# CS 240 – Data Structures and Data Management

## Module 4: Dictionaries

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Based on lecture notes by many previous cs240 instructors

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# Outline

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# Review: ADT Dictionary

**Dictionary**: A collection of items, each of which contains

- a *key*
- some *data* (the "value")

and is called a *key-value pair* (KVP). Keys can be compared and are (typically) unique.

Operations:

- *search*($k$) (also called *lookup*($k$))
- *insert*($k, v$)
- *delete*($k$) (also called *remove*($k$))
- optional: *successor*, *merge*, *is-empty*, *size*, *etc.*

Examples: symbol table, license plate database

# Review: Elementary realizations

Common assumptions:

- Dictionary has $n$ KVPs
- Each KVP uses constant space
  (if not, the "value" could be a pointer)
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- Dictionary is non-empty both before and after operation.

  (In a real-life implementation you would have to treat these special cases.)

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  (In a real-life implementation you would have to treat these special cases.)

Easy realizations:

| | *search* | *insert* | *delete* |
|---|---|---|---|
| unsorted list/array | $\Theta(n)$ | $\Theta(1)$ | $\Theta(1)$ |
| sorted array | $\Theta(\log n)$ | $\Theta(n)$ | $\Theta(n)$ |
| binary search tree | $\Theta(\mathit{height})$ | $\Theta(\mathit{height})$ | $\Theta(\mathit{height})$ |

# Review: Binary search

Only applies to a *sorted array*:

| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 30 | 40 | 70 | 90 | 100 | 120 | 140 |

*binary-search*$(A, n, k)$
$A$: Sorted array of size $n$, $k$: key

1. $\ell \leftarrow 0$, $r \leftarrow n-1$
2. **while** $(\ell \leq r)$
3. $\quad m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
4. $\quad$ **if** ($A[m]$ equals $k$) **then return** "found at $A[m]$"
5. $\quad$ **else if** $(A[m] < k)$ **then** $\ell \leftarrow m+1$
6. $\quad$ **else** $r \leftarrow m-1$
7. **return** "not found, but would be between $A[\ell-1]$ and $A[\ell]$"

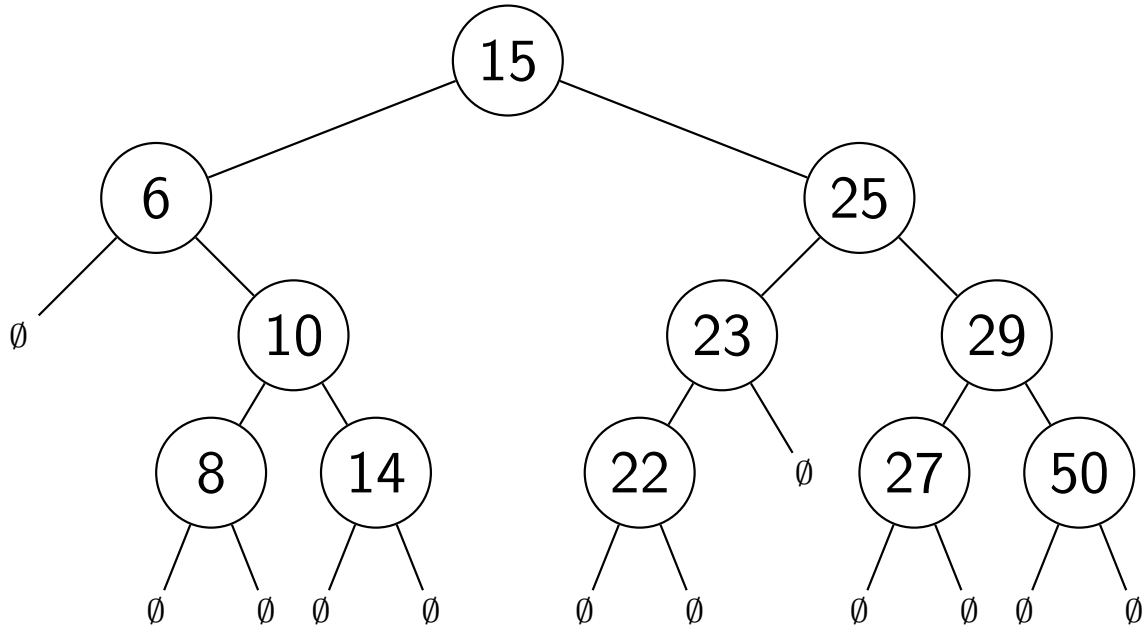
We will return to binary search (and sometimes improve it!) later.

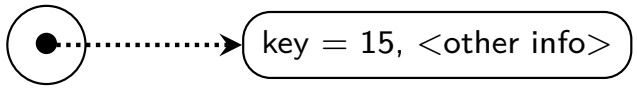
# Outline

# Review: Binary search trees

**Structure** Binary tree: all nodes have two (possibly empty) subtrees
Every node stores a KVP
Empty subtrees usually not shown

**Ordering** Every key $k$ in $T.left$ is less than the root key.
Every key $k$ in $T.right$ is greater than the root key.

