Math 237 course note

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Contents

1 Graphs of Scalar Functions

1.1 Scalar Functions

1. **Scalar function**
   A function $f : R^2 \to R$ is the scalar function with

   - Domain $D(f) \subseteq R^2$
   - Range: $R(f) \subseteq R \to$ codomain

2. Notation:
   We will use x for element in Rn

3. **Level curves**
   Let $f : R^2 \to R$, k$= f(x, y)$ where k is a constant in range of f

4. **Cross-section**
   Let z $= f(x, y)$, then:

   - z $= f(c, y)$
   - z $= f(x, d)$

   for constant c and d

5. **Level sets**
   $= \{\underline{x} \in R^n | f(\underline{x} = k\}$ where $k \in R(f)$

2 Limits

2.1 Definition of a limit

1. **Neighbourhood**
   An r-neighbourhood isof a point $(a, b) \in R_2$ is a set of
   $N_r(a, b) = \{(x, y) \in R^2 : ||(x, y) - (a, b)|| < r\}$ where $r \in R$

2. Remarks:

   - $||(x, y) - (a, b)|| = \sqrt{(x - a)^2 + (y - b)^2}$
   - When $r < 0$, $N_r(a, b)$ is a empty set, so we usually consider only non-negative values of r.

3. **Limit**
   Assume $f(x, y)$ is defined in a neighbourhood of (a, b), except possibly at (a, b).
   If $\forall\epsilon > 0, \exists\delta > 0$ such that
   $0<||(x, y) - (a, b)||<\delta$ implies $|f(x, y) - L| < \epsilon$
   Then, $\lim_{(x,y)\to(a,b)} f(x, y) = L$

2.2 Limit Theorems

1. Property of limit
   If $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ both exist, then:
   - $\lim_{(x,y)\to(a,b)}[f(x,y)+g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y)+\lim_{(x,y)\to(a,b)} g(x,y)$
   - $\lim_{(x,y)\to(a,b)}[f(x,y)g(x,y)] = [\lim_{(x,y)\to(a,b)} f(x,y)]+[\lim_{(x,y)\to(a,b)} g(x,y)]$
   - $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = frac\lim_{(x,y)\to(a,b)} f(x,y)\lim_{(x,y)\to(a,b)} g(x,y)$
2. **Uniqueness**
   If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then it is unique.

2.3 Proving a Limit Does Not Exist

1. Use Uniqueness to show that the limit is different at different y value
2. Let $y = f(x)$ show that the limit is depend on some variable, still against Uniqueness

2.4 Proving a Limit Exists

1. **Squeeze Theorem**
   If $\exists B(x,y)$ such that:

   - $\forall (x,y) \neq (a,b)$, $|f(x,y) - L| \leq B(x,y)$
   - $\lim_{(x,y)\to(a,b)} B(x,y) = 0$

   Then, $\lim_{(x,y)\to(a,b)} f(x,y) = L$

The following is a Algorithm for Determine whether $\lim_{(x,y)\to(a,b)} f(x,y)$ exists:

2.5 Appendix: Inequalities and Absolute Values

1. Property of Inequalities

   - Trichotomy: $\forall a, b \in R$ one and only one holds:
     (a) $a = b$
     (b) $a < b$
     (c) $b < a$
   - Transitivity: If $a < b$ and $b < c$, then $a < c$
   - Addition: If $a < b$, then $\forall c \in R, a + c < b + c$
   - Multiplication: if $a < b$ and $c < 0$, then $bc < ac$
   - Inverse Multiplicative: If $ab > 0$ and $a < b$, then $\frac{1}{b} < \frac{1}{a}$

2. Property of Absolute value

   - $|a| = \sqrt{a^2}$
   - $|a| < b \iff -b < a < b$
   - $\forall a, b \in R, |a + b| \leq |a| + |b|$
   - If $c > 0$, then $a < a + c$
   - $2|x||y| \leq x^2 + y^2$

3 Continuous Functions

3.1 Definition of Continuous Functions

1. **Continuous**
   A function f(x,y) is continuoues $\iff$
   $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$
   Additionly, if $\forall D \subset R^2$, f is coutinoues, we say f is continuous on D.

