CS 240 – Data Structures and Data Management

Module 4: Dictionaries

Arne Storjohann

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Fall 2025

version 2025-09-29 08:54

Outline

Outline

ADT Dictionary (review)

Dictionary: An ADT consisting of a collection of items, each of which contains

- a *key*
- some *data* (the "value")

and is called a *key-value pair* (KVP). Keys can be compared and are (typically) unique.

Operations:

- *search*(k) (also called *lookup*(k))
- *insert*(k, v)
- *delete*(k) (also called *remove*(k)))
- optional: *successor*, *join*, *is-empty*, *size*, *etc.*

Examples: symbol table, license plate database

Elementary Realizations (review)

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space
  (if not, the "value" could be a pointer)
- Keys can be compared in constant time

Unordered array or linked list

search $\Theta(n)$

insert $\Theta(1)$ (except array occasionally needs to resize)

delete $\Theta(n)$ (need to search)

Ordered array

search $\Theta(\log n)$ (via binary search)

insert $\Theta(n)$

delete $\Theta(n)$

Binary Search (review)

Only applies to a *sorted array*:

0	1	2	3	4	5	6
30	40	70	90	100	120	140

binary-search(A, n, k)
A: Sorted array of size n, k: key

```
ℓ ← 0, r ← n − 1
while (ℓ ≤ r)
        m ← ⌊(ℓ+r)/2⌋
        if (A[m] equals k) then return "found at A[m]"
        else if (A[m] < k) then ℓ ← m + 1
        else r ← m − 1
return "not found, but would be between A[ℓ−1] and A[ℓ]"
```

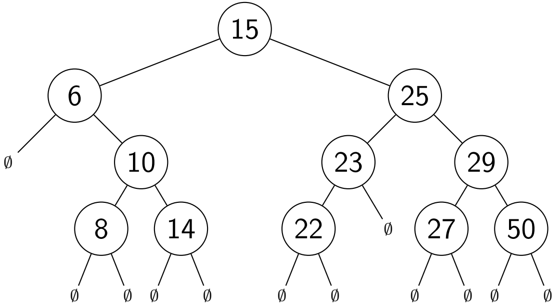
We will return to binary search (and sometimes improve it!) later.

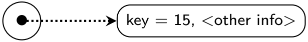
Outline

Binary Search Trees (review)

Structure Binary tree: all nodes have two (possibly empty) subtrees
Every node stores a KVP
Empty subtrees usually not shown

Ordering Every key k in $T.left$ is less than the root key.
Every key k in $T.right$ is greater than the root key.



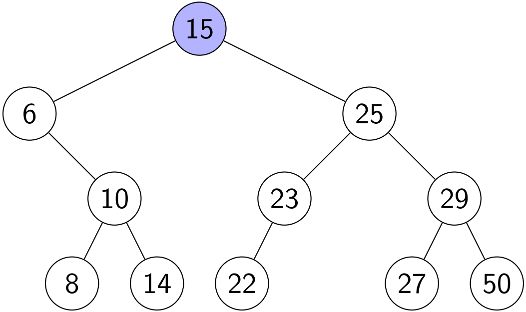
(In our examples we only show the keys, and we show them directly in the node. A more accurate picture would be 

)

BST as realization of ADT Dictionary (review)

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

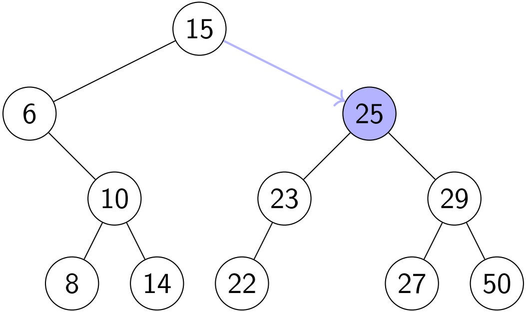
Example: *BST::search*(24)



BST as realization of ADT Dictionary (review)

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

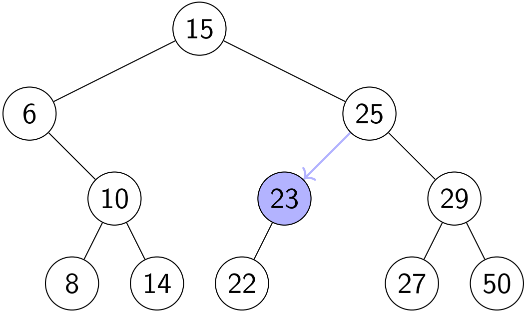
Example: *BST::search*(24)



BST as realization of ADT Dictionary (review)

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

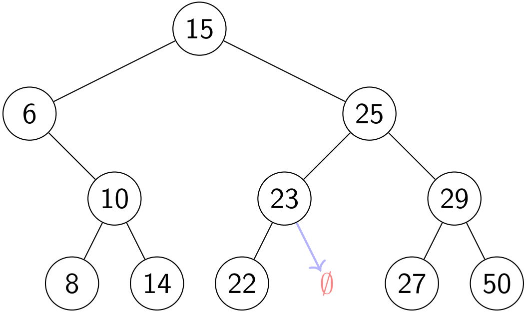
Example: *BST::search*(24)



BST as realization of ADT Dictionary (review)

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

Example: *BST::search*(24)

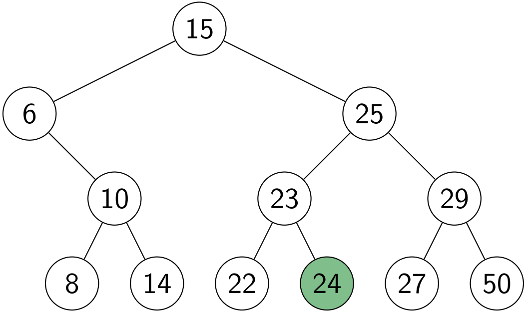


BST as realization of ADT Dictionary (review)

BST::search(k) Start at root, compare k to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

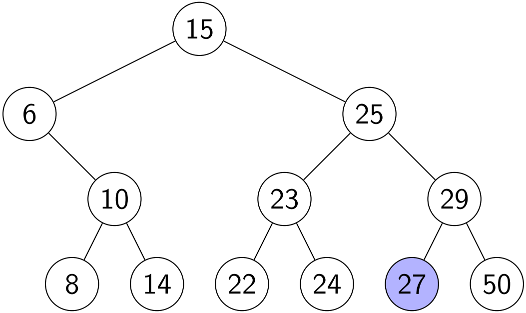
BST::insert(k, v) Search for k, then insert (k, v) as new node

Example: *BST::insert*$(24, v)$



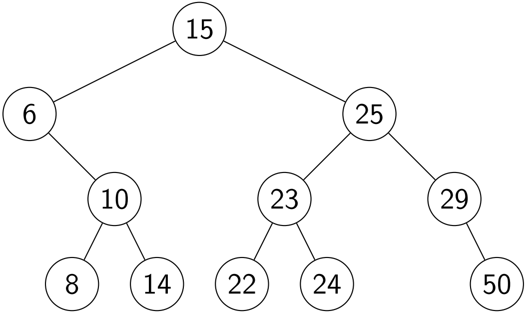
Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.