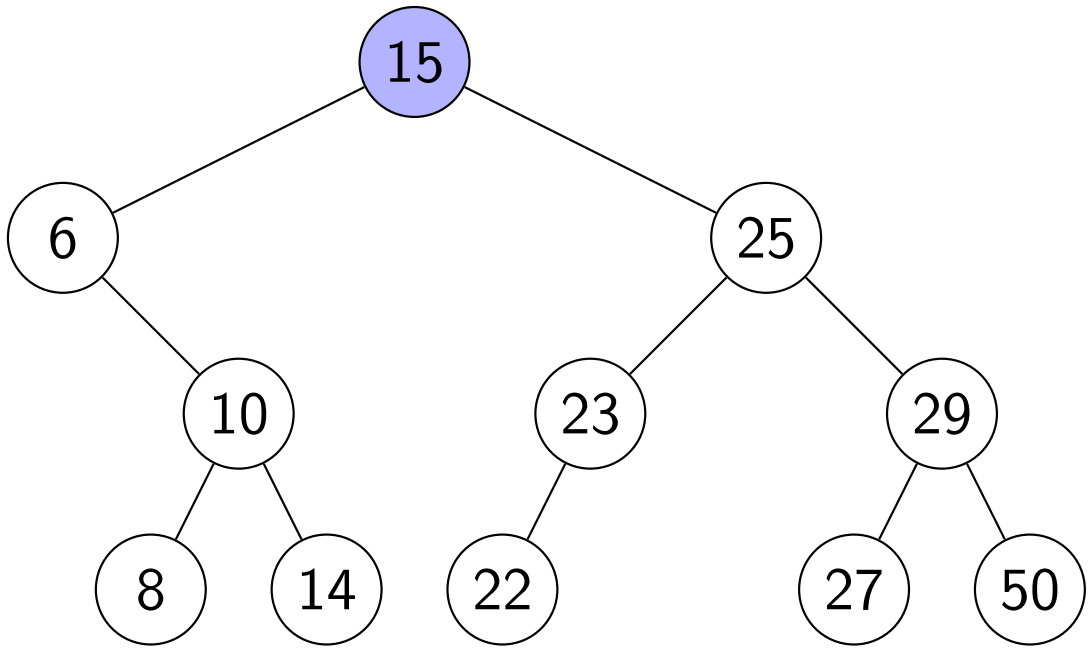



( In our examples we only show the keys, and we show them directly in the node. A more accurate picture would be  )

# Review: BST as realization of ADT Dictionary

*BST::search*($k$) Start at root, compare $k$ to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

Example: *BST::search*(24)



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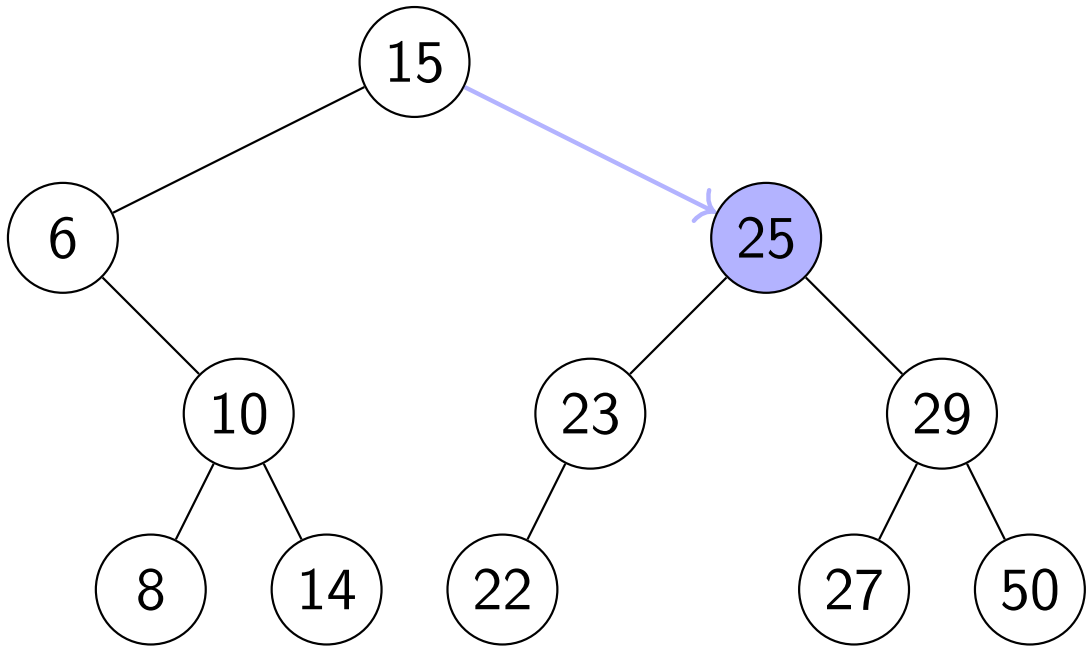

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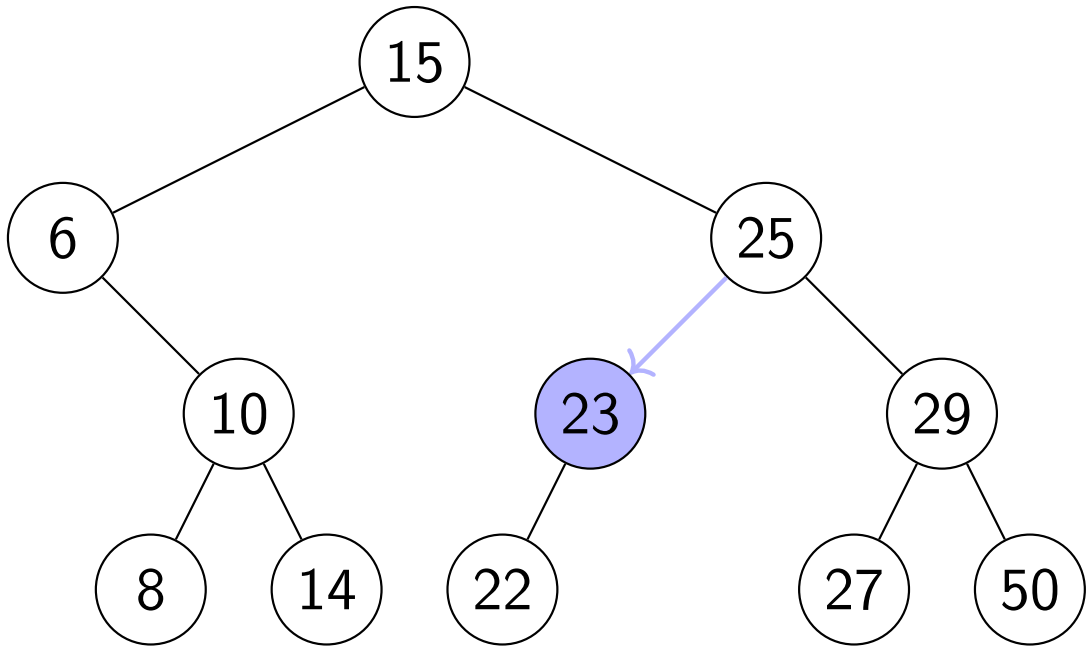

