# CS 240 – Data Structures and Data Management

## Module 1: Introduction and Asymptotic Analysis

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Based on lecture notes by many previous cs240 instructors

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# Outline

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# Course Objectives: What is this course about?

- Much of Computer Science is *problem solving*: Write a program that converts the given input to the expected output.
- When first learning to program, we emphasize *correctness*: does your program output the expected results?
- Starting with this course, we will also be concerned with *efficiency*: is your program using the computer's resources (typically processor time) efficiently?
- We will study efficient methods of *storing*, *accessing*, and *organizing* large collections of data.

  **Motivating examples:** Digital Music Collection, English Dictionary

  Typical operations include: *inserting* new data items, *deleting* data items, *searching* for specific data items, *sorting*.

# Course Objectives: What is this course about?

- We will consider various **abstract data types** (ADTs) and how to realize them efficiently using appropriate **data structures**.
- We will some problems in **data management** (sorting, pattern matching, compression) and how to solve them with efficient **algorithms**.
- There is a strong emphasis on mathematical analysis in the course.
- Algorithms are presented using pseudo-code and analyzed using order notation (big-Oh, etc.).

# Course Topics

1. background, big-Oh analysis
2. priority queues and heaps
3. efficient sorting, selection
4. binary search trees, AVL trees
5. skip lists
6. tries
7. hashing
8. quadtrees, kd-trees, range search
9. string matching
10. data compression
11. external memory

1 module $\approx$ 1 week per topic.

# CS Background

Topics covered in previous courses:

- arrays, linked lists
- strings
- stacks, queues
- abstract data types
- recursive algorithms
- binary trees
- basic sorting
- binary search
- binary search trees

Most are briefly reviewed in course notes, or consult any textbook (e.g. [Sedgewick,CLRS]).

# Useful Math Facts

**Logarithms:**

- $y = \log_b(x)$ means $b^y = x$. e.g. $n = 2^{\log n}$.
- $\log(x)$ (in this course) means $\log_2(x)$
- $\log(x \cdot y) = \log(x) + \log(y)$, $\log(x^y) = y \log(x)$, $\log(x) \leq x$
- $\log_b(a) = \frac{\log_c a}{\log_c b} = \frac{1}{\log_a(b)}$, $a^{\log_b c} = c^{\log_b a}$
- $\ln(x) = \text{natural log} = \log_e(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$

**Factorial:**

- $n! := n(n-1)(n-2)\cdots 2 \cdot 1 =$ # ways to permute $n$ elements
- $\log(n!) = \log n + \log(n-1) + \cdots + \log 2 + \log 1 \in \Theta(n \log n)$

(We will define $\Theta$ soon.)

**Probability:**

- $E[X]$ is the expected value of $X$.
- $E[aX] = aE[X]$, $E[X + Y] = E[X] + E[Y]$ (linearity of expectation)

# Useful Sums

**Arithmetic sequence:**
$\sum_{i=0}^{n-1} i = ???$

**Geometric sequence:**
$\sum_{i=0}^{n-1} 2^i = ???$

**Harmonic sequence:**
$\sum_{i=1}^{n} \frac{1}{i} = ???$

**A few more:**

$\sum_{i=1}^{n} \frac{i}{2^i} = ???$

$\sum_{i=1}^{n} i^k = ???$

# Useful Sums

**Arithmetic sequence:**

$\sum_{i=0}^{n-1} i = \frac{(n-1)n}{2}$ $\qquad$ $\sum_{i=0}^{n-1}(a+di) = na + \frac{dn(n-1)}{2} \in \Theta(n^2) \quad \text{if } d \neq 0.$

**Geometric sequence:**

$\sum_{i=0}^{n-1} 2^i = 2^n - 1$ $\qquad$ $\sum_{i=0}^{n-1} a\, r^i = \begin{cases} a\dfrac{r^n-1}{r-1} & \in \Theta(r^{n-1}) & \text{if } r > 1 \\ na & \in \Theta(n) & \text{if } r = 1 \\ a\dfrac{1-r^n}{1-r} & \in \Theta(1) & \text{if } 0 < r < 1. \end{cases}$