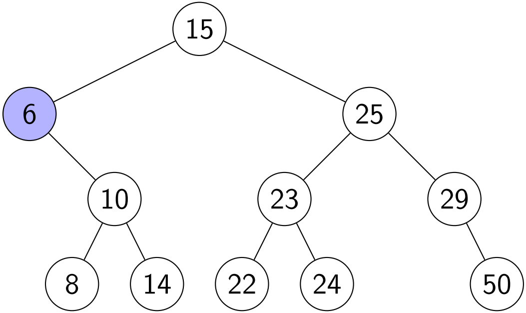
Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.



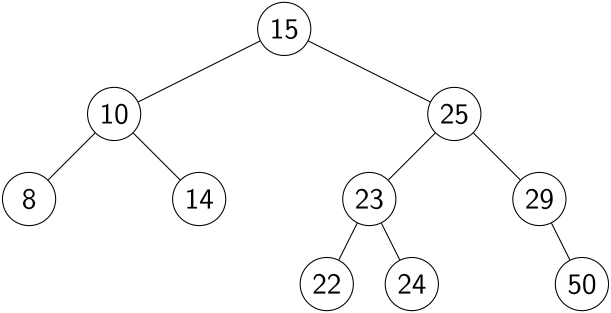
Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.
- If x has one non-empty subtree, move child up



Deletion in a BST

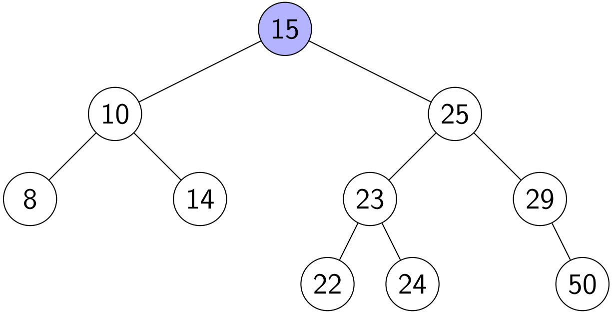
- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.
- If x has one non-empty subtree, move child up



Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.
- If x has one non-empty subtree, move child up
- Else, swap key at x with key at **successor** node and then delete that node.

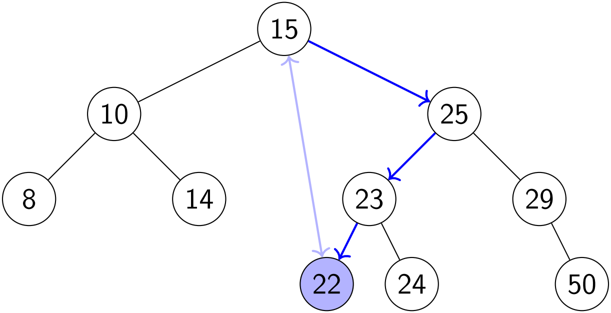
(Successor: next-smallest among all keys in the dictionary.)



Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.
- If x has one non-empty subtree, move child up
- Else, swap key at x with key at **successor** node and then delete that node.

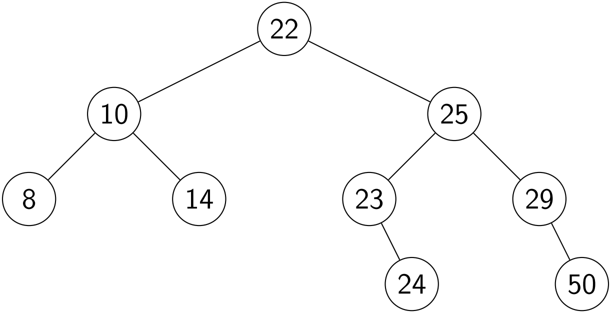
(Successor: next-smallest among all keys in the dictionary.)



Deletion in a BST

- First search for the node x that contains the key.
- If x is a **leaf** (both subtrees are empty), delete it.
- If x has one non-empty subtree, move child up
- Else, swap key at x with key at **successor** node and then delete that node.

(Successor: next-smallest among all keys in the dictionary.)



Height of a BST

BST::search, *BST::insert*, *BST::delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are inserted one-at-a-time, how big is h?

- Worst-case: $n - 1 = \Theta(n)$
- Best-case: $\Theta(\log n)$.
  Any binary tree with n nodes has height $h \geq \log(n+1) - 1$
  (Layer i has at most 2^i nodes. So $n \leq \sum_{i=0}^{h} 2^i = 2^{h+1} - 1$).

Height of a BST

BST::search, *BST::insert*, *BST::delete* all have cost $\Theta(h)$, where
$h =$ height of the tree $=$ max. path length from root to leaf

If n items are inserted one-at-a-time, how big is h?

- Worst-case: $n - 1 = \Theta(n)$
- Best-case: $\Theta(\log n)$.
  Any binary tree with n nodes has height $h \geq \log(n+1) - 1$
  (Layer i has at most 2^i nodes. So $n \leq \sum_{i=0}^{h} 2^i = 2^{h+1} - 1$).

Goal: Create subclasses of BSTs where the height is *always* good.

- Impose a structural property.
- Argue that the property implies logarithmic height.
- Discuss how to maintain the property during operatons.

Outline

4 Dictionaries and Balanced Search Trees
- ADT Dictionary
- Binary Search Trees
- AVL Trees
- Insertion in AVL Trees
- Restructuring a BST: Rotations
- AVL insertion revisited
- Deletion in AVL Trees

AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962, an **AVL Tree** is a BST with an additional **height-balance property** at every node:

The heights of the left and right subtree differ by at most 1.

Rephrase: If node v has left subtree L and right subtree R, then

$$\textbf{balance}(v) := \mathit{height}(R) - \mathit{height}(L) \text{ must be in } \{-1, 0, 1\}$$

$\mathit{balance}(v) = -1$ means v is *left-heavy*

$\mathit{balance}(v) = +1$ means v is *right-heavy*

AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962, an **AVL Tree** is a BST with an additional **height-balance property** at every node:

The heights of the left and right subtree differ by at most 1.

