $= \mathsf{size}(P) + \mathsf{size}(x) + \log t$

Claim

BoundedHalting is in **EXP**

Proof (sketch)

- use a **universal Turing machine**, run the simulation for t steps
- runtime polynomial can be made polynomial in $\mathsf{size}(P)$, t
- which is exponential in the input size

EXP-completeness

Claim

BoundedHalting is **EXP**-complete

Proof

- take decision problem Prob in **EXP**
- so there is a program / Turing machine P that decides Prob(x) using at most $2^{c\,\mathsf{size}(x)^k}$ operations (c, k constants)
- modify P to make it run forever if input **no**-instance, call P' the result
- **reduction:** on instance x, call BoundedHalting with input P', x and $t = 2^{c\,\mathsf{size}(x)^k}$
- P' is fixed, x is x and $\log t = c\,\mathsf{size}(x)^k$, so this is polynomial in $\mathsf{size}(x)$

Remark: because **NP** $\subset$ **EXP**, this shows BoundedHalting is **NP**-hard

Time hierarchy

Time hierarchy theorem (particular case)

$\mathbf{P} \neq \mathbf{EXP}$

Proof: take CS360

Consequence

$\textsc{BoundedHalting}$ is not in $\mathbf{P}$

Proof:

- if it was, using $\mathbf{EXP} \leq_P \textsc{BoundedHalting}$, we would get $\mathbf{P}{=}\mathbf{EXP}$

Even worse

HALTING

- **instance:** program / Turing maching P, input x to P
- **output:** does $P(x)$ stop on input x?
- **remark:** input size $=$ size(P) $+$ size(x)

1. **Undecidable** (CS245, CS360), so in particular not in **NP**

2. **RE-hard** (so in particular NP-hard, and **not** NP-complete)
   - take a **recursively enumerable** problem PROB
   - **meaning:** there is a program / Turing machine P st PROB(x) returns **true** for **yes**-instances, and either loops or returns **false** for **no**-instances
   - modify P to make it run forever if input **no**-instance, call P' the result
   - **reduction:** on instance x, call HALTING with input P', x
   - P' is fixed, x is x, so this is polynomial in size(x)

Variants of kSAT

Definitions

kSAT

- instance: a **boolean formula** in n variables $x_1, \ldots, x_n$ in **CNF**

$$(y_{\mathbf{1,1}} \vee \cdots \vee y_{\mathbf{1},\boldsymbol{k_1}}) \quad \wedge \quad \cdots \quad \wedge \quad (y_{\boldsymbol{\ell},\mathbf{1}} \vee \cdots \vee y_{\boldsymbol{\ell},\boldsymbol{k_\ell}})$$

  with literals $y_{i,j}$ of the form x_m, $\overline{x_m}$ and $k_i \leq k$

- **problem:** is there a choice of the variables that makes it true?

EXACT-kSAT

- same as above, but with **exactly** k literals (repetitions OK)

UNIQUE-kSAT

- same as above, but with **exactly** k literals and **no repeated variable**
- for $\boldsymbol{k} = \mathbf{3}$, $x \vee y \vee \overline{z}$ OK, $x \vee x \vee z$ not OK, $x \vee \overline{x} \vee z$ not OK

Equivalence

Claim

$\textsc{exact-k}\text{SAT} \leq_P \textsc{k}\text{SAT} \leq_P \textsc{exact-k}\text{SAT}$

Proof:

1. an $\textsc{exact-k}\text{SAT}$ instance is a $\textsc{k}\text{SAT}$ instance
2. transform $x \vee y$ into $x \vee x \vee y$

Claim

$\textsc{unique-k}\text{SAT} \leq_P \textsc{k}\text{SAT} \leq_P \textsc{unique-k}\text{SAT}$

Proof:

1. a $\textsc{unique-k}\text{SAT}$ instance is a $\textsc{k}\text{SAT}$ instance
2. transform $x \vee y$ into $(x \vee y \vee \text{dummy}) \wedge (x \vee y \vee \overline{\text{dummy}})$

2SAT and MAX-2SAT

2SAT is in P

Remark: any $\textsc{k}$SAT is in **NP**

- **instance:** formula F in kCNF
- **certificate $\boldsymbol{y}$:** boolean values for the variables that appear in F
- **algorithm $\boldsymbol{B(F, y)}$:** test if $F(y)$ is **true** (i.e. if all clauses are **true**)
- **NP**? yes! B runs in polynomial time, and F is satisfiable iff there exists a certificate of size $\leq \text{size}(F)$