**Harmonic sequence:**

$\sum_{i=1}^{n} \frac{1}{i} = ???$ $\qquad$ $H_n := \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + o(1) \in \Theta(\log n)$

**A few more:**

$\sum_{i=1}^{n} \frac{i}{2^i} = ???$ $\qquad$ $\sum_{i=1}^{n} \frac{i}{2^i} \in \Theta(1)$

$\sum_{i=1}^{n} i^k = ???$ $\qquad$ $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1}) \quad \text{for } k \geq 0$

# Outline

# Algorithms and Problems (review)

Let us clarify a few more terms:

**Problem:** Description of possible input and desired output. Example: Sorting problem.

Problem **Instance:** One possible input for the specified problem.

**Algorithm:** *Step-by-step process* (can be described in finite length) for carrying out a series of computations, given an arbitrary instance $I$.

**Solving a problem:** An Algorithm $A$ *solves* a problem $\Pi$ if, for every instance $I$ of $\Pi$, $A$ computes a valid output for the instance $I$ in finite time.

**Program:** A program is an *implementation* of an algorithm using a specified computer language.

In this course, our emphasis is on algorithms (as opposed to programs or programming). We do not use any particular computer language to describe them.

# Algorithms and Programs

**Pseudocode:** communicate an algorithm to another person.

In contrast, a program communicates an algorithm to a computer.

```
insertion-sort(A, n)
A: array of size n
for (i ← 1; i < n; i++) do
    for (j ← i; j > 0 and A[j−1] > A[j]; j−−) do
        swap A[j] and A[j − 1]
```

- sometimes uses English descriptions, e.g. ‘swap’,
- omits obvious details, e.g. $i$ is usually an integer
- has limited if any error detection, e.g. $A$ is assumed initialized
- should be precise about exit-conditions, e.g. in loops
- should use good indentation and variable-names

# Algorithms and Programs

From problem $\Pi$ to program that solves it:

1. Design an algorithm $\mathcal{A}$ that solves $\Pi$. $\rightarrow$ **Algorithm Design**
   A problem $\Pi$ may have several algorithms. Design many!
2. Assess *correctness* and *efficiency* of each $\mathcal{A}$. $\rightarrow$ **Algorithm Analysis**
   Correctness $\rightarrow$ CS245 (here informal arguments are enough).
   Efficiency $\rightarrow$ later
3. If acceptable (correct and efficient), implement algorithm(s).
   For each algorithm, we can have several implementations.
4. If multiple acceptable algorithms/implementations, run experiments to determine best solution.

CS240 focuses on the first two steps.
The main point is to avoid implementing obviously-bad algorithms.

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1. Introduction and Asymptotic Analysis
   - CS240 Overview
   - Algorithm Design
   - Analysis of Algorithms I
   - Asymptotic Notation
   - Rules for asymptotic notation
   - Analysis of Algorithms II
   - Example: Design and Analysis of *merge-sort*

# Efficiency of Algorithms/Programs (Review)

What do we mean by 'efficiency'?

- In this course, we are primarily concerned with the *amount of time* a program takes to run. $\rightarrow$ **Running Time**
- We also may be interested in the *amount of additional memory* the program requires. $\rightarrow$ **Auxiliary space**
- The amount of time and/or memory required by a program will usually depend on the given problem instance.
- So we express the time or memory requirements as a mathematical function of the instances (e.g. $T(I)$)
- But then aggregate over all instances $\mathcal{I}_n$ of size $n$ (e.g. $T(n)$).
- Do we take $\max, \min, \text{avg}$? ($\rightarrow$ later)

# Measuring Efficiency of Algorithms/Programs (Review)

What do we count as running time/space usage of an algorithm?