Rephrase: If node v has left subtree L and right subtree R, then

$$\textbf{balance}(v) := \textit{height}(R) - \textit{height}(L) \text{ must be in } \{-1, 0, 1\}$$

$\textit{balance}(v) = -1$ means v is *left-heavy*

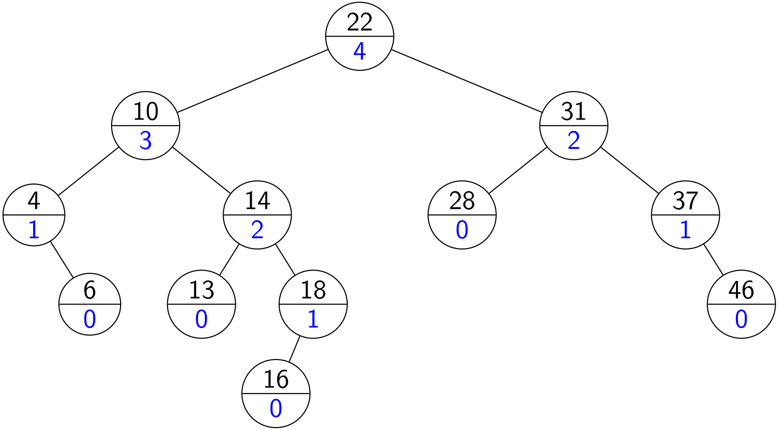
$\textit{balance}(v) = +1$ means v is *right-heavy*

- Need to store at each node v the height of the subtree rooted at it

(There are ways to implement AVL-trees where we only store $\textit{balance}(v)$, so fewer bits. But the code gets more complicated (no details).)

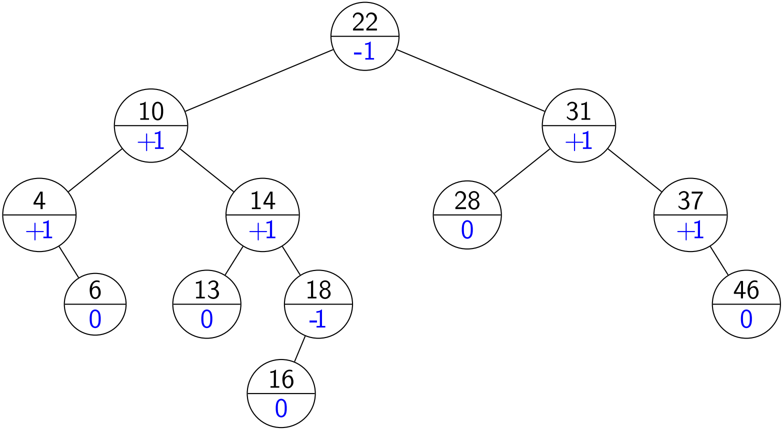
AVL tree example

(The lower numbers indicate the height of the subtree.)



AVL tree example

Alternative: store balance (instead of height) at each node.



Height of an AVL tree

Theorem: An AVL tree on n nodes has $\Theta(\log n)$ height.
$\Rightarrow$ *search*, *BST::insert*, *BST::delete* all cost $\Theta(\log n)$ in the *worst case!*

Proof:

- Define $N(h)$ to be the *least* number of nodes in a height-h AVL tree.
- What is a recurrence relation for $N(h)$?
- What does this recurrence relation resolve to?

Outline

AVL insertion

To perform *AVL::insert*(k, v):

- First, insert (k, v) with the usual BST insertion.
- We assume that this returns the new leaf z where the key was stored.
- Then, move up the tree from z.

  (We assume for this that we have parent-links. This can be avoided if *BST::insert* returns the full path to z.)

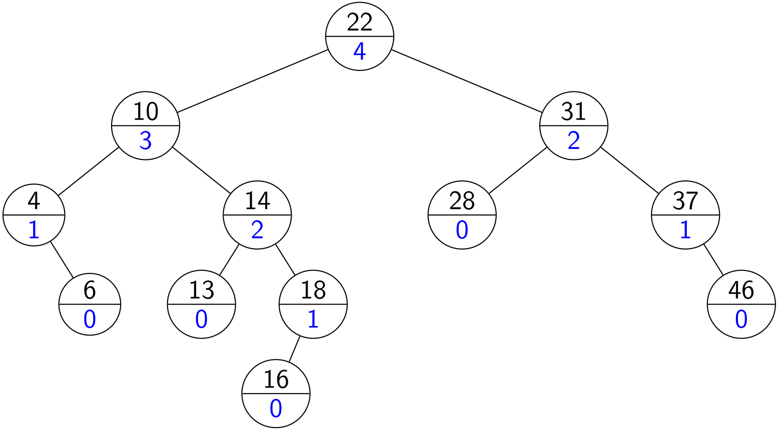
- Update height (easy to do in constant time):

  setHeightFromSubtrees(u)
  1. $u.height \leftarrow 1 + \max\{u.left.height, u.right.height\}$

- If the height difference becomes ± 2 at node z, then z is **unbalanced**. Must re-structure the tree to rebalance.

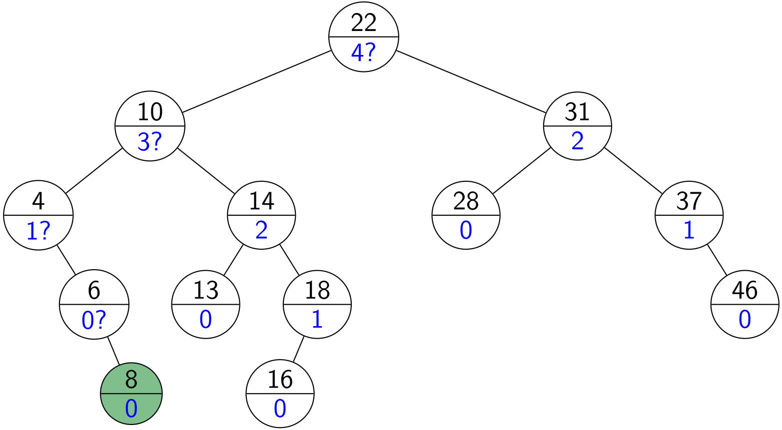
AVL Insertion Example

Example: *AVL::insert*(8)



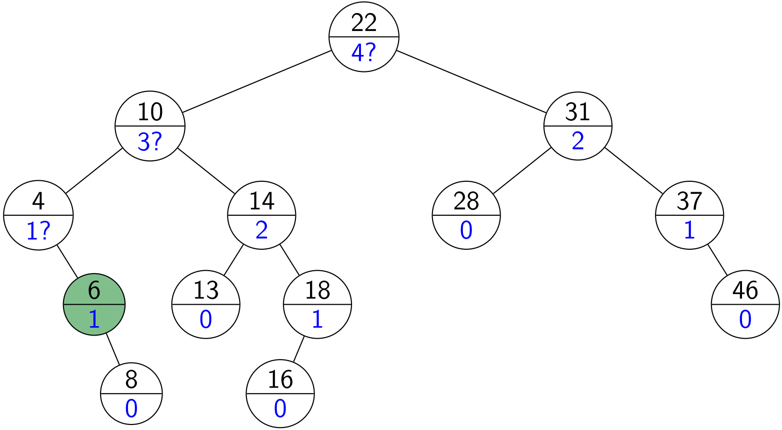
AVL Insertion Example

Example: *AVL::insert*(8)



