

The final rules

CS135 Lecture 10

Rules for recursion (L06 version)



1. Change one argument closer to termination while recurring. No other arguments can change.
2. When recurring on a natural number use `(sub1 n)` and test termination with `zero?`
3. When recurring on a list use `(rest 1st)` and test termination with `empty?`

In this version, the argument in #1 is always the same argument and no other arguments can change.

Together, #2 and #3 guarantee that the function will terminate because the argument always gets “smaller” until it reaches a base case.

Finalizing the Rules for Recursion



In this lecture we will make two final extensions to our rules:

1. We will allow recursion on more than one argument. Arguments involved in recursion should either be unchanged or get closer to the base case. At least one argument should be closer to the base case on each recursive call.
2. Arguments that aren't involved in recursion can change.

We'll make one extension at a time...

L10.0 Indexing a list



The n th item in a list

```
;; Produce the nth item in a list
;; index: Nat (listof Any) -> Any
(define (index n lst) ...)
```

Since n is a natural number, it's simplest to start indexing from 0:

```
(check-expect (index 4 (list 0 1 2 3 4 5)) 4)
(check-expect (index 2 (list 'a 'b 'c 'd 'e)) 'c)
```

What to do if n is \geq than the length of the list? Maybe just produce `empty`?

```
(check-expect (index 100 (list 0 1 2 3 4 5)) empty)
(check-expect (index 100 empty) empty)
```



Key observation

The solution to

```
(index 2 (list 'a 'b 'c 'd 'e))
```

is the same as the solution to

```
(index 1 (list 'b 'c 'd 'e))
```

is the same as the solution to

```
(index 0 (list 'c 'd 'e))
```

On each recursive call we **sub1** from n and recurse on the **rest** of the list.

Eventually either $n = 0$ or the list is empty. What then?

Recursion on a natural number and a list



If more than one argument changes we need to explicitly check all base cases. So the body of our index function will look like one of these:

```
(define (index n lst)
  (cond [(empty? lst) ...]
        [(zero? n) ...]
        ...))
```

```
(define (index n lst)
  (cond [(zero? n) ...]
        [(empty? lst) ...]
        ...))
```

Which one?



Recursion on two arguments

Let's think about what we do for each base case and decide

```
(define (index n lst)
  (cond [(empty? lst) empty]
        [(zero? n) (first lst)]
        ...))
```

```
(define (index n lst)
  (cond [(zero? n) (first lst)]
        [(empty? lst) empty]
        ...))
```

You have all the pieces to finish the function. Try to put them together yourself.

Generalizing `first` and `rest`



```
(define (first-n n lst)
  (cond [(or (empty? lst) (zero? n)) empty]
        [else (cons (first lst)
                     (first-n (sub1 n) (rest lst))))])
(check-expect (first-n 3 (list 1 2 3 4 5 6 7)) (list 1 2 3))

(define (rest-n n lst)
  (cond [(or (empty? lst) (zero? n)) lst]
        [else (rest-n (sub1 n) (rest lst))]))
(check-expect (rest-n 3 (list 1 2 3 4 5 6 7)) (list 4 5 6 7))
```

Rules for Recursion (almost the final version)



1. Change **at least** one argument closer to termination while recurring
2. When recurring on a natural number use **(sub1 n)** and test termination with **zero?**
3. When recurring on a list use **(rest 1st)** and test termination with **empty?**

L10.1 Recursion on two lists

Vectors



In mathematics, a **vector** is a set of numbers arranged in a specific order, often used to represent a point in space or a direction and magnitude.

For example, $\langle 3, 5, 2 \rangle$ is a vector with three elements. Each element corresponds to a coordinate or a value in a specific dimension.

In Racket, we represent a vector as a list of numbers:

```
(list 3 5 2)
```

```
(list -10.6 14.8 23.7 0.03 111.8)
```

For convenience, we define the empty vector with no entries, as `empty`.

Dot Product



A ***dot product*** is a way of multiplying two vectors.

To compute the dot product of two vectors, we multiply entries in corresponding positions (first with first, second with second, and so on) and sum the results.

$$\langle 1 \ 2 \ 3 \rangle \cdot \langle 4 \ 5 \ 6 \rangle = 1 \times 4 + 2 \times 5 + 3 \times 6 = 4 + 10 + 18 = 32$$

```
(check-expect (dot-product (list 1 2 3) (list 4 5 6)) 32)
```

What do we do if the lists are of different lengths?