First option: *experimental studies*

- Write a program implementing the algorithm.
- Run the program with inputs of varying size and composition and measure time and space.
- Plot/compare the results.

There are numerous shortcomings:

- Implementation may be complicated/costly.
- Outcomes are affected by many factors: *hardware* (processor, memory), *software environment* (OS, compiler, programming language), and *human factors* (programmer).
- We cannot test all instances; what are good *sample inputs*?

# Running Time of Algorithms/Programs

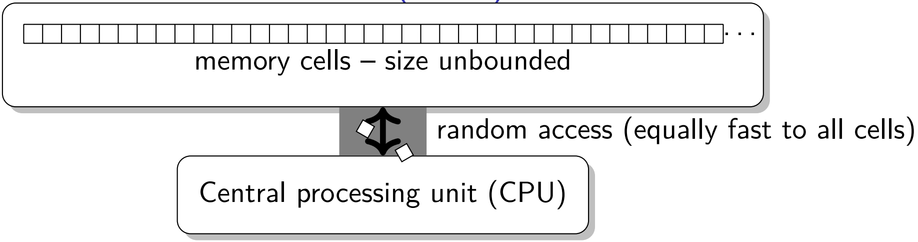
Better: theoretical analysis:

- Does not require implementing the algorithm (we work on *pseudo-code*).
- Is independent of the hardware/software environment (we work on an *idealized computer model*).
- Takes into account all input instances.

This is the approach taken in CS240.

We use experimental results only if theoretical analysis yields no useful results for deciding between multiple algorithms.

# Random Access Machine (RAM) model




- Each **memory cell** stores one (finite-length) datum, typically a number, character, or reference.
  Assumption: cells are big enough to hold the items that we store.
- Any **access to a memory location** takes constant time.
  (We will revisit this assumption late in the course.)
- Any **primitive operation** takes constant time.
  (Add, subtract, multiply, divide, follow a reference, ...)
  Not primitive: $\sqrt{n}$, anything involving irrational numbers

These assumptions may not be valid for a "real" computer.

# Running Time and Space

With this computer model, we can now formally define:

- The **running time** is the number of memory accesses plus the number of primitive operations.
- The **space** is the maximum number of memory cells ever in use.
- **Size($I$)** of instance $I$ is the number of memory cells that $I$ occupies.

The real-life time and space is proportional to this.

We compare algorithms by considering the **growth rate**: What is the behaviour of algorithms as size $n$ gets large?

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- **Example 1:** What is larger, $100n$ or $10n^2$?
- **Example 2 (Matrix multiplication, approximately):** What is larger: $4n^3$, $300n^{2.807}$, or $10^{67}n^{2.373}$?

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To simplify comparisons, use **order notation** (big-$O$ and friends).
Informally: ignore constants and lower order terms

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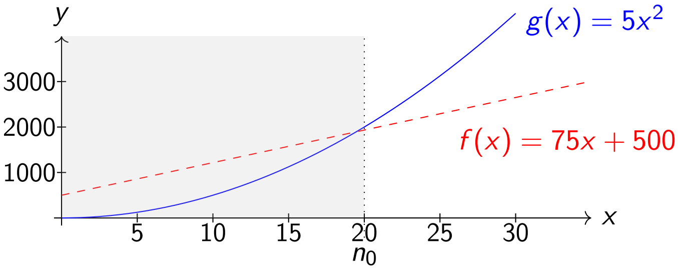
# Order Notation

Study relationships between *functions*.

**Example**: $f(x) = 75x + 500$ and $g(x) = 5x^2$ (e.g. $c = 1, n_0 = 20$)




***O*-notation:** $f(x) \in O(g(x))$ ($f$ is *asymptotically upper-bounded* by $g$) if there exist constants $c > 0$ and $n_0 \geq 0$ s.t. $|f(x)| \leq c\,|g(x)|$ for all $x \geq n_0$.