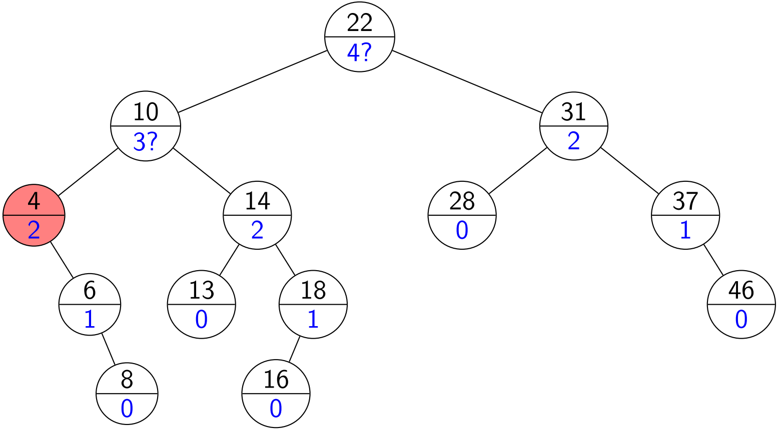
AVL Insertion Example

Example: *AVL::insert*(8)



AVL Insertion Example

Example: *AVL::insert*(8)

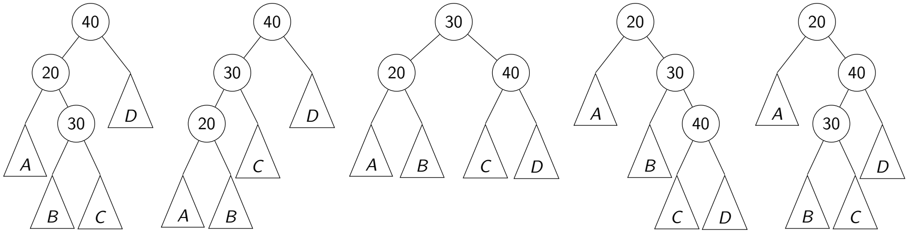


Outline

4 Dictionaries and Balanced Search Trees
- ADT Dictionary
- Binary Search Trees
- AVL Trees
- Insertion in AVL Trees
- **Restructuring a BST: Rotations**
- AVL insertion revisited
- Deletion in AVL Trees

Changing structure without changing order

Note: There are many different BSTs with the same keys.

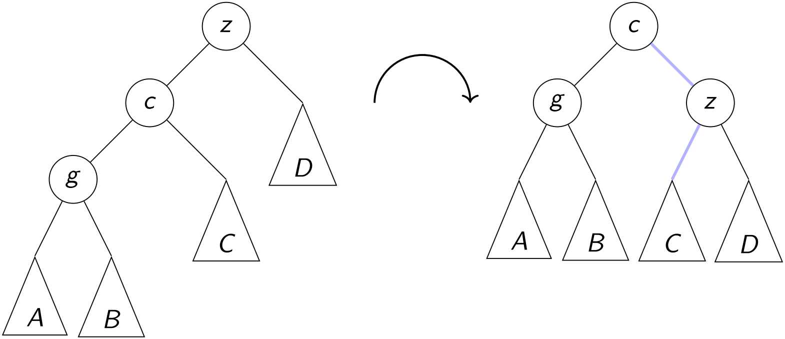


Goal: Change the *structure* locally nodes without changing the *order*.

Longterm goal: Restructure such the subtree becomes balanced.

Right Rotation

This is a **right rotation** on node z:

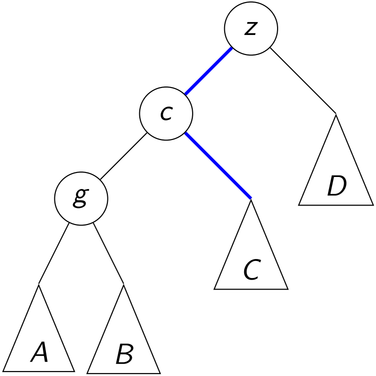


rotate-right(z)

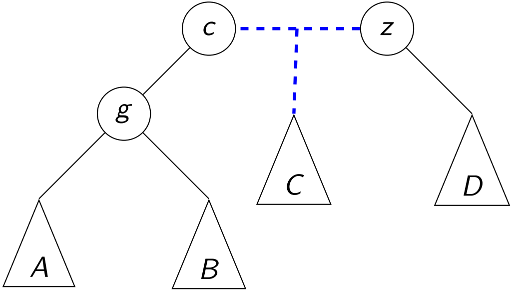
1. $c \leftarrow z.left$, $z.left \overset{p}{\leftarrow} c.right$, $c.right \overset{p}{\leftarrow} z$
2. *setHeightFromSubtrees*(z), *setHeightFromSubtrees*(c)
3. **return** c `// returns new root of subtree`

(Notation $\overset{p}{\leftarrow}$ means 'also change parent-reference of right-hand-side')

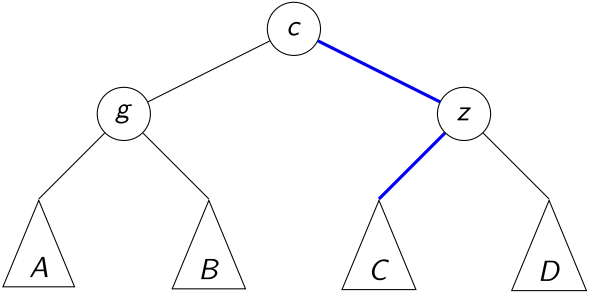
Why do we call this a rotation?



Why do we call this a rotation?

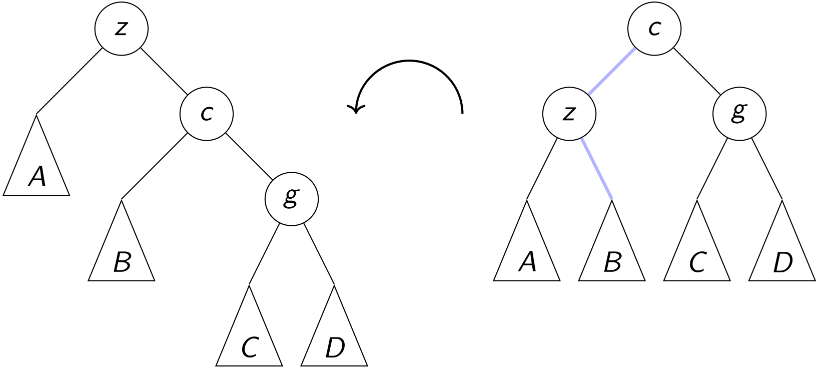


Why do we call this a rotation?



Left Rotation

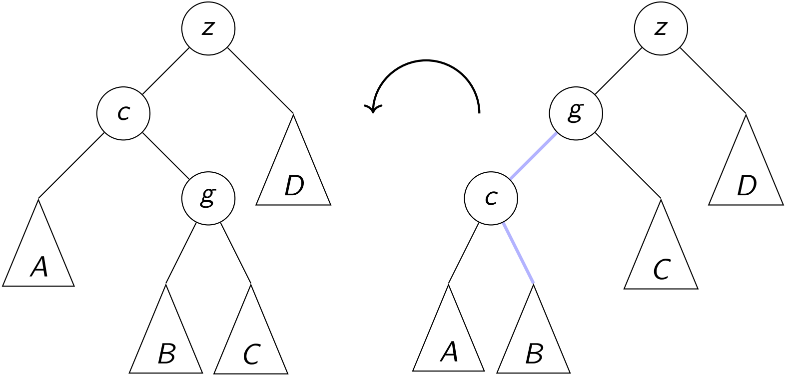
Symmetrically, this is a **left rotation** on node z:



Again, only two links need to be changed and two heights updated.
Useful to fix right-right imbalance.