Recursion on two lists

If more than one argument changes we need to explicitly check all base cases.

```
;; compute the dot product of two vectors
;; dot-product: (listof Num) (listof Num) -> Num
(define (dot-product lst1 lst2)
  (cond [(empty? lst1) 0]
        [(empty? lst2) 0]
        [...])
```



Recursion on two lists

Change at least one argument closer to termination while recurring:

```
;; compute the dot product of two vectors
;; dot-product: (listof Num) (listof Num) -> Num
(define (dot-product lst1 lst2)
  (cond [(empty? lst1) 0]
        [(empty? lst2) 0]
        [else (+ (* (first lst1) (first lst2))
                  (dot-product (rest lst1) (rest lst2))))]))
```

In this case both change. This is called recursion in “lockstep”.

The `index` function is an example of a natural number and list in lockstep.



Merging two sorted lists.

Design a function `merge` that consumes two lists of numbers. Each consumed list is sorted in ascending order. Produced list is in ascending order.

```
;; merge: (listof Num) (listof Num) -> (listof Num)
;; Requires: lists must be sorted in ascending order
(define (merge lst1 lst2)
  (cond [(empty? lst1) lst2]
        [(empty? lst2) lst1] ....)
  (check-expect (merge (list 2 4 6 8) (list 1 3 5 7))
                (list 1 2 3 4 5 6 7 8))
  (check-expect (merge (list 1 2 2 3 3) (list 1 2 3 3))
                (list 1 1 2 2 2 3 3 3 3)))
```

Merging two sorted lists



If neither list is empty, there are three cases:

Case #1 [`< (first lst1) (first lst2)`)

Case #2 [`> (first lst1) (first lst2)`)

Case #3 [`else ;; the first elements are equal`



Case #1 [`(< (first lst1) (first lst2))`

`(merge (list 1 3 3 7) (list 2 3 9 11))`



`[(< (first lst1) (first lst2))
(cons (first lst1) (merge (rest lst1) lst2))]`



`(cons 1 (merge (list 3 3 7) (list 2 3 9 11)))`



Case #2 [(> (first lst1) (first lst2))

(merge (list 3 3 7) (list 2 3 9 11))



[(> (first lst1) (first lst2))
(cons (first lst2) (merge lst1 (rest lst2))))]



(cons 2 (merge (list 3 3 7) (list 3 9 11))))

Case #3 [else



```
(merge (list 3 3 7) (list 3 9 11))
```



```
[else (cons (first lst1)
             (cons (first lst2)
                   (merge (rest lst1) (rest lst2)))))]
```



```
(cons 3 (cons 3 (merge (list 3 7) (list 9 11))))
```



Merging two sorted lists.

```
;; merge: (listof Num) (listof Num) -> (listof Num)
;; Requires: lists must be sorted in ascending order
(define (merge lst1 lst2)
  (cond [(empty? lst1) lst2]
        [(empty? lst2) lst1]
        [(< (first lst1) (first lst2))
         (cons (first lst1) (merge (rest lst1) lst2)))]
        [(> (first lst1) (first lst2))
         (cons (first lst2) (merge lst1 (rest lst2)))]
        [else (cons (first lst1)
                     (cons (first lst2)
                           (merge (rest lst1) (rest lst2))))]))
```

L10.2 Accumulators

Reversing a list in lecture L09



```
(define (attach element lst)
  (cond [(empty? lst) (cons element empty)]
        [else (cons (first lst) (attach element (rest lst))))]))
```



```
(define (rev lst)
  (cond [(empty? lst) empty]
        [else (attach (first lst) (rev (rest lst))))]))
```

Steps to reverse a list of length $n = 3.5n^2 + 6.5n + 4 = O(n^2)$

Can we do better?



Reversing a list

Reverse would straightforward if we could build a list during recursion.

List A

(list 1 2 3 4 5)

(list 2 3 4 5)

(list 3 4 5)

(list 4 5)

(list 5)

empty

List B

empty

(list 1)

(list 2 1)

(list 3 2 1)

(list 4 3 2 1)

(list 5 4 3 2 1)

Each step of the recursion we remove an element from List A and add it to List B.



Reversing a list

We can build a list during recursion if we allow arguments not involved in recursion to change. In this case, we will have a second argument that will increase.

```
(define (rev/accumulate lst accumulator)
  (cond [(empty? lst) accumulator]
        [else (rev/accumulate
                  (rest lst)
                  (cons (first lst) accumulator))]))
```

We still use `(rest lst)` and test termination with `empty?` so we know the function will terminate. The argument that increases is called an “accumulator”.