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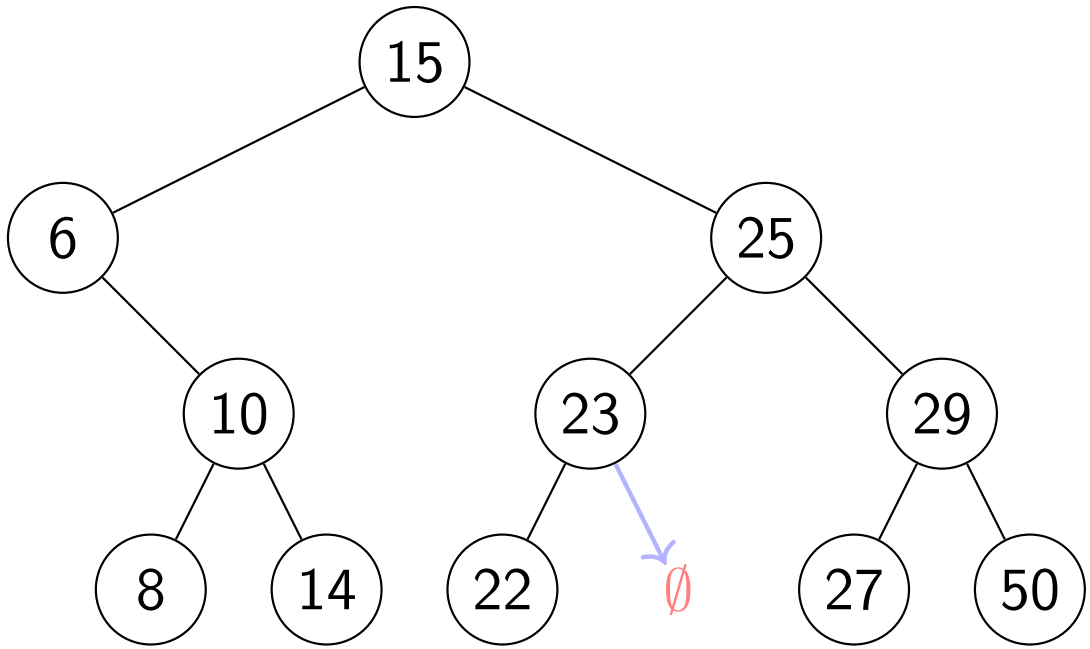
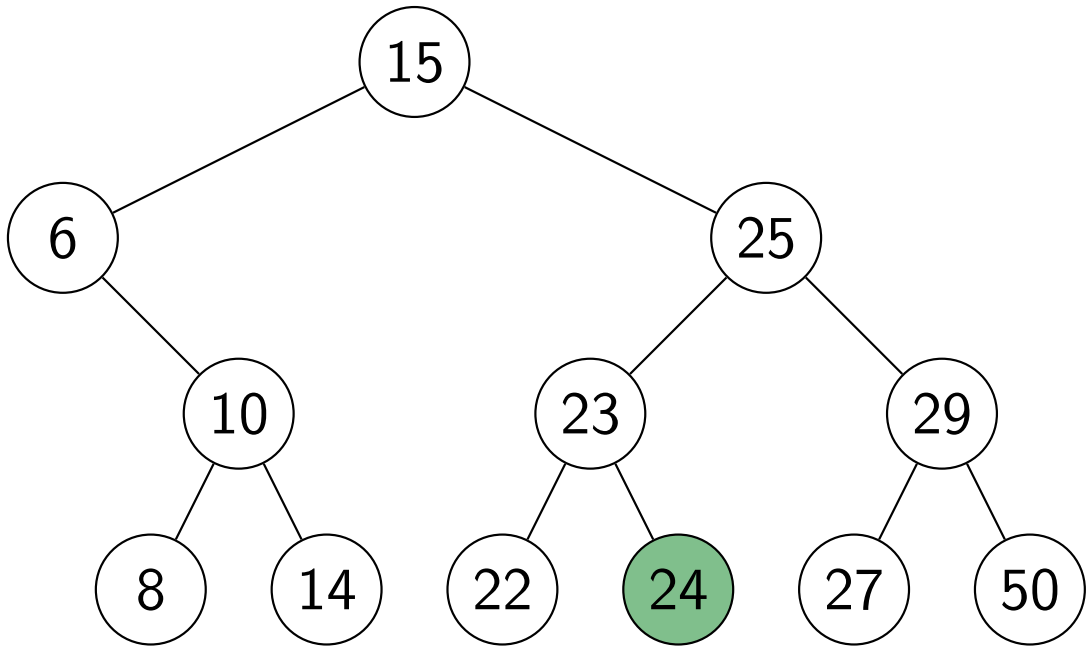


# Review: BST as realization of ADT Dictionary

*BST::search*$(k)$ Start at root, compare $k$ to current node's key.
Stop if found or subtree is empty, else recurse at subtree.