Double Right Rotation

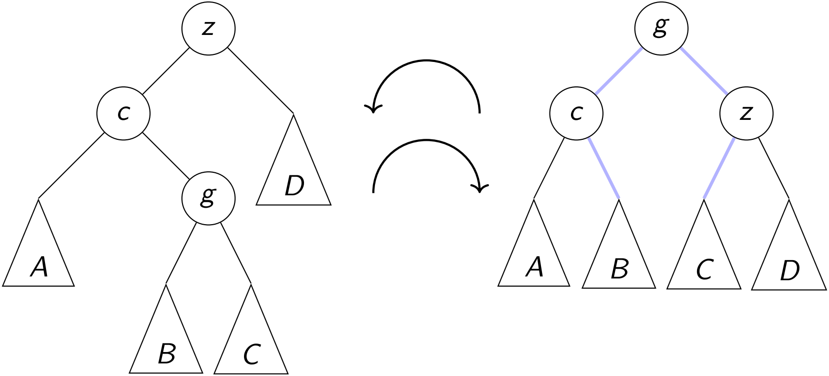
This is a **double right rotation** on node z:



First, a left rotation at c.

Double Right Rotation

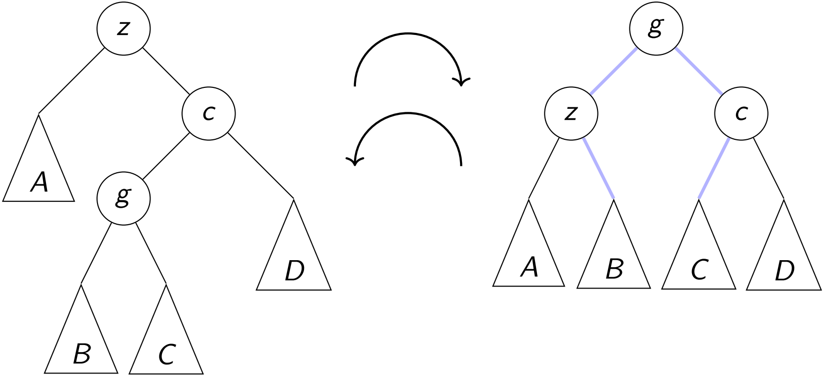
This is a **double right rotation** on node z:



First, a left rotation at c.
Second, a right rotation at z.

Double Left Rotation

Symmetrically, there is a **double left rotation** on node z:

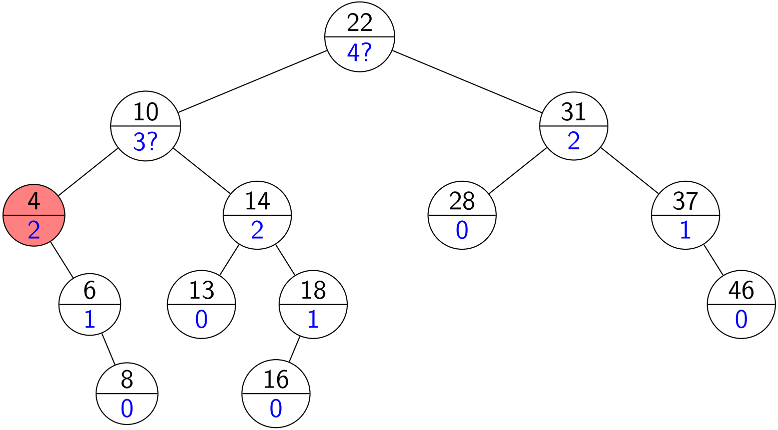


First, a right rotation at c.
Second, a left rotation at z.

Outline

AVL Insertion Example revisited

Example: *AVL::insert*(8)



AVL insertion revisited

- Imbalance at z: do (single or double) rotation
- Choose c as child where subtree has bigger height.

```
AVL::insert(k, v)
    z ← BST::insert(k, v)   // leaf where k is now stored
    while (z is not NULL)
        if (|z.left.height − z.right.height| > 1) then
            Let c be taller child of z
            Let g be taller child of c (so grandchild of z)
            restructure(g, c, z) // see later
            break        // can argue that we are done
        setHeightFromSubtrees(z)
        z ← z.parent
```

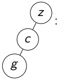
Can argue: For insertion *one* rotation restores all heights of subtrees.

$\Rightarrow$ No further imbalances, can stop checking.

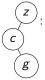
Fixing a slightly-unbalanced AVL tree

restructure(g, c, z)
node g is child of c which is child of z

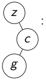
1. $p \leftarrow z.parent$ `// save for later`
2. **case** { *left*



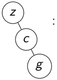
: `// Right rotation`
$u \leftarrow$ *rotate-right*(z)



: `// Double-right rotation`
$z.\text{left} \overset{p}{\leftarrow}$ *rotate-left*(c)
$u \leftarrow$ *rotate-right*(z)



: `// Double-left rotation`
$z.\text{right} \overset{p}{\leftarrow}$ *rotate-right*(c)
$u \leftarrow$ *rotate-left*(z)



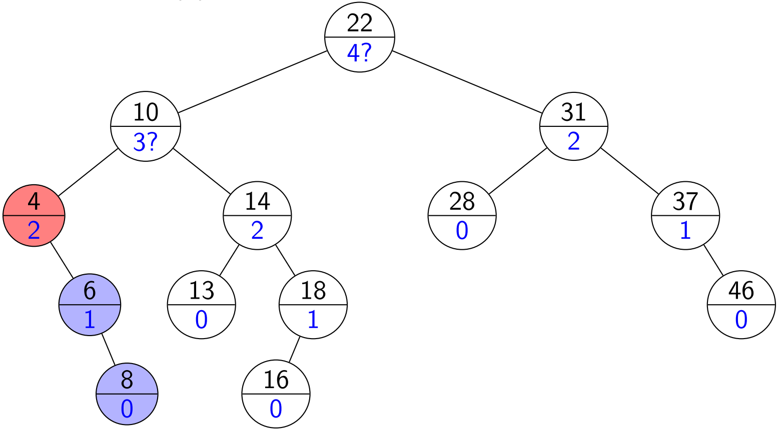
: `// Left rotation`
$u \leftarrow$ *rotate-left*(z)

3. make u the appropriate child of p and **return** u

Rule: The middle key of g, c, z becomes the new root.

