CS 231
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Math session solutions

Question 1: You have a list of n integers that you wish to split into equal-size pieces, or rather, as-close-as-possible-to-equal-size pieces, as the size of the list may not be divisible by 3. Express the lengths of the biggest and the smallest pieces as integers.

Solution to Question 1:

We will use floors and ceilings to express the sizes of the pieces.

If $n/3$ is an integer, then $n \bmod 3 \equiv 0$. The other two possibilities are $n \bmod 3 \equiv 1$ and $n \bmod 3 \equiv 2$, as the remainder cannot be three or greater.

Case 1: $n \bmod 3 \equiv 0$

In this case, all pieces are of size $n/3 = \lceil n/3 \rceil = \lfloor n/3 \rfloor$.

Case 2: $n \bmod 3 \equiv 1$

In this case, two of the pieces will have $\lfloor n/3 \rfloor$ items and one will have one more item, or $\lceil n/3 \rceil$.

Case 3: $n \bmod 3 \equiv 2$

In this case, one of the pieces will have $\lfloor n/3 \rfloor$ items and two will each have one more item, or $\lceil n/3 \rceil$.

In all cases, the smallest piece has $\lfloor n/3 \rfloor$ items and the biggest has $\lceil n/3 \rceil$ items.

Question 2: Suppose you wish to form a list of k items from a set of n distinct items, where each item can appear at most once in the list. Two lists are different if they either do not contain the same items or if they have the same items in a different order. How many different lists can you form?

Solution to Question 2:

We calculate the number of lists by first determining the number of subsets of k items chosen from a set of n items and then determining how many orderings there are for each subset.

The number of subsets of size k is simply $\binom{n}{k}$, which is equal to $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. The number of orderings of a subset of k items is $k!$. Multiplying the two quantities together, we obtain the result $\frac{n!}{(n-k)!}$.

Question 3: Show that $\sum_{i=2}^{n} \log i \in \Theta(n \log n)$ *without* determining an exact value for the sum.

Solution to Question 3:

To determine an upper bound, we use the idea of replacing each term by an upper bound on the term. Because the sum consists of $n-1$ terms, each of size at most $\log n$, we can show the sum is in $O(n \log n)$.

To determine a lower bound, we can first drop all terms smaller than $\log\lceil n/2 \rceil$. After dropping those terms, we will have $\lceil n/2 \rceil$ remaining terms, and each will have size at least $\log\lceil n/2 \rceil$. Because logarithms of bigger numbers are no smaller than logarithms of smaller numbers, we know that $\log\lceil n/2\rceil \geq \log(n/2)$, and because of the formula $\log(x/y) = \log x - \log y$, we know that $\log(n/2) = \log n - \log 2 = \log n - 1$.

We can then form a lower bound on our $\lceil n/2\rceil$ terms of size at least $\log\lceil n/2\rceil \geq \log(n/2) = \log n - 1$ by multiplying the number of terms by the lower bound on their size. We thus have $\lceil n/2\rceil(\log n - 1) = \lceil n/2\rceil \log n - \lceil n/2\rceil \geq (n \log n)/2 - \lceil n/2\rceil$.

We observe that when $n \geq 2^4 = 16$, $(\log n)/4 \geq 1$. Thus, $(n \log n)/4 \geq n \geq \lceil n/2\rceil$ for any value of $n \geq 16$; the total $(n \log n)/2 - \lceil n/2\rceil \geq (n \log n)/4 + ((n \log n)/4 - \lceil n/2\rceil) \geq (n \log n)/4 = c\dot{n} \log n$ for $c = 1/4$ and any $n \geq n_0 = 16$. Summarizing the steps, we have shown that $\sum_{i=2}^{n} \log i \in \Omega(n \log n)$, by showing the following for $c = 1/4$ and any $n \geq n_0 = 16$:

$$\sum_{i=2}^{n} \log i \geq \sum_{i=\lceil n/2\rceil}^{n} \log i \geq \lceil n/2\rceil \log\lceil n/2\rceil \geq (n \log n)/2 - \lceil n/2\rceil \geq (n \log n)/4 = c\dot{n} \log n$$

Putting the two results together, we have shown that the sum is in $\Theta(n \log n)$.

Question 4: Express $\log((mn)!)$ in order notation as a function of m and n.

Solution to Question 4:

We can use the fact that $\log((mn)!) = \log(mn \times (mn - 1) \times \cdots \times 1)$ and the formula $\log xy = \log x + \log y$ to show that $\log((mn)!) = \sum_{i=1}^{mn} \log i$. Then, using the result of Question 3, we conclude that $\log((mn)!) \in \Theta(mn \log mn) = \Theta(mn(\log m + \log n))$.