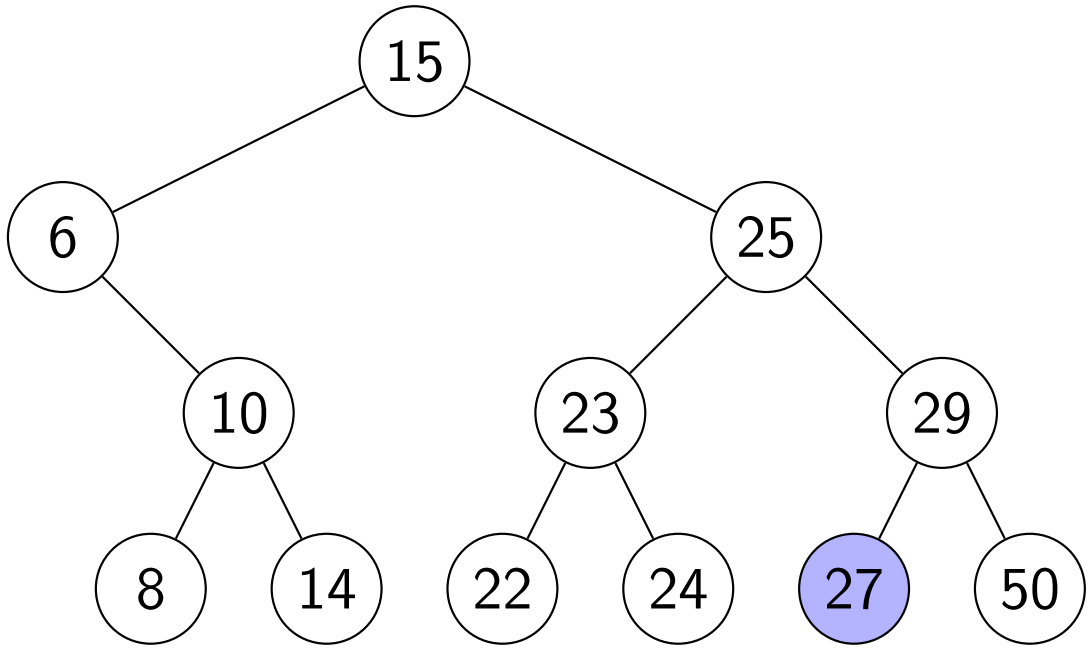
*BST::insert*$(k, v)$ Search for $k$, then insert $(k, v)$ as new node

Example: *BST::insert*$(24, v)$



# Deletion in a BST

- First search for the node $x$ that contains the key.
- If $x$ is a **leaf** (both subtrees are empty), delete it.



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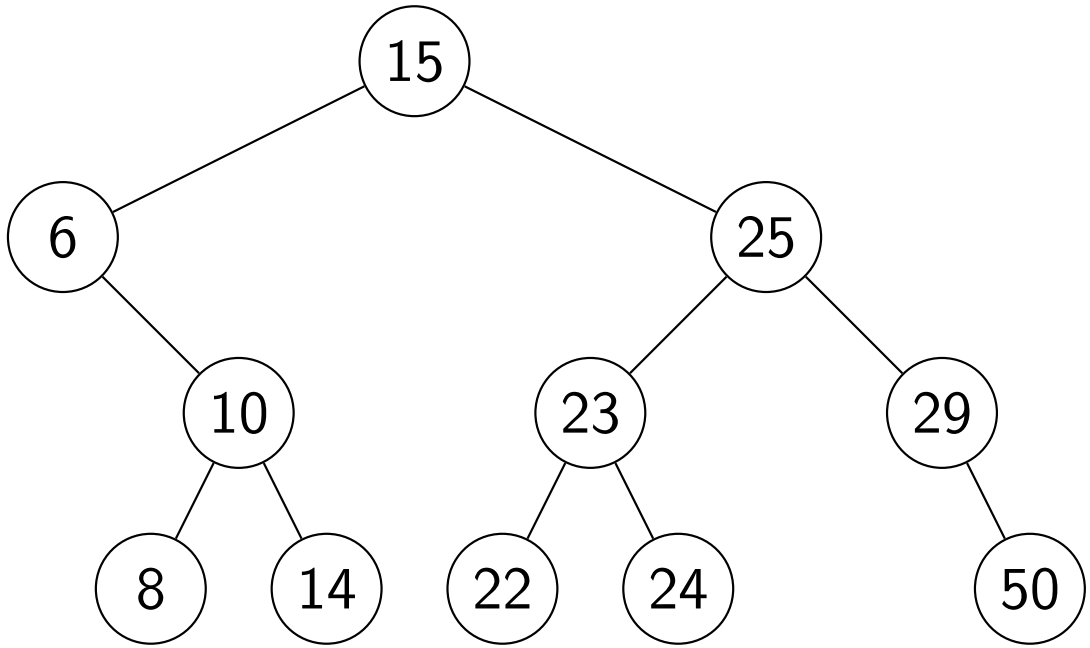