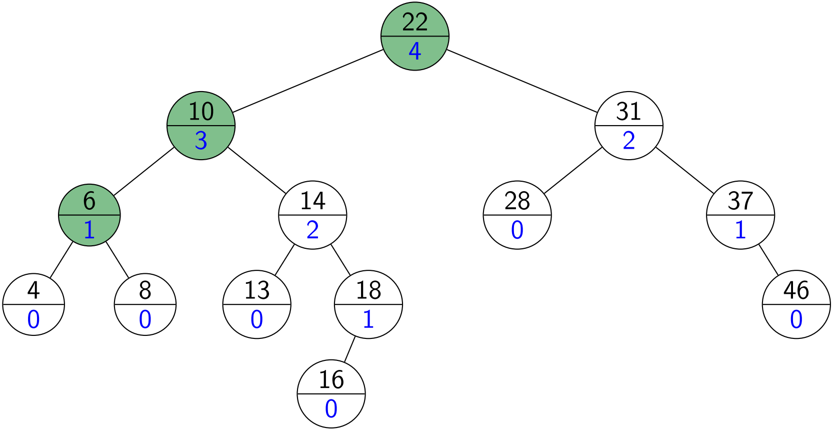
AVL Insertion Example revisited

Example: *AVL::insert*(8)



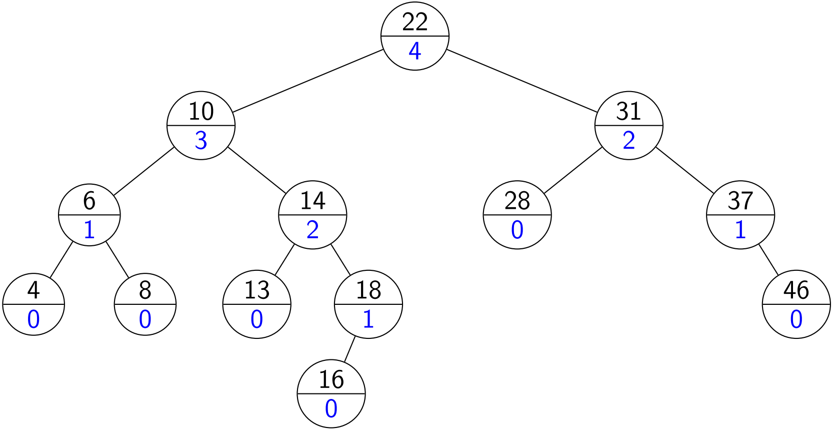
AVL Insertion Example revisited

Example: *AVL::insert*(8)



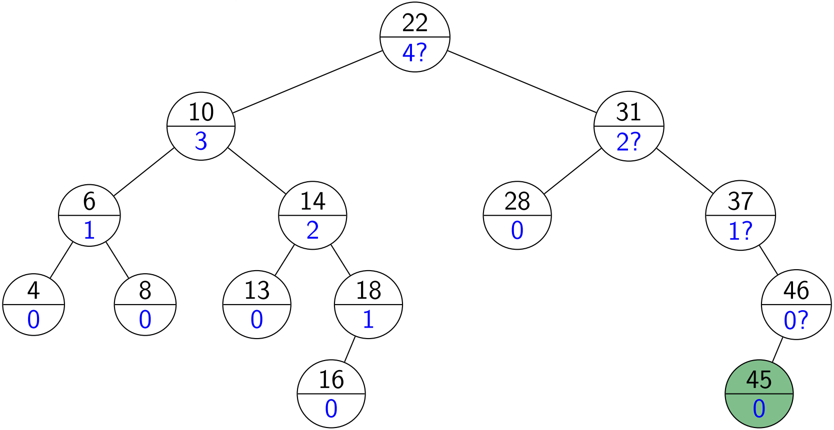
AVL Insertion: Second example

Example: *AVL::insert*(45)



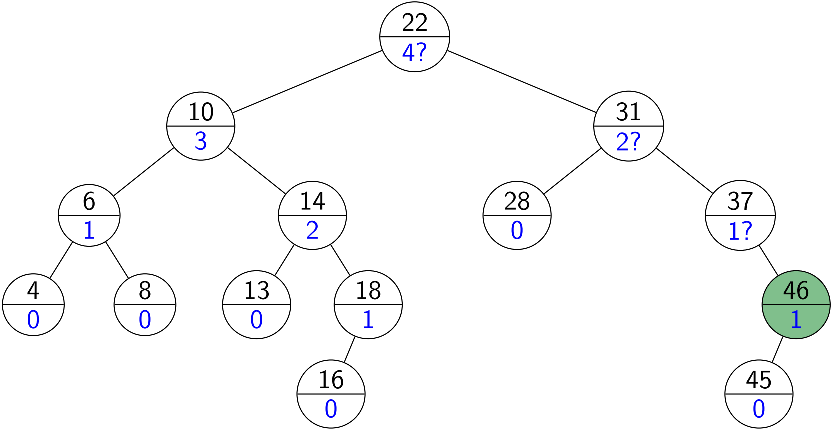
AVL Insertion: Second example

Example: *AVL::insert*(45)



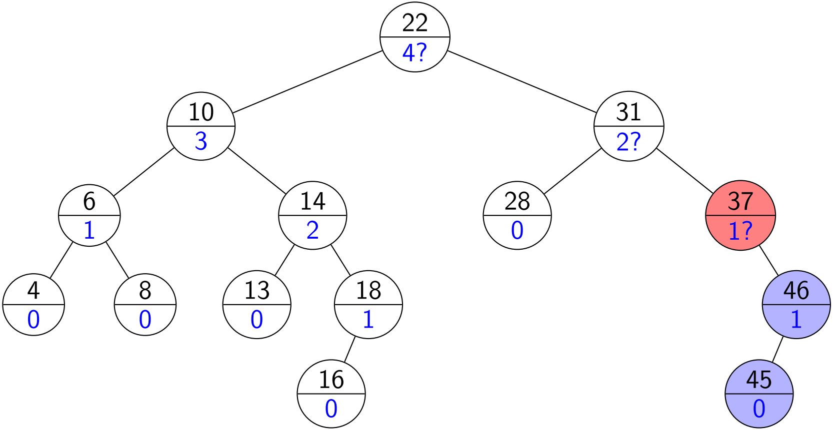
AVL Insertion: Second example

Example: *AVL::insert*(45)



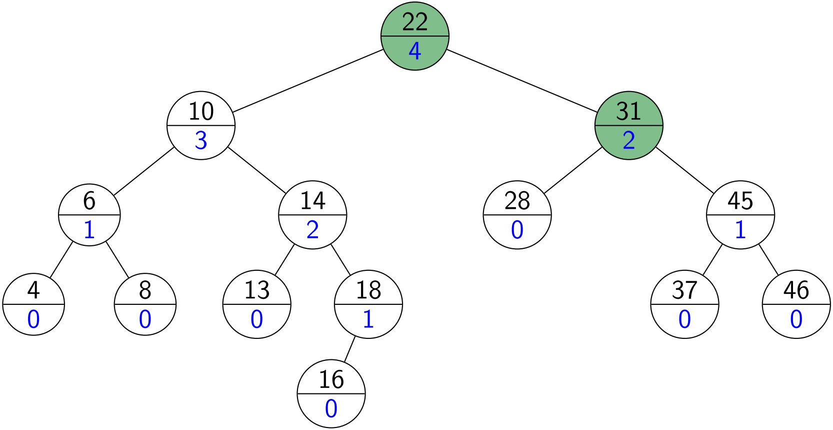
AVL Insertion: Second example

Example: *AVL::insert*(45)



AVL Insertion: Second example

Example: *AVL::insert*(45)



Outline

AVL Deletion

Remove the key k with *BST::delete*.