2. The following three requirements must meet:

   - $\lim_{(x,y)\to(a,b)} f(x,y)$ exist
   - f is defined at (a,b)
   - $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

3.2 The continuity Theorems

1. Basic Functions The following function continuous on their domains:
   - The constant function: $f(x,y) = k$
   - The power functions: $f(x,y) = ax^m + by_n$
   - The logarithm function: $ln(\cdots)$
   - THe exponential function: $e^{(\cdots)}$
   - The trignometic functions: $sin(\cdots), cos(\cdots), \ldots$
   - The inverse trig functions: $sin^{-1}(\cdots), \ldots$
   - THe absolute value function $|\cdots|$
2. Operation on Functions
   - **Sum** $f+g$ is defined by
     $(f+g)(x,y) = f(x,y) + g(x,y)$
   - **Product** fg is defined by
     $(fg)(x,y) = f(x,y)g(x,y)$
   - **Quotient** $\frac{f}{g}$ is defined by
     $\frac{f}{g}(x,y) = \frac{f(x,y)}{g(x,y)}$, given $g(x,y) \neq 0$
3. Composite Function
   Let $g : R \to R$ and $F : R^n \to R$ be 2 funciton
   The function $g \circ f$ is defined by $(g \circ f)(x,y) = g(f(x,y)$
   Where $g \circ f : R^n \to R$ and domain $D(g \circ f) = \{(x,y) \in D(f) : f(x,y) \in D(g)\}$

4. **Continuity Theorem**
   Assume f and g are both continuous at (a, b), then:

   - $f + g$ and fg are continuous at (a, b)
   - $\frac{f}{g}$ is continuous at (a, b)
   - $g \circ f$ is continuous at (a, b)

4 The Linear Approximation and Partial Derivatives

4.1 Partial Derivatives

1. A scalar Function $f(x,y)$ can be differentiated in two natural ways:
   - Treat y as a constant, we obtain $\frac{df}{dx}$
   - Treat x as a constant, we obtain $\frac{df}{dy}$

2. **Partial Derivatives**
   The partial derivatives of $f(x,y)$ are defined by
   - $\frac{df}{dx} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$
   - $\frac{df}{dy} = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$

   and these limits always exist

3. Operator notation
   - $D_1 f = \frac{df}{dx} = f_x$
   - $D_2 f = \frac{df}{dy} = f_y$

4.2 Higher-Order Partial Derivatives

1. Second Partal Derivatives
   There are four types of second derivatives
   - $\frac{d^2f}{dx^2} = \frac{d}{dx}(\frac{df}{dx}) = f_{xx} = D_1^2 f$
   - $\frac{d^2f}{dydx} = \frac{d}{dy}(\frac{df}{dx}) = f_{xy} = D_2 D_1 f$
   - $\frac{d^2f}{dxdy} = \frac{d}{dx}(\frac{df}{dy}) = f_{yx} = D_1 D_2 f$
   - $\frac{d^2f}{dy^2} = \frac{d}{y}(\frac{df}{dy}) = f_{yy} = D_2^2 f$
2. **Clairaut's Theorem**
   If Both f_{xy} and f_{yx} are defined in some neighborhood of (a, b) and both continuous, then
   $$f_{xy}(a, b) = f_{yx}(a, b)$$

4.3 The tangent Plane

1. Tangent Plane
   Let $z = f(x, y)$, the tangent plane at point $(a, b, f(a, b)$ is

$$z = f(a, b) + \frac{df}{dx}(a, b) \times (x - a) + \frac{df}{dy}(a, b) \times (y - b)$$

4.4 Linear Approximation for $z = f(x, y)$

1. Review for 1D
   Let $y = f(x)$, then Linear approximation at point $(a, f(a))$ is:

$$L_a(x) = f(a) + f^{'}(a)(x - a)$$

2. For 2D
   Let $z = f(x, y)$, the linearization of point (a, b) is

$$L_{(a,b)}(x, y) = f(a, b) + \frac{df}{dx}(a, b) \times (x - a) + \frac{df}{dy}(a, b) \times (y - b)$$

3. Increment form of Linear Approximation
   Let $z = f(x, y)$ and suppose we know $f(a, b)$ Let $\Delta x = x - a, \Delta y = y - b$ and
   $\Delta f = f(x, y) - f(a, b) = \frac{df}{dx}(a, b) \times (x - a) + \frac{df}{dy}(a, b) \times (y - b)$

4.5 Linear Approximation in Higher Dimensions

1. Linear Approximation in R^3
   Let $f(x, y, z)$ be a function, we define the linearization of g at $\vec{v} = (a, b, c)$ by

   $$L_{\vec{v}}(x, y, z) = f(\vec{v}) + f_x(\vec{v}) \times (x - a) + f_y(\vec{v}) \times (y - b) + f_z(\vec{v}) \times (z - c)$$