**In CS240**: Parameter is usually an integer (write $n$ rather than $x$).
$f(n), g(n)$ usually positive for sufficiently big $n$ (omit absolute value signs).

# Example 1: Order Notation

In order to prove that $2n^2 + 3n + 11 \in O(n^2)$ **from first principles** (i.e., directly from the definition), we need to find $c$ and $n_0$ such that the following condition is satisfied:

$$2n^2 + 3n + 11 \leq c\, n^2 \text{ for all } n \geq n_0.$$

Many, but not all, choices of $c$ and $n_0$ will work.

# Aymptotic Lower Bound

- We have $2n^2 + 3n + 11 \in O(n^2)$.
- But we also have $2n^2 + 3n + 11 \in O(n^{10})$.
- We want a *tight* asymptotic bound.

**$\Omega$-notation:** $f(x) \in \Omega(g(x))$ ($f$ is *asymptotically lower-bounded* by $g$) if there exist constants $c > 0$ and $n_0 \geq 0$ s.t. $c\,|g(x)| \leq |f(x)|$ for all $x \geq n_0$.

**Example:** Prove that $f(n) = 2n^2 + 3n + 11 \in \Omega(n^2)$ from first principles.

**Example:** Prove that $\frac{1}{2}n^2 - 5n \in \Omega(n^2)$ from first principles.

# Aymptotic Tight Bound

**$\Theta$-notation:** $f(x) \in \Theta(g(x))$ ($f$ is *asymptotically tightly-bounded* by $g$) if there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ such that

$$c_1 \, |g(x)| \leq |f(x)| \leq c_2 \, |g(x)| \text{ for all } x \geq n_0.$$

Equivalently: $f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

We also say that *the growth rates of $f$ and $g$ are the same*. Typically, $f(x)$ may be "complicated" and $g(x)$ is chosen to be a very simple function.

**Example:** Prove that $\log_b(n) \in \Theta(\log n)$ for all $b > 1$ from first principles.

# Common Growth Rates

Commonly encountered growth rates in analysis of algorithms include the following:

- $\Theta(1)$ (*constant*),
- $\Theta(\log n)$ (*logarithmic*),
- $\Theta(n)$ (*linear*),
- $\Theta(n \log n)$(*linearithmic*),
- $\Theta(n \log^k n)$, for some constant $k$ (*quasi-linear*),
- $\Theta(n^2)$ (*quadratic*),
- $\Theta(n^3)$ (*cubic*),
- $\Theta(2^n)$ (*exponential*).

These are sorted in *increasing order* of growth rate.

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How do we define 'increasing order of growth rate'?

# How Growth Rates Affect Running Time

It is interesting to see how the running time is affected when the size of the problem instance *doubles* (i.e., $n \rightarrow 2n$).

- constant complexity: $T(n) = c$
- logarithmic complexity: $T(n) = c \log n$
- linear complexity: $T(n) = cn$
- linearithmic $\Theta(n \log n)$: $T(n) = c\, n \log n$
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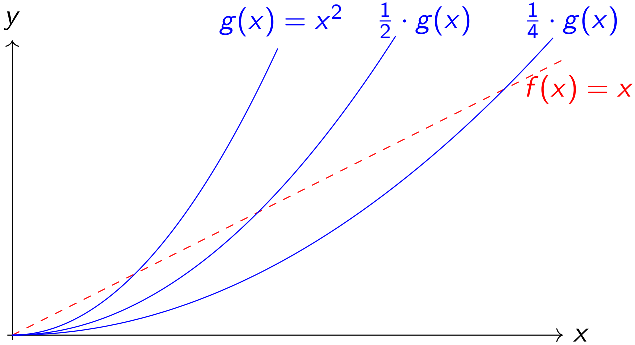
- cubic complexity: $T(n) = cn^3$ $\rightsquigarrow$ $T(2n) = 8T(n)$.
- exponential complexity: $T(n) = c\, 2^n$ $\rightsquigarrow$ $T(2n) = (T(n))^2/c$.