Find node where *structural* change happened.

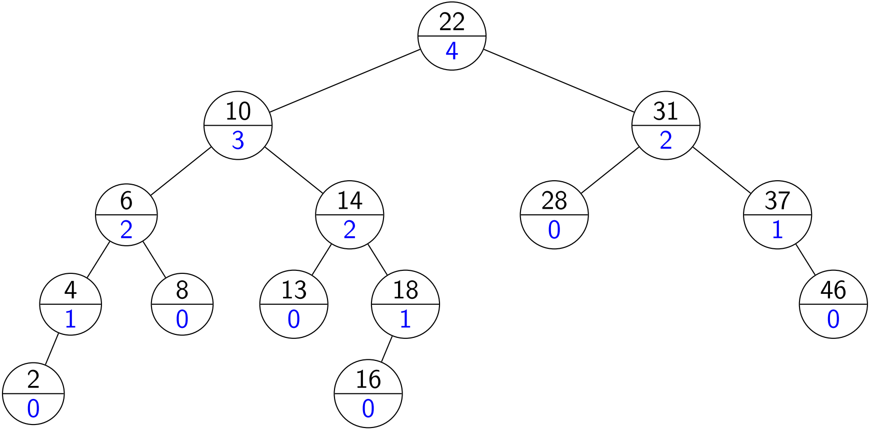
(This is not necessarily near the node that had k.)

Go back up to root, update heights, and rotate if needed.

```
AVL::delete(k)
    z ← BST::delete(k)
    // Assume z is the parent of the BST node that was removed
    while (z is not NULL)
        if (|z.left.height − z.right.height| > 1) then
            Let c be taller child of z
            Let g be taller child of c (break ties to avoid double rotation)
            z ← restructure(g, c, z)
        // Always continue up the path
        setHeightFromSubtrees(z)
        z ← z.parent
```

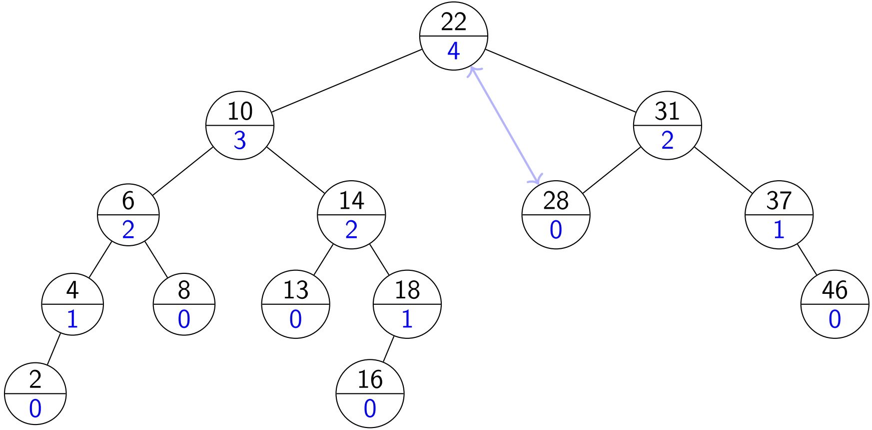
AVL Deletion Example

Example: *AVL::delete*(22)



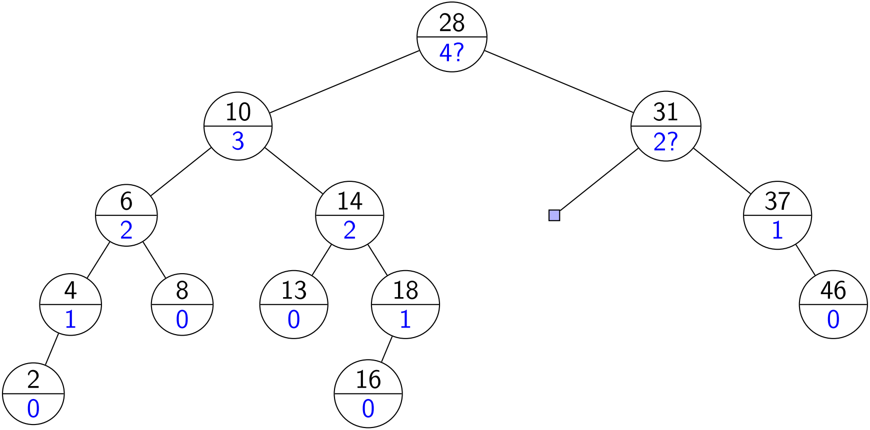
AVL Deletion Example

Example: *AVL::delete*(22)



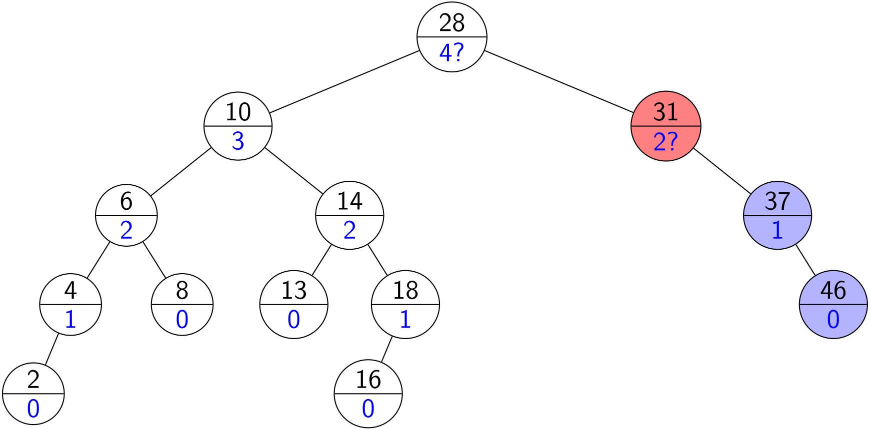
AVL Deletion Example

Example: *AVL::delete*(22)