2. **Gradient**
   Suppose that $f(x, y, z)$ have partial derivatives at $\vec{a} \in R^3$, then
   The gradient of f at $\vec{a}$ is

   $$\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$$

3. Linearization and Approximation
   Let $f(\vec{x}), \vec{x} \in R^3$ has partial derivatives at $\vec{a} \in R^3$, then

   - The linearization of f at $\vec{a}$ is

     $$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \times (\vec{x} - \vec{a})$$

   - The linear apprioximation of f at $\vec{a}$ is

     $$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a}) \times (\vec{x} - \vec{a})$$

5 Differentiable Function

5.1 Definition of Differentiability

1. Differentiability for Functions in One Variable

   - Error of linear approximation:

   $$R_{1,a}(x) = g(x) - L_a(x)$$

   $$= g(x) - g(a) - g^{'}(a)(x-a)$$

   - Theorem 1
     If $g^{'}(a)$ exists, then $\lim_{x \to a} \frac{|R_{1,a}(x)|}{|x-a|} = 0$

2. Differentiability for Functions in Two Variables

   - Error of linear approximation

   $$R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$$

   - **Differentiable**
     A function f(x, y) is differentiable at (a, b) if

   $$\lim_{(x,y) \to (a,b)} \frac{|R_{1,(a,b)}(x,y)|}{||(x,y)-(a,b)||} = 0$$

     where

   $$R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$$

   - Theorem 2
     If a function f(x, y) satisfies

   $$\lim_{(x,y) \to (a,b)} \frac{|f(x,y) - f(a,b) - c(x-a) - d(y-b)|}{||(x,y)-(a,b)||} = 0$$

     for some constants c and d then $c = f_x(a,b)$ and $d = f_y(a,b)$

3. Tangent Plane
   Let function $f(x,y)$ be differentiable at (a,b), the tangent plane of $z = f(x,y)$ at $(a,b,f(a,b))$ is the graph of linearization which

   $$z = f(a,b) + \frac{df}{dx}(a,b)(x-a) + \frac{df}{dy}(a,b)(y-b)$$

5.2 Differentiability and Continuity

1. Theorem 1
   if $f(x, y)$ is differentiable at (a, b), then f is continuous at(a, b).

2. And that's it. Why this is a seperate lecture????????

5.3 Continuous Partial Derivatives and Differentiability

1. The Mean Value Theorem
   If $f(x)$ is

   - Continuoues on closed interval $[x_1, x_2]$
   - Differentiable on the open interval (x_1, x_2)

   Then, $\exists x_0 \in (x_1, x_2)$ such that

   $$f(x_2) - f(x_1) = f^{'}(x_0)(x_2 - x_1)$$

2. Theorem 2
   If partial derivatives $\frac{df}{dx}$ and $\frac{df}{dx}$ are both continuous at (a, b), then $f(x, y)$ is defferentiable at (a, b)

3. Differetiability for $f : R^n \to R$
   A function $f : R^n \to R$ is differentiable at a point $\vec{a} = (a_1, \ldots, a_n)$ if:

   $$\lim_{\vec{x} \to \vec{a}} \frac{|f(\vec{x}) - f(\vec{a}) - L_{\vec{a}}(\vec{x} - \vec{a})|}{||\vec{x} - \vec{a}||} = 0$$

   where $L : R^n \to R$ is a linear transformation

6 Chain Rule

6.1 Basic Chain Rule in Two Dimensions

1. Basic chain rule for R

$$T^{'}(t) = f^{'}(x(t))x^{'}(t)$$

   or

$$\frac{dT}{dt} = \frac{dT}{dx} * \frac{dx}{dt}$$

2. Chain rule for $f(x(t), y(t))$
   Let $G(t) = f(x(t), y(t))$ and Let $a = x(t_0)$ and $b = y(t_0)$
   If f is differentiable at (a, b) and $x^{'}(t_0)$ and $y^{'}(t_0)$
   Then $G^{'}(t_0)$ exists and equals

$$G^{'}(t_0) = f_x(a, b)x^{'}(t_0) + f_y(a, b)y^{'}(t_0)$$

3. If $g(s, t) = f(x, y)$ and $x(s, t) = \dots$ and $y(s, t) = \dots$
   then, $g_s(s, t) = f_x(u(s, t), v(s, t))u_s(s, t) + f_y(u(s, t), v(s, t))v_s(s, t)$