# Strictly smaller asymptotic bounds

- We have $f(n) = n \in \Theta(n)$.
- How to express that $f(n)$ grows slower than $n^2$?




***o*-notation:** $f(x) \in o(g(x))$ ($f$ is *asymptotically strictly smaller* than $g$) if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $|f(x)| \leq c\,|g(x)|$ for all $x \geq n_0$.

# Strictly smaller/larger asymptotic bounds

**Example:** Prove that $n \in o(n^2)$ from first principles.

# Strictly smaller/larger asymptotic bounds

**Example:** Prove that $n \in o(n^2)$ from first principles.

- Main difference between $o$ and $O$ is the quantifier for $c$.
- $n_0$ will depend on $c$, so it is really a function $n_0(c)$.
- We also say 'the growth rate of $f$ is *less than* the growth rate of $g$'.
- Rarely proved from first principles (instead use limit-rule $\rightsquigarrow$ later).

**$\omega$-notation:** $f(x) \in \omega(g(x))$ ($f$ is *asymptotically strictly larger* than $g$) if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $|f(x)| \geq c\,|g(x)|$ for all $x \geq n_0$.

- Symmetric, the growth rate of $f$ is *more than* the growth rate of $g$.

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# The Limit Rule

Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Suppose that

$$L = \lim_{n \to \infty} \frac{f(n)}{g(n)} \quad \text{(in particular, the limit exists).}$$

Then

$$f(n) \in \begin{cases} o(g(n)) & \text{if } L = 0 \\ \Theta(g(n)) & \text{if } 0 < L < \infty \end{cases}$$

If the fraction goes towards $\infty$ then $f(n) \in \omega(g(n))$.

The required limit can often be computed using *l'Hôpital's rule*. Note that this result gives *sufficient* (but not necessary) conditions for the stated conclusion to hold.

# Application 1: Logarithms vs. polynomials

Compare the growth rates of $f(n) = \log n$ and $g(n) = n$.

Now compare the growth rates of $f(n) = (\log n)^c$ and $g(n) = n^d$ (where $c > 0$ and $d > 0$ are arbitrary numbers).

# Application 2: Polynomials

Let $f(n)$ be a polynomial of degree $d \geq 0$:

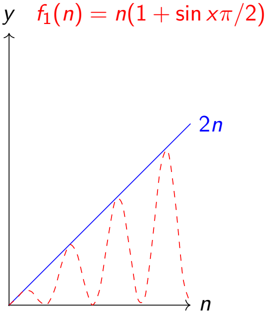
$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

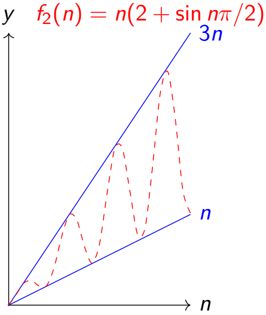
for some $c_d > 0$.

Then $f(n) \in \Theta(n^d)$:

# Example: Oscillating functions

Consider two oscillating functions $f_1, f_2$ for which $\lim_{n\to\infty} \frac{f_i(n)}{n}$ does not exist. Are they in $\Theta(n)$?




So no limit $\rightsquigarrow$ must use other methods to prove asymptotic bounds.

# Order Notation Summary

**$O$-notation:** $f(x) \in O(g(x))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $|f(x)| \leq c\,|g(x)|$ for all $x \geq n_0$.

**$\Omega$-notation:** $f(x) \in \Omega(g(x))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $c\,|g(x)| \leq |f(x)|$ for all $x \geq n_0$.

**$\Theta$-notation:** $f(x) \in \Theta(g(x))$ if there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ such that $c_1\,|g(x)| \leq |f(x)| \leq c_2\,|g(x)|$ for all $x \geq n_0$.

**$o$-notation:** $f(x) \in o(g(x))$ if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $|f(x)| \leq c\,|g(x)|$ for all $x \geq n_0$.