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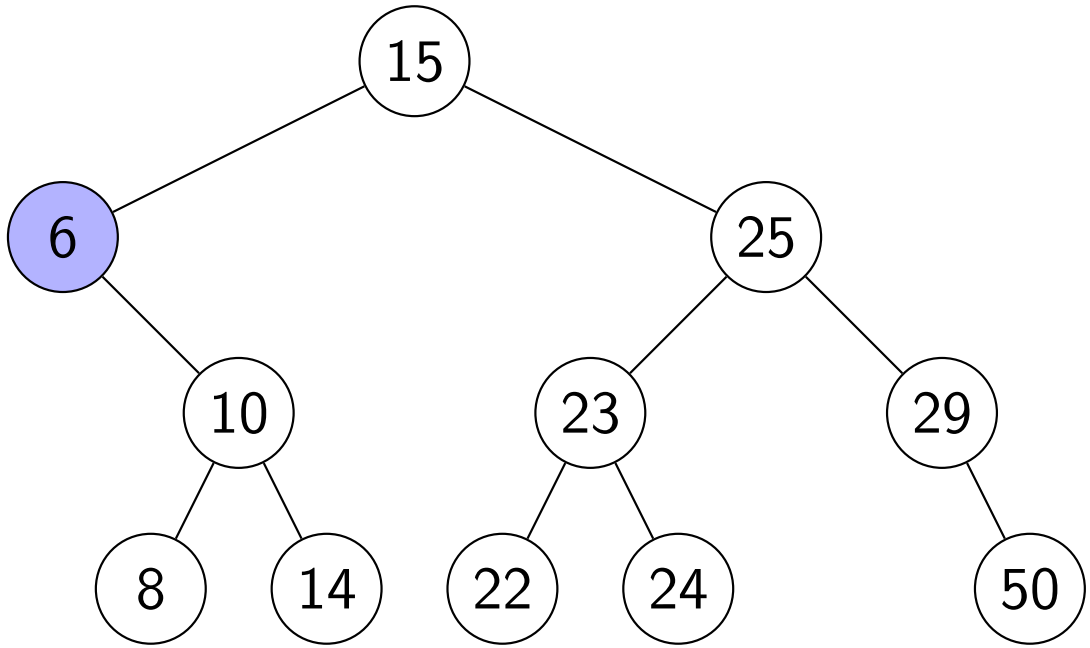

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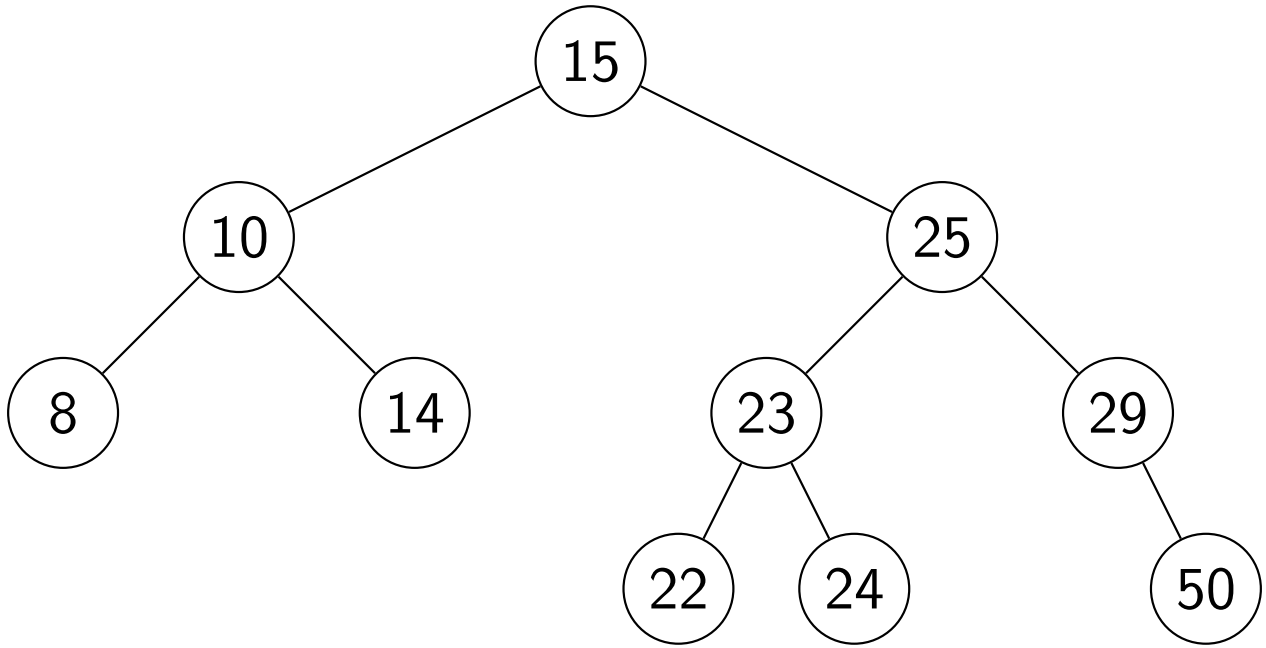