4. Vector From of basic chain rule
   Let $f(x, y), x(t)$ and $y(t)$ all be differentiable, define$T(t) = f(x(t), y(t))$, then

$$\frac{dT}{dt} = \nabla f * \frac{d\vec{x}}{dt}$$

   and

$$\frac{d}{dt}f(\vec{x}(t)) = \nabla f(\vec{x}(t)) * \frac{d\vec{x}}{dt}(t)$$

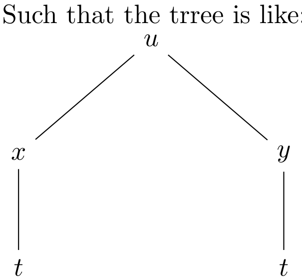
   wirh $\vec{x}(t) = (x_1(t), x_2(t))$

6.2 Extensions of the Basic Chain rule

1. Dependence Tree with one independent variable
   Let $u = f(x, y)$ with differentiable function $x(t)$ and $y(t)$ then

   - Dependent variable: u
   - Intermediate variables: x, y
   - Independent variables: t

   Such that the trree is like:



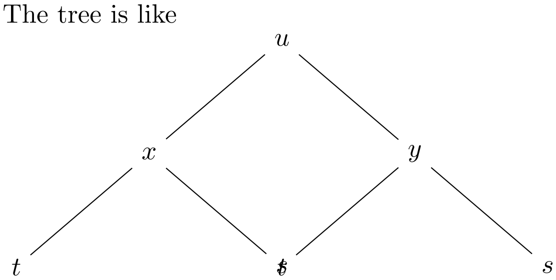
   and the formula is

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt}$$

2. Dependence Tree with more independent variable
   Let $u = f(x, y)$ with $x = x(s, t)$ and $y = y(s, t)$ have first order partial derivatives at (s, t) and f is differentiable at $(x, y) = (x(s, t)y, (s, t))$

   - Dependent variable: u
   - Intermediate variables: x, y
   - Independent variables: t, s

   The tree is like



   where

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} + \frac{du}{dy}\frac{dy}{ds} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt}$$

6.3 The Chain Rule for Second Partial Derivatives

3. Laplace's equation

$$u_{xx} + u_{yy} = 0$$

4. Remark
   Let $z = f(x)$ and $x = e^u$
   - $z^{'}(u) = f^{'}(e^u)e^u$
   - $z^{''}(u) = x^2 f^{''}(x) + x f^{'}(x)$

7 Directional Derivatives and Gradient Vector

7.1 Directional Derivatives

1. Directional Derivatives
   The directional derivative of $f(x, y)$ at poinrt (a, b) in the direction of a unit vector $\vec{u} = (u_1, u_2)$ where $||\vec{u}|| = 1$

$$D_{\vec{u}}f(a, b) = \frac{d}{ds}f(a + su_1, b + su_2)|_{s=0}$$

2. Directional Derivative Theorem
   If f(x, y) is differentiable at (a, b) and $\vec{u} = (u_1, u_2)$ where $||\vec{u}|| = 1$, then

$$D_{\vec{u}}f(a, b) = \nabla f(a, b)\vec{u}$$

   where the middle part is a dot produce.

7.2 The gradient Vector in Two Dimensions

1. The greatest rate of change theorem (GRC)
   If $f(x, y)$ is differentiable at (a, b) and $\nabla f(a, b) \neq (0, 0)$, then the largest value of
   $D_{\vec{u}} f(a, b)$ is $||\nabla f(a, b)||$ and orrurs when $\vec{u}$ is in the direction of $\nabla f(a, b)$

2. Orthogonality Theorem
   If $f(x, y) \in C^1$ in a neighborhood of (a, b) and $\nabla f(a, b) \neq (0, 0)$ then $\nabla f(a, b)$ is orthogonal to the level curve of $f(x, y) = k$ through (a, b, k)

7.3 Graident Vector in THree Dimensions

1. Orthgonality Theorem in Three Dimensions
   If $f(x, y, z) \in C_1$ in the neighborhood of (a, b, c) and $\nabla f(a, b, c) \neq (0, 0, 0)$, then it is orthogonal to the level surface $f(x, y, z) = k$ through (a, b, c)

8 Taylor Polynomials and Taylor's Theorem

8.1 Taylor Polynomial of Degree 2

1. Single Variable Case for Taylor
   For degree 2 point a taylor polynomial

   $$P_{2,a}(x) = f(a) + f^{'}(a)(x-a) + \frac{1}{2}f^{''}(a)(x-a)^2 = L_a(x) + \frac{1}{2}f^{''}(a)(x-a)^2$$