**$\omega$-notation:** $f(x) \in \omega(g(x))$ if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $c\,|g(x)| \leq |f(x)|$ for all $x \geq n_0$.

# Algebra of Order Notations

Many rules are easily proved from first principle (exercise).

**Identity rule:** $f(n) \in \Theta(f(n))$

**Transitivity:**

- If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$.
- If $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$ then $f(n) \in \Omega(h(n))$.
- If $f(n) \in O(g(n))$ and $g(n) \in o(h(n))$ then $f(n) \in o(h(n))$.
- ...

**Maximum rules:** Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Then:

- $f(n) + g(n) \in O(\max\{f(n), g(n)\})$
- $f(n) + g(n) \in \Omega(\max\{f(n), g(n)\})$

Key proof-ingredient: $\max\{f(n), g(n)\} \leq f(n)+g(n) \leq 2\max\{f(n), g(n)\}$

# Relationships between Order Notations

- $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$

- $f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \notin \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \notin O(g(n))$

**Example:** Fill the following table with TRUE or FALSE:

| | | Is $f(n) \in \ldots(g(n))$? | | | |
|---|---|---|---|---|---|
| $f(n)$ | $g(n)$ | $o$ | $O$ | $\Omega$ | $\omega$ |
| $\log n$ | $\sqrt{n}$ | | | | |

# Abuse of Notation

- Normally, we say $f(n) \in \Theta(g(n))$ because $\Theta(g(n))$ is a set.
- Avoid doing arithmetic with asymptotic notations—it can go badly wrong ($\rightarrow$ later) Do **not** write $O(n) + O(n) = O(n)$.
  (CS136 allowed you to be sloppy here. CS240 does not, mostly because it can go badly wrong with recursions.)
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    - more precise about constants than "$O(n^2)$"
  - But use this very sparingly (typically only for stating the final result)
  - Similarly $f(n) = n^2 + o(1)$ means "$n^2$ plus a vanishing term."

# Outline

# Techniques for Run-time Analysis

- Goal: Use asymptotic notation to simplify run-time analysis.
- Running time of an algorithm depends on the *input size* $n$.

*print-pairs*$(A, n)$
1. **for** $i \leftarrow 0$ **to** $n-1$ **do**
2.     **for** $j \leftarrow 0$ **to** $i-1$ **do**
3.         print 'the next pair is $\{A[i], A[j]\}$'

- Identify *primitive operations* that require $\Theta(1)$ time.
  (For doing arithmetic, assume they require $c$ time for some $c > 0$.)
- The complexity of a loop is expressed as the *sum* of the complexities of each iteration of the loop.
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For *print-pairs*: The run-time is $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} c$.

# Techniques for Run-time Analysis

Two general strategies are as follows.

**Strategy I:** Use $\Theta$-bounds *throughout the analysis* and obtain a $\Theta$-bound for the complexity of the algorithm.
For *print-pairs*:

**Strategy II:** Prove a $O$-bound and a *matching* $\Omega$-bound *separately*.
Use upper bounds (for $O$) and lower bounds (for $\Omega$) early and frequently.
This may be easier because upper/lower bounds are easier to sum.
For *print-pairs*:

# Complexity of Algorithms

- Algorithm can have different running times on two instances of the same size.

*insertion-sort*$(A, n)$
$A$: array of size $n$

```
1.  for (i ← 1; i < n; i++) do
2.      for (j ← i; j > 0 and A[j−1] > A[j]; j−−) do
3.          swap A[j] and A[j − 1]
```

Let $T_{\mathcal{A}}(I)$ denote the running time of an algorithm $\mathcal{A}$ on instance $I$.

Study this value for the worst-possible, best-possible and ‘typical’ (average) instance $I$.