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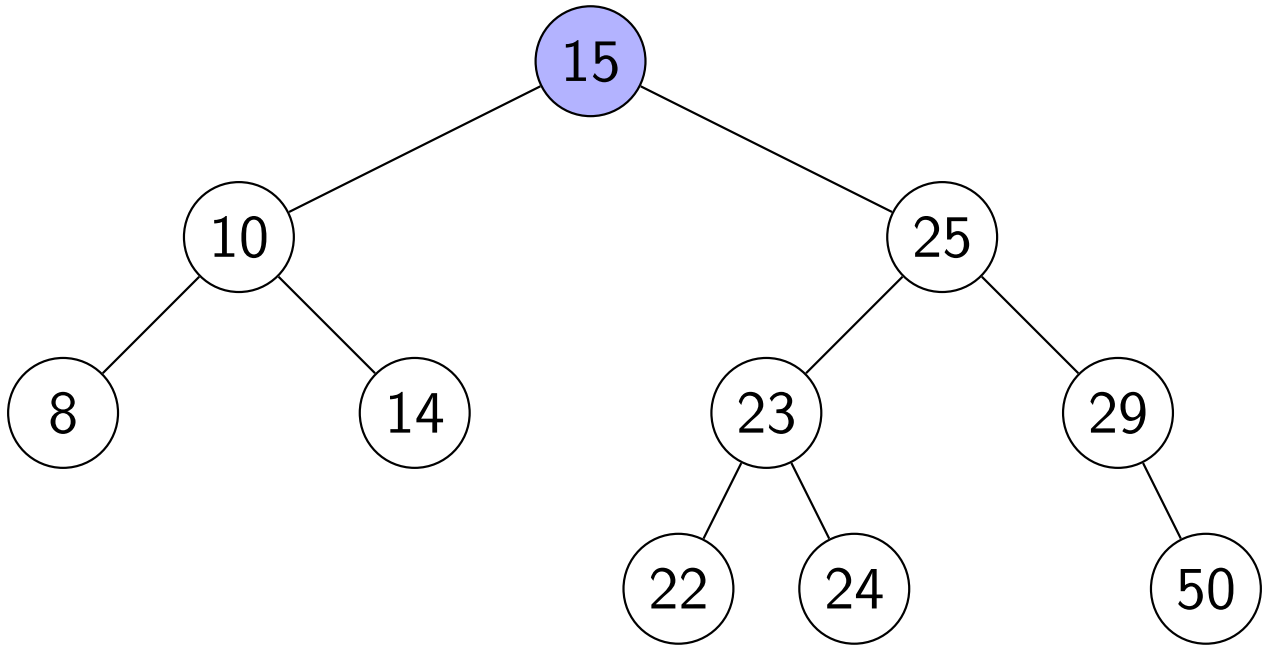
(Successor: next-smallest among all keys in the dictionary.)



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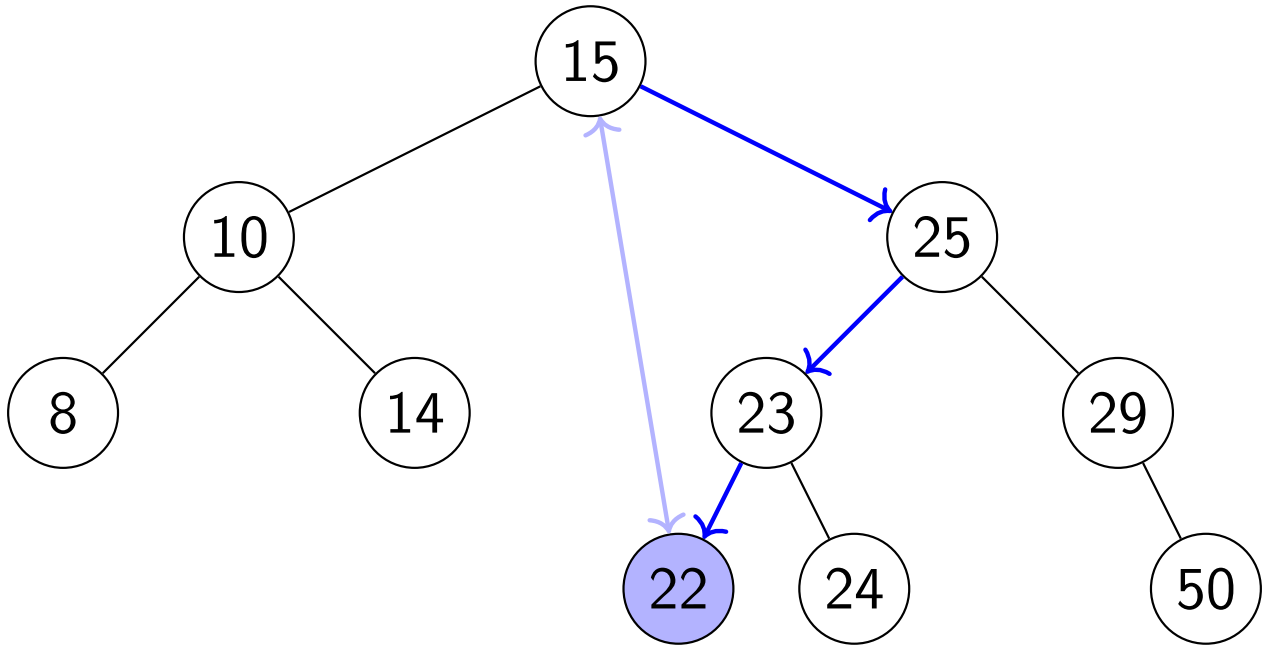