2. Two variable Case for Taylor Polynomial

   $$P_{2,(a,b)}(x,y) = f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)+\frac{1}{2}[f_{xx}(a,b)(x-a)^2+2f_{xy}(a,b)(x-a)(y-b)$$
   $$+f_{yy}(a,b)(y-b)^2]$$

3. Hessian Matrix

   $$Hf(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}$$

8.2 Taylor's formula with Second Degree Remainder

1. Taylor Remainder for single Variable Function
   if $f^{''}(x)$ exists on [a, x], then there exists a number c between a and x such that
   $$f(x) = f(a) + f^{'}(a)(x-a) + R_{1,a}(x)$$
   where
   $$R_{1,a}(x) = \frac{1}{2} f^{''}(c)(x-a)^2$$

2. Taylor's Theorem for Function of Two Variables
   If $f(x,y) \in C^2$ in some neighborhood N(a, b) of (a, b), then for all $(x,y) \in N(a,b)$ there exists a point (c, d) on the line segment joining (a, b) and (x, y) such that
   $$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_{1,(a,b)}(x,y)$$
   where
   $$R_{1,(a,b)}(x,y) = \frac{1}{2}[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2]$$

3. Corollary
   If $f(x,y) \in C^2$ in some closed neighborhood N(a, b) of (a, b), then there exists a positive constant M such that
   $$|R_{1,(a,b)}(x,y)| \leq M||(x,y)-(a,b)||^2, \forall (x,y) \in N(a,b)$$

8.3 Grneralizations of the Taylor Polynomial

1. Multi-Index Notation
   If $f \in C^k$ is a function of n variables, we can drite k th order partial derivative of $f(x_1, \ldots, x_n)$ is

$$d^a f = (\frac{d}{dx_1})^{a_1} \times \ldots \times (\frac{d}{dx_n})f$$

   where a is multi-index and $a = (a_1 \ldots a_n)$, the order is $k = \sum a_i = |a|$ and $a! = a_1! \times \ldots \times a_n!$

2. k-th degree Taylor polynomial
   is defined as

$$P_{k,(a,b)}(x, y) = \sum_{|a| \leq k} (d^a f)(a, b) \frac{[(x, y) - (a, b)]^a}{a!}$$

3. Theorem 1L Taylor's Theorem of order k
   If $f(x, y) \in C^{k+1}$ in some neighbourhood $N(a, b)$, then $\forall (x, y) \in N(a, b), \exists$ a point (c, d) on the line segment between(a, b) and (x, y) such that

$$f(x, y) = P_{k,(a,b)}(x, y) + R_{k,(a,b)}(x, y)$$

   where

$$R_{k,(a,b)}(x, y) = \sum_{|a|=k+1} d^a f(c, d) \frac{[(x, y) - (a, b)]^a}{a!}$$

4. Corollary
   If $f(x, y) \in C^k$ then

$$\lim_{(x,y) \to (a,b)} \frac{|f(x, y) - P_{k,(a,b)}(x, y)|}{||(x, y) - (a, b)||^k} = 0$$

9 Critical Points

9.1 Local Extrema and Critical Points

1. Local Max and Min
   A point (a, b) is

   - Local max if $f(x, y) \leq f(a, b) \forall (x, y)$ in neightborhood of (a, b)
   - Local min if $f(x, y) \geq f(a, b) \forall (x, y)$ in neighborhood of a, b

2. Thoerem 1
   If (a, b) is a local extrema, then $f_x(a, b) = f_y(a, b) = (0 or DNE)$

3. Critical POint
   A point(a, b) in the domain of $f(x, y)$ is called a critical point of f if
   $f_x(a, b) = 0 = f_y(a, b)$ or
   one of f_x or f_y DNE at (a, b)

4. Saddle Point
   A critical point (a, b) of $f(x, y)$ is called a saddle point of f if in every neighborhood of (a, b) there exists points(x_1, y_1) and(x_2, y_2) such
   $f(x_1, y_1) > f(a, b)$ and $f(x_2, y_2) < f(a, b)$

9.2 Second Derivative Test

1. Quadratic Froms
   A function Q of the form $Q(u,v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$ where a_{ii} is constants

2. Derterminant and Quadratic forms

   - Positive definite if $det(A) > 0$ and $a_{11} > 0$
   - Negative definite if $det(A) > 0$ and $a_{11} < 0$
   - Indifinite if $det(A) < 0$
   - Semidefinite if $det(A) = 0$