# Complexity of Algorithms

**Worst-case (best-case) complexity of an algorithm:** The *worst-case (best-case) running time* of an algorithm $\mathcal{A}$ is a function $T : \mathbb{Z}^+ \to \mathbb{R}$ mapping $n$ (the input size) to the *longest (shortest)* running time for any input instance of size $n$:

$$T_{\mathcal{A}}^{\mathrm{worst}}(n) = \max_{I \in \mathcal{I}_n} \{ T_{\mathcal{A}}(I) \}$$

$$T_{\mathcal{A}}^{\mathrm{best}}(n) = \min_{I \in \mathcal{I}_n} \{ T_{\mathcal{A}}(I) \}$$

**Average-case complexity of an algorithm:** The average-case running time of an algorithm $\mathcal{A}$ is a function $T : \mathbb{Z}^+ \to \mathbb{R}$ mapping $n$ (the input size) to the *average* running time of $\mathcal{A}$ over all instances of size $n$:

$$T_{\mathcal{A}}^{avg}(n) = \sum_{I \in \mathcal{I}_n} T_{\mathcal{A}}(I) \cdot \left(\text{relative frequency of } I\right)$$

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Goal in cs240: For a problem, find an algorithm that solves it and whose tight bound on the worst-case running time has the smallest growth rate.

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  *No!* The worst-case run-time of $\mathcal{A}_1$ may only be achieved on some instances. Possibly $\mathcal{A}_1$ is better on most instances.

  Also, the hidden constants may be so large that $\mathcal{A}_1$ is better on all but unrealistically big $n$.

# Outline

# Explaining the solution of a problem

To give an algorithm that 'solves a problem', we usually do four steps. We illustrate this here on *merge-sort*.

**Step 1: Describe the overall idea**

**Input:** Array $A$ of $n$ integers

1. We split $A$ into two subarrays $A_L$ and $A_R$ that are roughly half as big.
2. *Recursively* sort $A_L$ and $A_R$
3. After $A_L$ and $A_R$ have been sorted, use a function *merge* to merge them into a single sorted array.

# Explaining the solution of a problem

## Step 2: Give pseudo-code or detailed description.

```
merge-sort(A, n)
A: array of size n
    if (n ≤ 1) then return
    else
        m = ⌊(n − 1)/2⌋
        merge-sort(A[0..m], m + 1)
        merge-sort(A[m + 1..n−1], r)
        merge(A[0..m], A[m + 1..n−1])
```

(pseudo-code for *merge* to come)

Two tricks to reduce constant in the run-time and auxiliary space:

- Do not pass array $A$ by value, instead indicate the range of the array that needs to be sorted.
- *merge* needs an auxiliary array $S$. Allocate this only *once*.

# Explaining the solution of a problem

**Step 2: Give pseudo-code or detailed description.**

```
merge-sort(A, n, ℓ ← 0, r ← n − 1, S ← NULL)
A: array of size n, 0 ≤ ℓ ≤ r ≤ n − 1
if S is NULLdo initialize it as array S[0..n − 1]
if (r ≤ ℓ) then
    return
else
    m = ⌊(r + ℓ)/2⌋
    merge-sort(A, n, ℓ, m, S)
    merge-sort(A, n, m + 1, r, S)
    merge(A, ℓ, m, r, S)
```

- This would be much better for an efficient implementation.
- But the idea is much harder to understand.
- CS240 pseudocode will often prefer clarity over improved constants.

# Sub-routine *merge*

Idea: Always extract from each sub-array the value that is smaller and append it to the output.

*merge*$(A, \ell, m, r, S)$
$A[0..n-1]$ is an array, $A[\ell..m]$ is sorted, $A[m+1..r]$ is sorted
$S[0..n-1]$ is an array

copy $A[\ell..r]$ into $S[\ell..r]$
$(i_L, i_R) \leftarrow (\ell, m+1)$; `// start-indices of subarrays`
**for** $(k \leftarrow \ell; k \leq r; k{+}{+})$ **do** `// fill-index for result`
    **if** $(i_L > m)$ $A[k] \leftarrow S[i_R{+}{+}]$
    **else if** $(i_R > r)$ $A[k] \leftarrow S[i_L{+}{+}]$
    **else if** $(S[i_L] \leq S[i_R])$ $A[k] \leftarrow S[i_L{+}{+}]$
    **else** $A[k] \leftarrow S[i_R{+}{+}]$

# Analysis of merge-sort

**Step 3:** Argue correctness.

- Typically state loop-invariants, or other key-ingredients, but no need for a formal (CS245-style) proof by induction.
- Sometimes obvious enough from idea-description and comments.