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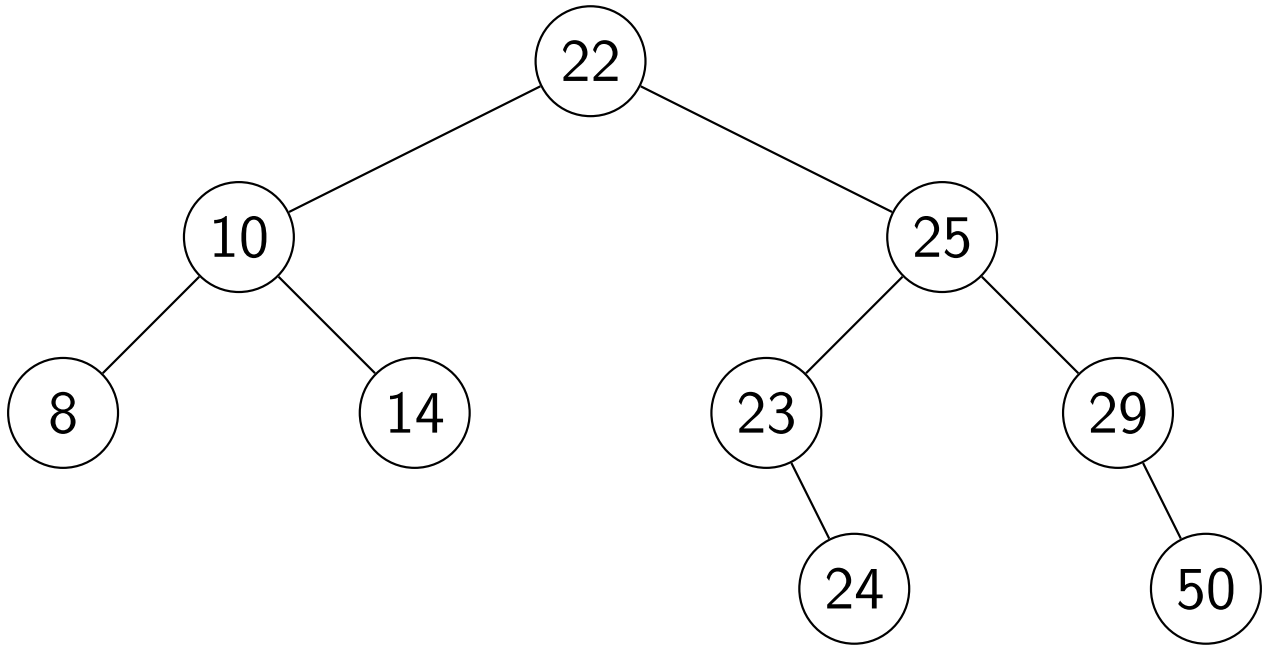

# Height of a BST

*BST::search*, *BST::insert*, *BST::delete* all have cost $\Theta(h)$, where $h$ = maximum number for which level $h$ contains nodes. Single-node tree has height 0, empty tree has height -1

If $n$ items are inserted one-at-a-time, how big is $h$?

- Worst-case: $n - 1 = \Theta(n)$
- Best-case: $\Theta(\log n)$.
  Any binary tree with $n$ nodes has height $h \geq \log(n+1) - 1$ (See Module 2).

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**Goal:** Create subclasses of BSTs where the height is *always* good.

- Impose a structural property.
- Argue that the property implies logarithmic height.
- Discuss how to maintain the property during operatons.

# Outline

# AVL trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962, an **AVL tree** is a BST with an additional **height-balance property** at every node:

*The heights of the left and right subtree differ by at most 1.*

Rephrase: If node $z$ has left subtree $L$ and right subtree $R$, then

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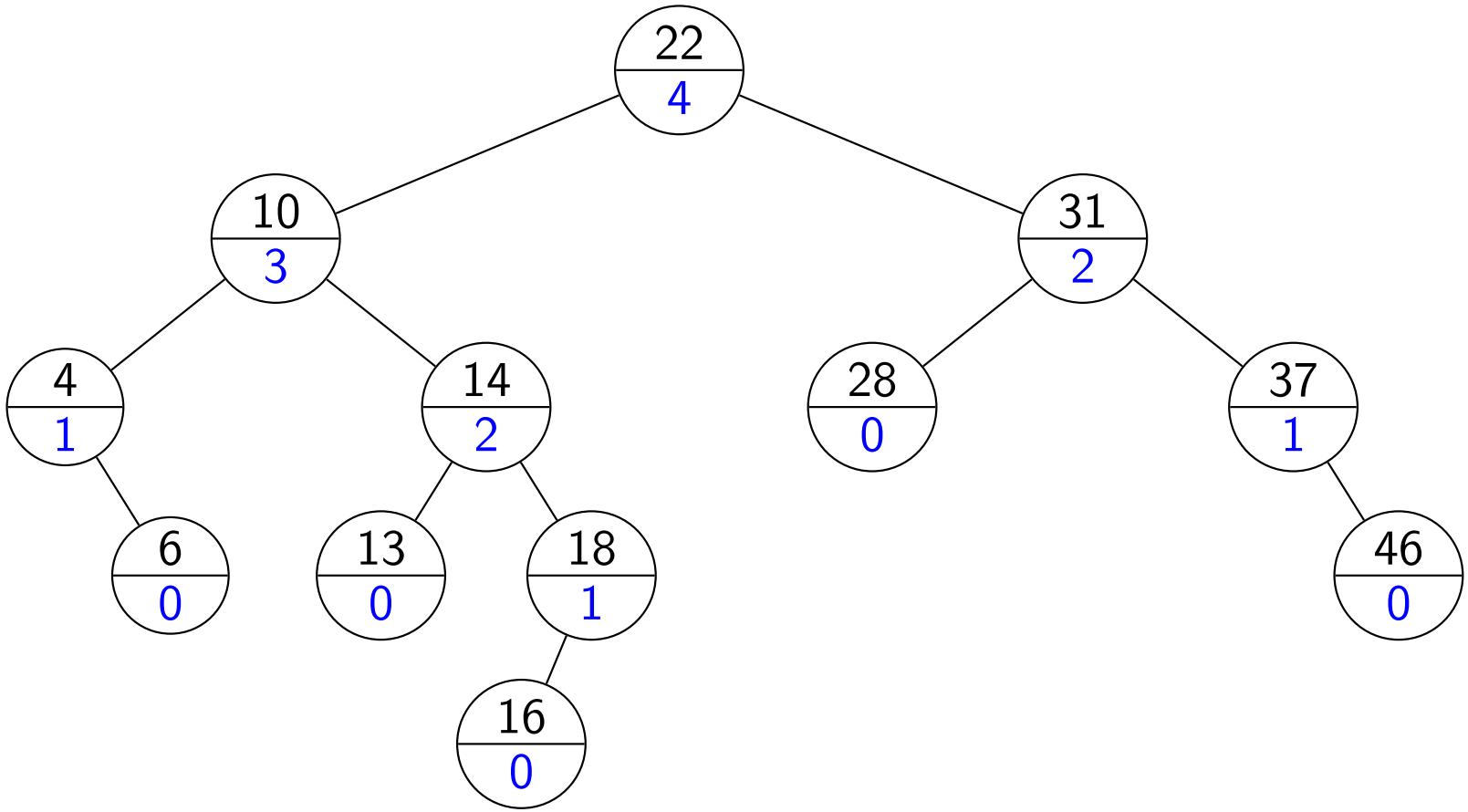
**Things to show:**

- This *structural condition* implies logarithmic height.
- After *insert* and *delete*, we can restore the structural condition within logarithmic time.
  - For this, we need to store at each node $z$ the height of the subtree rooted at it.