3. Hessian Matrix Hf(a, b) = $\begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{pmatrix}$

4. Determinant and Quadratic Forms
   $Q(u,v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$

   - Positive definite : if $det(A) > 0$, and $a_{11} > 0$
   - Negative definite: $det(A) > 0$ and $a_{11} < 0$
   - Indefinite: $det(A) < 0$
   - Semidefinite: $det(A) = 0$

5. Second Partical Derivatives Test
   Suppose that $f(x,y) \in C^2$ in some neighborhood of (a, b) and that $f_x(a,b) = 0 = f_y(a,b)$

   - if $Hf(a,b)$ is positive definite, then(a, b) is a local minimum
   - if $Hf(a,b)$ is negative definite, then (a, b) is a local maximum
   - if $Hf(a,b)$ is indifinite, then (a, b) is a saddle point
   - if $Hf(a,b)$ is semidefinite, then the test is inconclusive

9.3 Convex Functions

1. Convex and stritly convex functions of one variable
A twice differentiable function $f(x)$ is convex if $f^{''}(x) \geq 0$ for all x and f is strictly convec if $f^{''}(x) > x$ for all x, which means concave up

2. Theorem1 :Properties of convex functions of one variable
If $f(x) \in C^2$ and is strictly convec, then

   - $f(x) > L_a(x) = f(a) + f^{'}(x)(x-a)$ for all $x \neq a$, for any $a \in R$
   - For $a < b$, $f(x) < f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ for $x \in (a, b)$

3. Convec and strictly convex functions of two variables
Let f(x, y) have continuous second partial derivatives. We say that f is convex if $Hf(x, y)$ is positive semi-definite for all (x, y) and
f is strictly convex if $Hf(x, y)$ is positve definte for all (x, y).

4. Theorem 2: Properties of convex functions of two variables
If $f(x, y)$ has continuous second partial derivatives and is strictly convex, then

   - $f(x, y) > L_{(a,b)}(x, y)$ for all $(x, y) \neq (a, b)$ and
   - $f(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2)) < f(a_1, a_2) + t[f(b_1, b_2) - f(a_1, a_2)]$ for 0 ¡

5. Theorem 3: Critical Points of convex and strictly convex functions

   - if $f(x, y) \in C^2$ is convec, then every critical point(c, d) satisfies $f(x, y) \geq f(c, d)$ for all $(x, y) \neq (c, d)$
   - If $f(x, y) \in C^2$ is strictly convec and has a critical point(c ,d) then $f(x, y) \geq f(c, d)$ for all $(x, y) \neq (c, d)$ and f has no other critical point

9.4 Proof of the Second Partial Derivative Test

1. Lemma 1
   Let
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
   be a positive definite matrix, if $|a'-a|, |b'-b|, |c'-c|$ are sufficiently small, then
   $$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$
   is positive definite

10 Optimization Problems

10.1 The extreme Value Theorem

1. Absolute Max and Min for one variable

   - Absolute Max of f on I for a point $x = c \in I$ if $\forall x \in I, f(x) \leq f(c)$
   - Absolute Min of f on I for a point $x = c \in I$ if $\forall x \in I, f(x) \geq f(c)$

2. The extreme Value theorem for one variable
   If f(x) is continuous on finite closed intervial I, then $\exists c_1, c_2 \in I$ such
   $f(c_1) \leq f(x) \leq f(c_2) \forall x \in I$

3. Absolute Max and Min for two variable

   - Absolute Max of f on I for a point $x = (a, b) \in I$ if $\forall x \in I, f(x, y) \leq f(a, b)$
   - Absolute Min of f on I for a point $x = (a, b) \in I$ if $\forall x \in I, f(x, y) \geq f(a.b)$

4. Bounded Set
   A set $S \in R^2$ is said to be bounded $\iff$ it contain some neighbourhood of the origin

5. Boundary Point
   Given a set $S \subseteq R^2$, a point $(a, b) \in S^2$ is said to be a boundary point of S $\iff$ every neighbourhood contain at least one point in S and one point not in S

6. Boundary of S
   B(S) contain all Boundary Point of S

7. Closed Set
   A set $S \subseteq R^2$ is said to be closed if S contain all boundary points

8. Extreme valud theorem for two variables
   If f(x, y) is continouous on a closed and bounded set $S \subseteq S^2$, then there exist points $(a, b), (c, d) \in S$such
   $f(a, b) \leq f(x, y) \leq f(c, d) \forall (x, y) \in S$