Question 5: In a *d-ary* tree, each internal node can have at most d children.

1. What is the maximum number of leaves in a d-ary tree of height h?
2. What is the minimum number of leaves in a d-ary tree of height h?
3. What is the maximum number of nodes in a d-ary tree of height h?
4. What is the minimum number of nodes in a d-ary tree of height h?
5. What is the maximum height of a d-ary tree with ℓ leaves?
6. What is the minimum height of a d-ary tree with ℓ leaves?

Solution to Question 5:

We obtain the maximum number of leaves by ensuring that each internal node has d children and that all the leaves are at the same level. Because the number of nodes goes up by a factor of d at each subsequent level, there are d^h nodes at level h, and hence d^h leaves.

As there is no minimum number of children required for an internal node, in a tree with at least one node, each internal node can have exactly one child, resulting in a minimum of one leaf.

The maximum number of nodes is reached when each internal node has d children and all leaves are at the same level. The total number of nodes in the tree is $\sum_{i=0}^{h} d^i = \sum_{j=1}^{h+1} d^{j-1} = \frac{(1-d^{h+1})}{(1-d)}$. For d a constant, this number is in $\Theta(d^h)$.

The minimum number of nodes will be one per level, for a total of $h + 1$ nodes.

There is no maximum height for a d-ary tree with ℓ leaves. To see why, even if $\ell = 1$, we can create a tree with one child for each internal node and make the height as big as we like, always retaining a single leaf at the last level.

If the number of leaves is a power of d, we can obtain the minimum height by giving each internal node d children and filling the last level with leaves, for a height of $\log_d \ell$. Otherwise, we put leaves on the last two levels, so that the height is $\lceil \log_d \ell \rceil$. When d is a constant, then $\Theta(\log_d \ell)$ is equivalent to $\Theta(\log \ell)$, using the formula $\log_a x = \log_b x / \log_b a$ and the fact that Θ notation ignores constant factors.

Question 6: In all of the following subquestions, $P(x)$ is the predicate "The integer x is odd" and $Q(x)$ is the predicate "The integer x is greater than 10."

1. Prove that the following statement is false: "There exists an x such that $P(x)$ is true and $P(x+1)$ is true."

2. Prove that the following statement is true: "There exists an x such that $Q(x)$ is true and $Q(x-1)$ is true."

3. What kind of statement is the following? "For every number x greater than 5, $Q(x)$ is true." Prove or disprove it.

Solution to Question 6:

1. To show that the statement is false, we show that the negation of the statement is true. The negation of the statement is "For all x, if $P(x)$ is true then $P(x+1)$ is false." That is, if x is odd, then we know $x+1$ is even, and hence $P(x+1)$ is false.

2. To show the statement is true, we need to demonstrate a single value of x such that both $Q(x)$ and $Q(x-1)$ are true. There are many possible choices; in fact, the only x for which $Q(x)$ is true and $Q(x-1)$ is false is the value $x = 11$. For any other example, such as $x = 12$, such that $Q(x)$ is true, $Q(x-1)$ is also true.

3. The statement is a universal statement. To disprove it, we prove its negation, which is "There exists a number greater than 5 such that $Q(x)$ is false." We can prove the statement by providing any of the numbers in the set $\{6, 7, 8, 9, 10\}$.

Question 7: Suppose you wished to use induction to prove that a tree with height n has at most 2^n leaves.

1. What is the base case?

2. Provide a proof of the base case.

3. What is the induction step?

4. Provide a proof of the induction step.

Solution to Question 7:

1. For the base case, $n = 0$, resulting in a tree with a single node.
2. In a tree with a single node, the single node is a leaf. The number of leaves is thus $2^n = 2^0 = 1$.
3. For the induction step, we assume that a tree of height i has at most 2^i nodes for any $i \leq k$ and show that this implies that a tree of height $k + 1$ has at most 2^{k+1} nodes.
4. A tree of height $k + 1$ consists of a root with at most two children, each of which is the root of a subtree of height at most k. By our assumption, each of the at most two subtrees has at most 2^k leaves, for a total of at most $2 \cdot 2^k = 2^{k+1}$.

Question 8: Determine the expected value of the weight of an edge in Sample graph 4, where an event is the selection of an edge, each edge is equally likely, and the value of an event is the weight of the selected edge.

Solution to Question 8:

We calculate the expected value by taking the sum of the products of the probabilities of each event and the value of each event. As there are six edges in the graph, there are six possible events, each with weight 1/6. Thus, the total cost is $1/6 \times 10 + 1/6 \times 20 + 1/6 \times 30 + 1/6 \times 40 + 1/6 \times 5 + 1/6 \times 6 = 1/6 \times 111 = 111/6$.