Reversing a list

In the stepper, we see:

```
(rev/accumulate (list 1 2 3 4 5) empty) ⇒ ... ⇒  
(rev/accumulate (list 2 3 4 5) (list 1)) ⇒ ... ⇒  
(rev/accumulate (list 3 4 5) (list 2 1)) ⇒ ... ⇒  
(rev/accumulate (list 4 5) (list 3 2 1)) ⇒ ... ⇒  
(rev/accumulate (list 5) (list 4 3 2 1)) ⇒ ... ⇒  
(rev/accumulate empty (list 5 4 3 2 1))
```

The base case is: [(empty? lst) accumulator]

The base case produces the accumulator.

Wrapping it up



Accumulators need to be initialized (in this case to `empty`). This initialization happens in a *wrapper function*.

```
(define (rev/accumulate lst accumulator)
  (cond [(empty? lst) accumulator]
        [else (rev/accumulate
                  (rest lst)
                  (cons (first lst) accumulator))])))

(define (rev lst)
  (rev/accumulate lst empty))
```

We have already seen other examples of wrapper functions this term.



Efficiency

`(rev empty)` \Rightarrow 5 steps

`(rev (list 1))` \Rightarrow 12 steps

`(rev (list 1 2))` \Rightarrow 19 steps

`(rev (list 1 2 3))` \Rightarrow 26 steps

...

`(rev (list 1 2 3 4 5 6 7 8 9 10))` \Rightarrow 75 steps

Total steps (where n is the length of the list): $7n + 5 = O(n)$, i.e., linear.

Built-in `reverse`



DrRacket has a built-in function `reverse` that you can now use as appropriate.

Like the built-in functions `length` and `append`, you should think of the efficiency of `reverse` as linear in the size of the consumed list even though in the stepper it only takes one step.

Recursion using an accumulator can be tricky. If you write a function where the answer comes out “backwards” it can be better to `reverse` the answer in a wrapper, rather than trying get an accumulator working.

Be super-careful to avoid `reverse` in the body of a recursive function.

Accumulative recursion



An *accumulator* is a function parameter that keeps track of partial results during recursion.

Instead of waiting until the recursion reaches the end and then combining results on the way back up, an accumulator carries the needed information along as the recursion goes down. This can make the function more efficient and easier to reason about.

Accumulators can take some time and examples to understand, so we'll go slow. We will have more examples in coming lectures.

Counting down



The Rules for Recursion make it easy to generate a list of natural numbers in decreasing order.

```
(define (count-down n)
  (cond [(zero? n) (list 0)]
        [else (cons n (count-down (sub1 n))))])
(check-expect (count-down 9) (list 9 8 7 6 5 4 3 2 1 0))
```

But to generate a list of numbers in increasing order, while following the rules, we either need to use `reverse`, do some arithmetic with `n`, or count up with an accumulator.

Counting up with an accumulator



```
;; Produce a strictly increasing list of natural numbers by
;; counting up from a natural number n times
;; count-up: Nat Nat -> (listof Nat)
(define (count-up from n)
  (cond [(zero? n) empty]
        [else (cons from (count-up (add1 from) (sub1 n))))]))
```



```
(check-expect (count-up 100 5)
              (list 100 101 102 103 104))
(check-expect (count-up 9999999 0) empty)
```

Rules for recursion (final version)



1. Change at least one argument closer to termination while recurring
2. When recurring on a natural number use `(sub1 n)` and test termination with `zero?`
3. When recurring on a list use `(rest 1st)` and test termination with `empty?`
4. Arguments that aren't involved in recursion can change.

Lecture 10 Summary

L10: You should know



- The final “Rules for Recursion”. You must stick to these rules for now.
- Recursion on a natural number and a list.
- Recursion on two lists, including the “lockstep” pattern.
- Accumulators and accumulative recursion.
- How to reverse a list using an accumulator.



L10: Allowed constructs

Newly allowed constructs:

`reverse`

Rules for Recursion (final version)

Previously allowed constructs:

`() [] + - * / = < > <= >= ,`

`abs acos add1 and append asin atan boolean? check-expect
check-within cond cons cons? cos define e else empty empty?
even? exp expt false first inexact? integer? length list
list? log max min not number? odd? or pi quotient rational?
remainder rest second sin sqr sqrt sub1 symbol? symbol=? tan
third true zero?`
`listof Any anyof Bool Int Nat Num Rat Sym`