10.2 Algorighm for Extreme Values

1. Check if $S \subseteq R^2$ is closed and bounded
2. check it $f(x, y)$ is continouous
3. Next Finad all critical points of S
4. evaluate f at each point
5. Fined max and min value s of f aon B(S)
6. The max of f on S is the largetst value found in step 4 and 5 and min is the same

10.3 Lagrage Multiplier Algorithm

Assume that f(x, y) is a differentiable function and $g \in C^1$. To find the max and min of f subject ot the constraint g(x, y) = k
Evaluate f(x, y) at all points(a, b) which satisfy one of the following

- $\nabla f(a,b) = \lambda \nabla g(a,b)$
- $\nabla g(a,b) = (0,0)$
- (a,b) is an end point of $g(x,y) = k$

11 Coordinate System

11.1 PolarC oordinates

1. Definitions
   - Pole: the origin of a polar plane
   - Polar axis: a ray drawed from the pole
   - (r, θ) is a coordinate from the polar plane
2. Relationship with Cartesian Coordinates
   - $x = r * cos\theta$
   - $y = r * sin\theta$
   - $r = \sqrt{x^2 + y^2}$
   - $\theta = tan^{-1}(\frac{y}{x})$
3. Area of sector

$$= \frac{1}{2} * r^2 * (\theta_2 - \theta_1)$$

11.2 Cylindrical Coordinates

12 Mapping of R^2 into R^2

12.1 The Geometry of Mappings

1. Vector-valued Function
   A function Whose domain is a subset of R^n and whose codomain is R^m is called a vector -valued function

2. Mapping
   A $R^n - > R^n$ vector calued function is a mapping

13 Jacobians and Inverse Mappings

13.1 The inverse Mapping Theorem

1. Invertible Mapping and Inverse Mapping
   Let F be a mapping from set D_{xy} on set D_{uv}. If there exists a mapping of F^{-1}, called the inverse of F, which maps D_{uv} onto D_{xy} such that
   $(x, y) = F^{-1}(u, v) \iff (u, v) = F(x, y)$
   then F is invertible on D_{xy}

2. One-to-One(injective)
   A mapping F from $R^2 \to R^2$ is said to be one-to-one (or injective) on a set $D_{xy} \iff F(a, b) = F(c, d)$ implies $(a, b) = (c, d) \forall (a, b), (c, d) \in D_{xy}$

3. One-to-One implies Invertible
   If F is one-to-one, then F is intervible

4. Theorem 2: Inverse of the derivative Matrix
   Consider a mapping F which maps D_{xy} onto D_{uv}
   If F has continuous partial derivatives at $\vec{x} \in D_{xy}$ and there exists an inverse mapping F^{-1} of F which has contiuous partial derivatives at
   $\vec{u} = F(\vec{x}) \in D_{uv}$, then
   $DF^{-1}(\vec{u})DF(\vec{x}) = I$

5. The jacobian
   The Jacobian of a mapping $(u, v) = F(x, y) = (u(x, y), v(x, y))$ is denoted $\frac{d(u,v)}{d(x,y)}$, and is defined by
   $\frac{d(u,v)}{d(x,y)} = \frac{du}{dx} \times \frac{dv}{dy} - \frac{du}{dy} \times \frac{dv}{dx}$

6. Corollary 3

14 Double Integrals

14.1 Definition of Double Intergrals

1. Integrable function
   Let $D \subseteq R^2$ be closed and bounded. Let P be a partition of D, and $|\Delta P|$ be the length of the longest side of all rectangle in P.
   A function $f(x, y)$ which is bounded on D is integrable on D if all riemann sum approach the same value as $|\Delta P| \to 0$

2. Double Integral
   If$f(x, y)$ is integrable on a closed bounded set D, then we define the double integral of F on D as
   $\int\int_D f(x,y)dA = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$

3. Theorem 1 Linearity
   If $D \subseteq R^2$ is a closed and bounded set and f and g are two integrable functions on D, then for any constatn c:

   - $\int\int_D (f + g)dA = \int\int_D f dA + \int\int_D g dA$
   - $\int\int_D cf dA = c\int\int_D f dA$

4. Thoerem 2 Basic Inequality
   if $\forall(x, y) \in D, f(x, y) \leq g(x, y)$
   then $\int\int_D f dA \leq \int\int_D g dA$