We know: 3SAT NP-complete (and so $\textsc{k}$SAT as well, for $k \geq 3$)

Claim:

2SAT in **P**

Proof: we start from a formula F in 2CNF that has $\boldsymbol{s}$ clauses

assume all clauses have 2 literals

Introducing a graph

Idea: $x_i \vee x_j$ is equivalent to

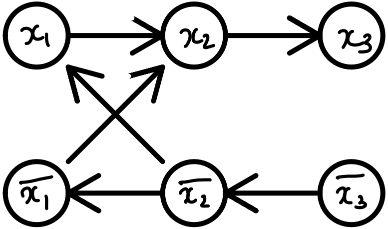
$$\overline{x_i} \implies x_j \quad \text{and to} \quad \overline{x_j} \implies x_i$$

- we can chain these implications to eventually find out a satisfiable solution
- so we put them in a directed graph G (with vertices labeled x_i and $\overline{x_i}$)

Example

$$(x_1 \vee x_2) \wedge (x_2 \vee \overline{x_1}) \wedge (x_3 \vee \overline{x_2})$$

gives



How to use the graph

Observation: suppose booleans $y_1, \ldots, y_n$ satisfy F

- assigns boolean values to all vertices
- if vertex v is true and $v \to w$ edge, w true — because $\overline{v} \vee w$ clause in F
- so if v is true and $v \rightsquigarrow w$ path, w true

Consequence: if some $x_i, \overline{x_i}$ are in the same SCC of G, F not satisfiable

Decision algorithm:

- construct G (at most $2s$ vertices and $2s$ edges)
- find the SCCs of G (= put indices on vertices)
- if any $x_i, \overline{x_i}$ that appear in F have the same index, return **false**
- else, return **true**

Runtime: $\boldsymbol{O(s)}$ in the word RAM model, **polynomial** in $\boldsymbol{s \log n}$ in the bit model

Proof + finding satisfying assignments

Algorithm, cont. (assuming **true**)

- **contract** the SCCs of G to obtain a DAG G'
- find a topological order o on G'
- for $i = 1, \ldots, n$
  - if $o(x_i) < o(\overline{x_i})$, take $y_i =$ **false**
  - if $o(\overline{x_i}) < o(x_i)$, take $y_i =$ **true**
  - if $o(x_i)$ undefined, y_i arbitrary

(still polynomial time)

Claim: $F(y_1, \ldots, y_n) =$ **true**

Proof: suppose that $x_i \vee x_j$ clause not satisfied, so x_i and x_j assigned **false**

- so $o(x_i) < o(\overline{x_i})$ and $o(x_j) < o(\overline{x_j})$
- $(\overline{x_i}, x_j)$ edge, so $o(\overline{x_i}) \leq o(x_j)$ and $\boldsymbol{o(\overline{x_i}) < o(\overline{x_j})}$
- $(\overline{x_j}, x_i)$ edge, so $o(\overline{x_j}) \leq o(x_i)$ and $\boldsymbol{o(\overline{x_j}) < o(\overline{x_i})}$

contradiction

MAX-kSAT

k-terms conjonctive formula satisfiability, optimization version:

- **instance:** a boolean formula $\boldsymbol{F}$ in n variables $x_1, \dots, x_n$ in kCNF
- **problem:** find the maximal number of clauses that can be satisfied simultaneously

Decision version: MAX-κSAT

- **instance:** F as above, and an integer K
- **problem:** is there a choice of the variables that satisfies at least K clauses?
- **certificate:** boolean values for the variables that appear in F
- **algorithm $\boldsymbol{B}$:** count if at least K clauses in $F(y)$ are true

We prove: MAX-2SAT NP-complete

Exercise

we already could tell that MAX-κSAT NP-complete for $k \geq 3$

3SAT $\leq_P$ MAX-2SAT

Preliminaries:

- consider a clause $C = x \vee y \vee z$ (repeated variables OK)
- introduce a new variable t, and the 10 clauses

$$x,\ y,\ z,\ t,\ \overline{x} \vee \overline{y},\ \overline{y} \vee \overline{z},\ \overline{z} \vee \overline{x},\ x \vee \overline{t},\ y \vee \overline{t},\ z \vee \overline{t}$$