**Step 4:** Analyze the run-time.

- First analyze work done outside recursions.
- If applicable, analyze subroutines separately.
- If there are recursions: how big are the subproblems? The run-time then becomes a recursive function.

Let $T(n)$ denote the time to run *merge-sort* on an array of length $n$.

1. (initialize array) takes time $\Theta(n)$
2. (recursively call *merge-sort*) takes time $T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor)$
3. (call *merge*) takes time $\Theta(n)$

# The run-time of *merge-sort*

- The **recurrence relation** for $T(n)$ is as follows (constant factor $c$ replaces $\Theta$):

$$T(n) = \begin{cases} T\left(\lceil \frac{n}{2} \rceil\right) + T\left(\lfloor \frac{n}{2} \rfloor\right) + c\,n & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

- The following is the corresponding **sloppy recurrence** (it has floors and ceilings removed):

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

- When $n$ is a power of 2, then the exact and sloppy recurrences are *identical* and can easily be solved by various methods.
  E.g. prove by induction that $T(n) = cn\log(2n) \in \Theta(n \log n)$.
- It is possible to show that $T(n) \in \Theta(n \log n)$ *for all $n$* by analyzing the exact recurrence.

# Asymptotics and Arithmetic revisited

**Recall:** You should not intermix asymptotics and arithmetic.

- Writing $O(n) + O(n) = O(n)$ is very bad style.
- It even occasionally leads to *incorrect* results.
- Example: What is wrong with the following proof?

# Asymptotics and Arithmetic revisited

**Recall:** You should not intermix asymptotics and arithmetic.

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- It even occasionally leads to *incorrect* results.
- Example: What is wrong with the following proof?

**Claim (false!)**: If $T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$ then $T(n) \in O(n)$.

"**Proof**": Use induction on $n$.

- In the base case ($n = 1$) we have $T(n) = c \in O(1) = O(n)$.
- Assume the claim holds for all $n'$ with $n' < n$.
- Step: We have

$$T(n) = 2T\left(\tfrac{n}{2}\right) + cn \overset{IH}{\in} 2O\left(\tfrac{n}{2}\right) + O(n) = O(n) + O(n) = O(n)$$

# Some Recurrence Relations

| Recursion | resolves to | example |
|---|---|---|
| $T(n) \leq T(n/2) + O(1)$ | $T(n) \in O(\log n)$ | binary-search |
| $T(n) \leq 2T(n/2) + O(n)$ | $T(n) \in O(n \log n)$ | merge-sort |
| $T(n) \leq 2T(n/2) + O(\log n)$ | $T(n) \in O(n)$ | heapify (*) |
| $T(n) \leq cT(n-1) + O(1)$ for some $c < 1$ | $T(n) \in O(1)$ | avg-case analysis (*) |
| $T(n) \leq 2T(n/4) + O(1)$ | $T(n) \in O(\sqrt{n})$ | range-search (*) |
| $T(n) \leq T(\sqrt{n}) + O(\sqrt{n})$ | $T(n) \in O(\sqrt{n})$ | interpol. search (*) |
| $T(n) \leq T(\sqrt{n}) + O(1)$ | $T(n) \in O(\log \log n)$ | interpol. search (*) |

- Once you know the result, it is (usually) easy to prove by induction.
- These bounds are tight if the upper bounds are tight.
- Many more recursions, and some methods to find the result, in CS341.

(*) These may or may not get used later in the course.