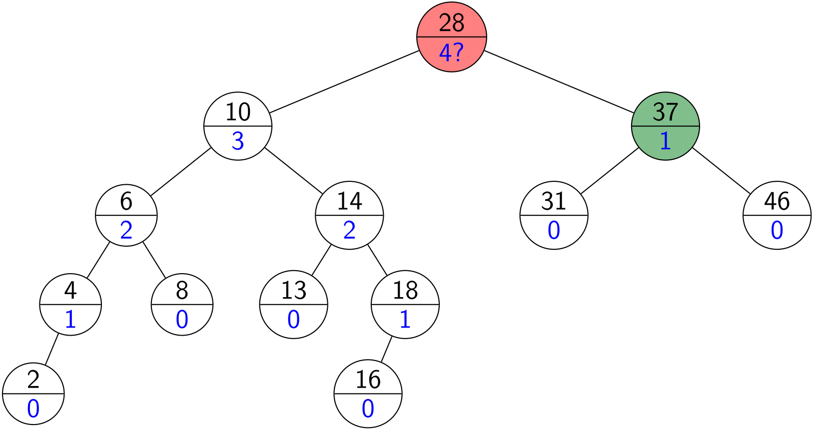
AVL Deletion Example

Example: *AVL::delete*(22)



AVL Deletion Example

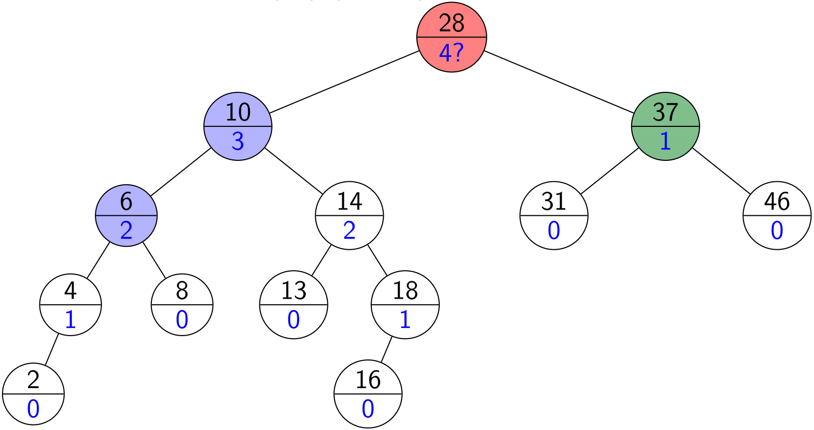
Example: *AVL::delete*(22)



A single *restructure* is not enough to restore all balances.

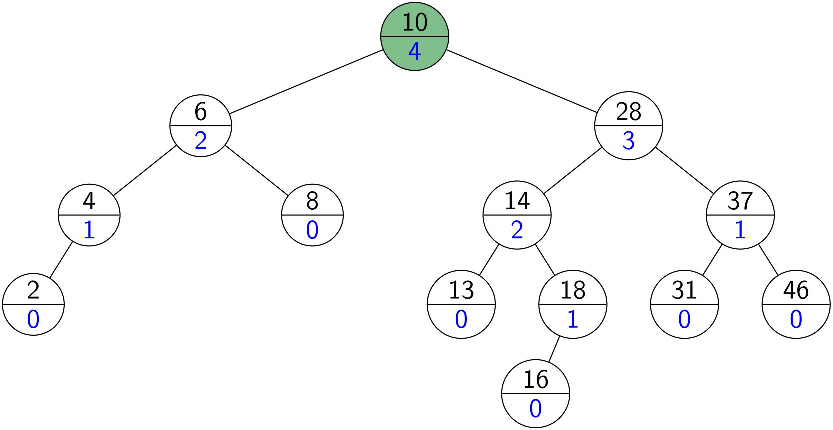
AVL Deletion Example

Example: *AVL::delete*(22) (cont'd)



AVL Deletion Example

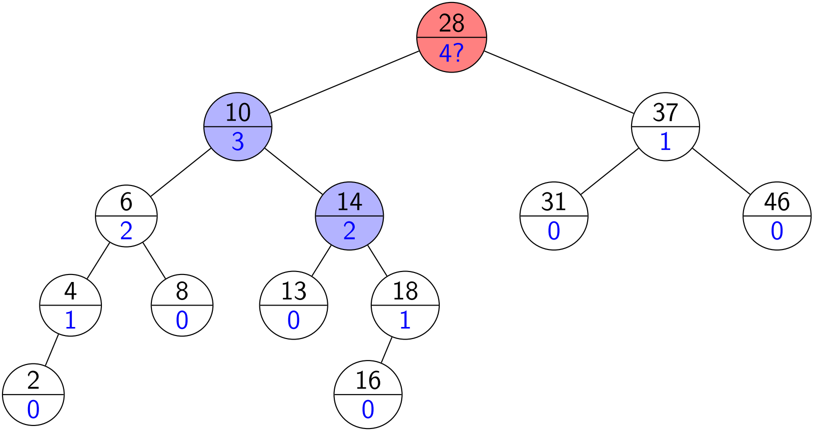
Example: *AVL::delete*(22) (cont'd)



AVL Deletion Example

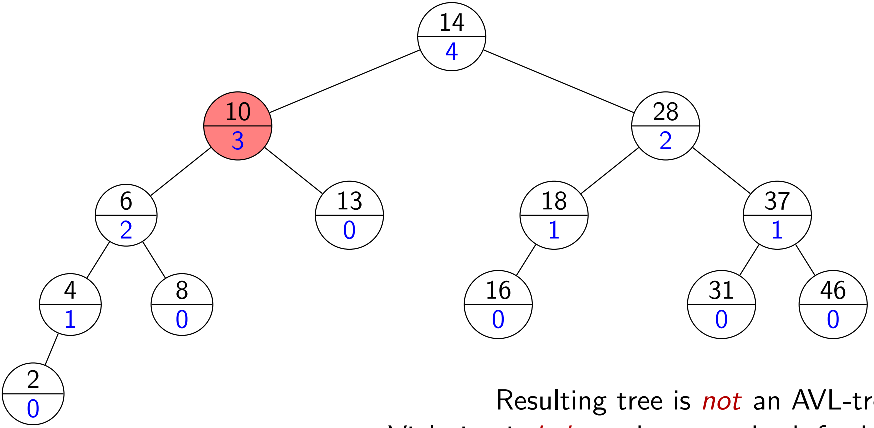
Important: Ties *must* be broken to avoid double rotation.

Consider again the above example. If we applied double-rotation:



AVL Deletion Example

Important: Ties *must* be broken to avoid double rotation.
Consider again the above example. If we applied double-rotation:



Resulting tree is *not* an AVL-tree.
Violation is *below* where we check further.

AVL Tree Summary

search: Just like in BSTs, costs $\Theta(\textit{height})$

insert: *BST::insert*, then check & update along path to new leaf

- total cost $\Theta(\textit{height})$
- *restructure* will be called *at most once*.

delete: *BST::delete*, then check & update along path to deleted node

- total cost $\Theta(\textit{height})$
- *restructure* may be called $\Theta(\textit{height})$ times.

Worst-case cost for all operations is $\Theta(\textit{height}) = \Theta(\log n)$.

- In practice, the constant is quite large.
- Other realizations of ADT Dictionary are better in practice ($\rightarrow$ later)