Claim

- you cannot satisfy more than 7 of these new clauses
- a boolean assignment of x, y, z, t that satisfies 7 clauses makes C **true**
- given a boolean assignment for x, y, z that makes C **true**, you can find a value for t that satisfies 7 clauses

case discussion (discuss whether $0, 1, 2$ or 3 of x, y, z are **true**)

3SAT $\leq_P$ MAX-2SAT

Reduction. Given a family F of k clauses that form a 3SAT problem, introduce

- one new variable t_i per clause in F,
- the 10 clauses as seen before (per clause in F)
- $K = 7k$

(takes polynomial time)

Correctness:

- you cannot satisfy more than $7k$ of these new clauses
- you satisfy $7k$ of them simultaneously if and only if you can satisfy all k input clauses simultaneously

Conclusion: MAX-2SAT is NP-complete

Randomization and approximation

Using randomization (for the optimization problem)

MAX-UNIQUE-3SAT

- **input:** F in 3CNF, with 3 distinct variables per clause (works for any k)
- **problem:** find the maximal number of clauses that can be satisfied simultaneously
- decision version NP-complete

Claim

using in expected polynomial time in n, s, we can find an assignment that satisfies at least 87.5% of the clauses

RandomAssignment(F)
1. F formula in 3CNF, 3 distinct variables per clause, $\boldsymbol{s}$ clauses
2. **repeat**
3. pick $x_1, \ldots, x_n$ uniformly at random in $\{0, 1\}$
4. **until** at least $7s/8$ clauses are satisfied
5. **return** $x_1, \ldots, x_n$

Analysing a single assignment

Definition: for $i = 1, \ldots, s$, let X_i be the indicator random variable

- $X_i = 0$ if ith clause is not satisfied
- $X_i = 1$ if ith clause is satisfied

Analysis:

- clause i has 3 variables and out of the 8 possibilities, only 1 makes it **false**
- so $p(X_i = 1) = 7/8$
- so $E[X_i] = 7/8$

Looking at all clauses:

- the number N of satisfied clauses is $\sum_{i \leq s} X_i$
- so $E[N] = 7s/8$

Overall runtime

Defining p

- let $\boldsymbol{p}$ be the probability that a random assignment satisfies at least $7s/8$ clauses
- then the **expected number of attempts** is

$$p + 2p(1-p) + 3p(1-p)^2 + \cdots = \frac{\mathbf{1}}{\boldsymbol{p}}$$

- and the expected runtime is $\boldsymbol{O((n+s)/p)}$ (in the word RAM model)

Introducing $p_0, \ldots, p_s$ and s'

- for $j = 0, \ldots, s$, let $\boldsymbol{p_j}$ be the probability that we satisfy $\boldsymbol{j}$ clauses
- let $\boldsymbol{s'}$ be the largest integer **less than** $7s/8$

Consequences

- $\boldsymbol{s' \leq 7s/8 - 1/8}$
- $\boldsymbol{p = \sum_{j \geq s'+1} p_j}$

Overall runtime

$$
\begin{aligned}
\frac{7}{8}s &= E[N] \\
&= \sum_j j p_j \\
&= \sum_{j \le s'} j p_j + \sum_{j \ge s'+1} j p_j \\
&\le \sum_{j \le s'} s' p_j + \sum_{j \ge s'+1} s p_j && j \le s',\ j \le s \\
&= s'(1-p) + sp && \text{previous slide} \\
&\le s' + sp && 1 - p \le 1 \\
&\le \frac{7}{8}s - \frac{1}{8} + sp && \text{previous slide}
\end{aligned}
$$

Finally: $1/8 \le sp$ so $\boldsymbol{1/p \le 8s}$

Bonus

Medium: derandomize the algorithm

- assign one variable at a time
- at the beginning,

$$\frac{7}{8}s = E[N] = \frac{1}{2}E[N|x_1 = 0] + \frac{1}{2}E[N|x_1 = 1]$$

  so one of $E[N|x_1 = 0]$ and $E[N|x_1 = 1]$ must be at least $\frac{7}{8}s$
- both can be computed in polynomial time, choose the better one and continue

Extra hard: beat 7/8

- if there is a polynomial-time algorithm that finds a fraction $7/8 + \varepsilon$ of the optimal, then **P=NP**