5. Theorem 3 Absolute Value Inequality
   $|\int\int_D f dA| \leq \int\int_D |f| dA$

6. THeorem 4 Decomposition
   Let $D_1 + D_2 = D$, then
   $\int\int_D f dA = \int\int_{D_1} f dA + \int\int_{D_2} f dA$

14.2 Iterated Integrals

1. Iterated Integrals
   Let $D \subseteq R^2$ be defined by $y_l(x) \leq y \leq y_u(x)$, and $x_l \leq x \leq x_u$ where both y are continuous , if $f(x, y)$ continuous on D, then
   $\int\int_D f(x,y)dA = \int_{x_l}^{x_u} \int_{y_l(x)}^{y_u(x)} f(x,y)dydx$

14.3 The Change of Varibale Theorem

1. The Theorem
   Let each of D_{uv} and D_{xy} be closed bounded set
   Let $(x, y) = F(u, v) = (f(u, v), g(u, v))$
   be one to one mapping of D_{uv} onto D_{xy} with $f, g \in C^1$
   and $\frac{d(x,y)}{d(u,v)} \neq 0$ expect for possibly on a finite collection of piecewise-smooth curves in D_{uv}
   If $G(x, y)$ is continouous on D_{xy}, then
   $\int\int_{D_{xy}} G(x, y)dxdy = \int\int_{D_{uv}} G(f(u, v), g(u, v))|\frac{d(x,y)}{d(u,v)}|dudv$

15 Triple Integrals

15.1 Definition of Triple Integrals

1. Integrable
   A function $f(x, y, z)$ which is bounded on a closed bounded set $D \subset R^3$ is said to be integrable on D $\iff$ all Riemann sums approach the same value as $\Delta P \to 0$

2. Triple Integral
   if $f(x, y, z)$ is integrable on a closed bounded set D, then we define the triple integral of f over D as
   $\int\int\int_D f(x, y, z)dV = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i$

3. Average Value
   Let $D \subset R^3$ be closed and bounded with volume $V(D) \neq 0$, and let $f(x, y, z)$ be a bounded and integrable function on D
   The average value of f over D is defined by
   $f_{avg} = \frac{1}{V(D)} \int\int\int_D f(x, y, z)dV$

4. Properties of Triple Integral

   - Linearty
     if $D \subset R^3$ is a closed and boueded set, c is constant, and f and g are two integrable functions on D, then
     $\int\int\int_D (f + g)dV = \int\int\int_D fdV + \int\int\int_D gdV$
     $\int\int\int_D cfdV = c\int\int\int_D fdV$
   - Basic Inequality
     If $D \subset R^3$ is a closed and bounded set and f and g are two integrable functions on D
     such that $f(x, y, z) \leq g(x, y, z)$ for all $(x, y, z) \in D$, then
     $\int\int\int_D fdV \leq \int\int\int gdV$
   - Absolute Value Inequality
     if $D \subset R^3$ is a closed and bounded set and f is an integrable function on D, then
     $|\int\int\int_D fdV| \leq \int\int\int_D |f|dV$
   - Decomposition
     Assume $D \subset R^3$ is a closed and bounded set and f is an integrable function on D.
     If D is decomposed in two D_1 and D_2, then
     $\int\int\int_D fdV = \int\int\int_{D_1} fdV + \int\int\int_{D_2} fdV$

15.2 Iterated Integrals

Let $D \subset R^3$ defined by $z_l(x, y) \leq x \leq z_u(x, y)$ and $(x, y) \in D_{xy}$
where z_l and z_u are continuous functinos on D_{xy} and D_{xy} is closed bounded subset in R^2
If $f(x, y, z)$ is continuous, then
$\int \int \int_D f(x, y, z) dV = \int \int_{D_{xy}} \int_{z_l(x,y)}^{z_u(x,y)} f(x, y, z) dz dA$

15.3 The change of Variable Theorem

1. Change of Variable Theorem
   Let $x = f(u, v, w), y = g(u, v, w), z = h(u, v, w)$
   be a one-to-one mapping of D_{uvw} onto D_{xyz}, with f, g, h having continuous partials, and
   $\frac{d(x,y,z)}{d(u,v,w)} \neq 0$ on D_{uvw}
   If $G(x, y, z)$ is continuous on D_{xyz}
   $\int\int\int_D G(x, y, z)dV = \int\int\int_{D_{uvw}} G(f(u, v, w), g(u, v, w), h(u, v, w))|\frac{d(x,y,z)}{d(u,v,w)}|dV$