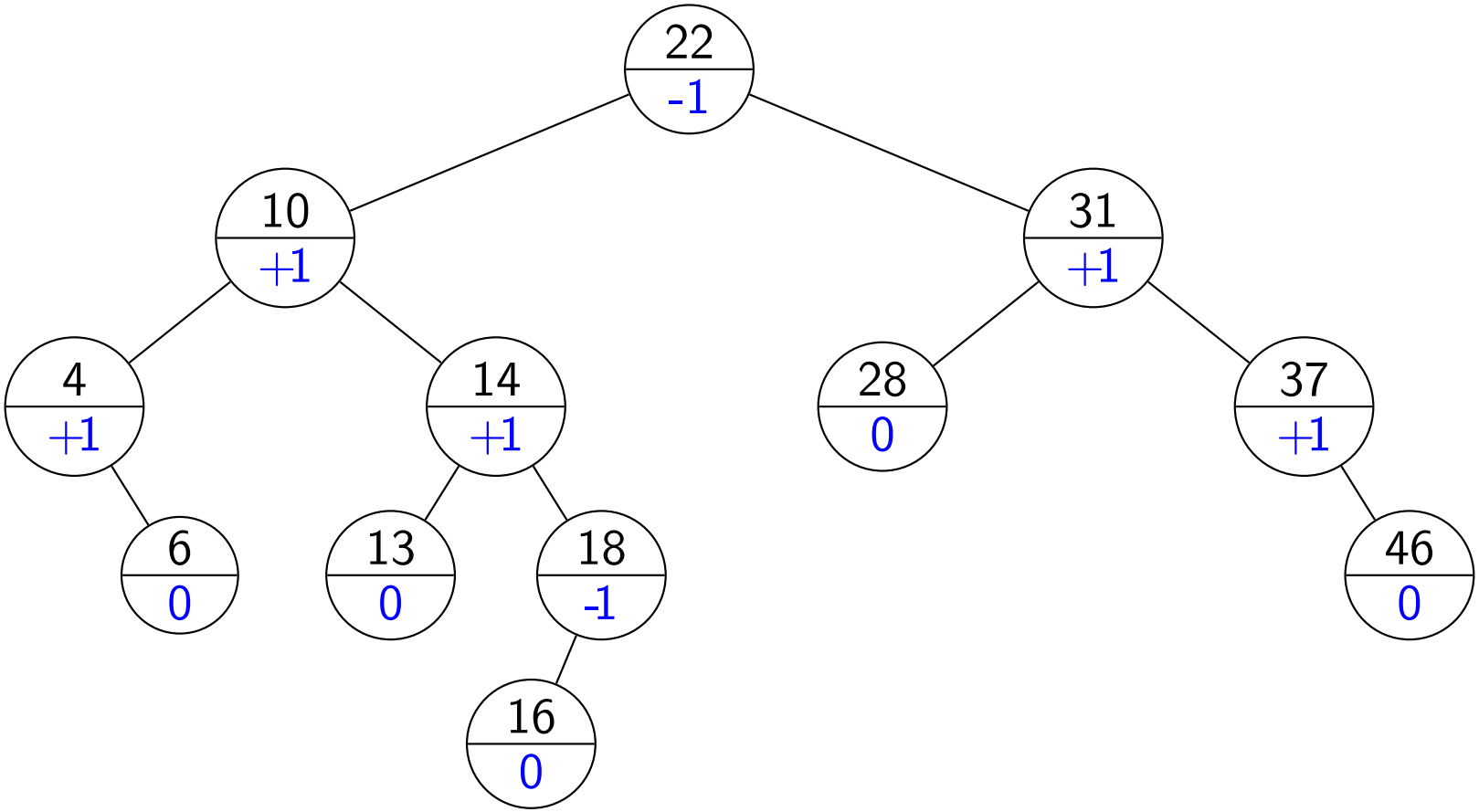
# AVL tree example

(The lower numbers indicate the height of the subtree.)



# AVL tree example

Alternative: store height-difference (instead of height) at each node.




- Saves space (2 bits vs. 1 integer per node)
- Pseudo-code gets a lot more complicated $\rightsquigarrow$ not done here

# AVL trees: Height

**Theorem:** The height of an AVL tree on $n$ nodes is in $\Theta(\log n)$
$\Rightarrow$ *search*, *BST::insert*, *BST::delete* all cost $\Theta(\log n)$ in the *worst case!*

**Proof:**

- Define $N(h)$ to be the *least* number of nodes in a height-$h$ AVL tree.
- What is a recurrence relation for $N(h)$?
- What does this recurrence relation resolve to?

# Outline

# AVL trees: Insertion

To perform *AVL::insert*$(k, v)$:

- First, insert $(k, v)$ with the usual BST insertion.
- We assume that this returns the new leaf $z$ where the key was stored.
- Then, move up the tree from $z$.

$$\begin{pmatrix}\text{We assume for this that we have parent-links. This can be}\\ \text{avoided if } \textit{BST::insert} \text{ returns the full path to } z.\end{pmatrix}$$

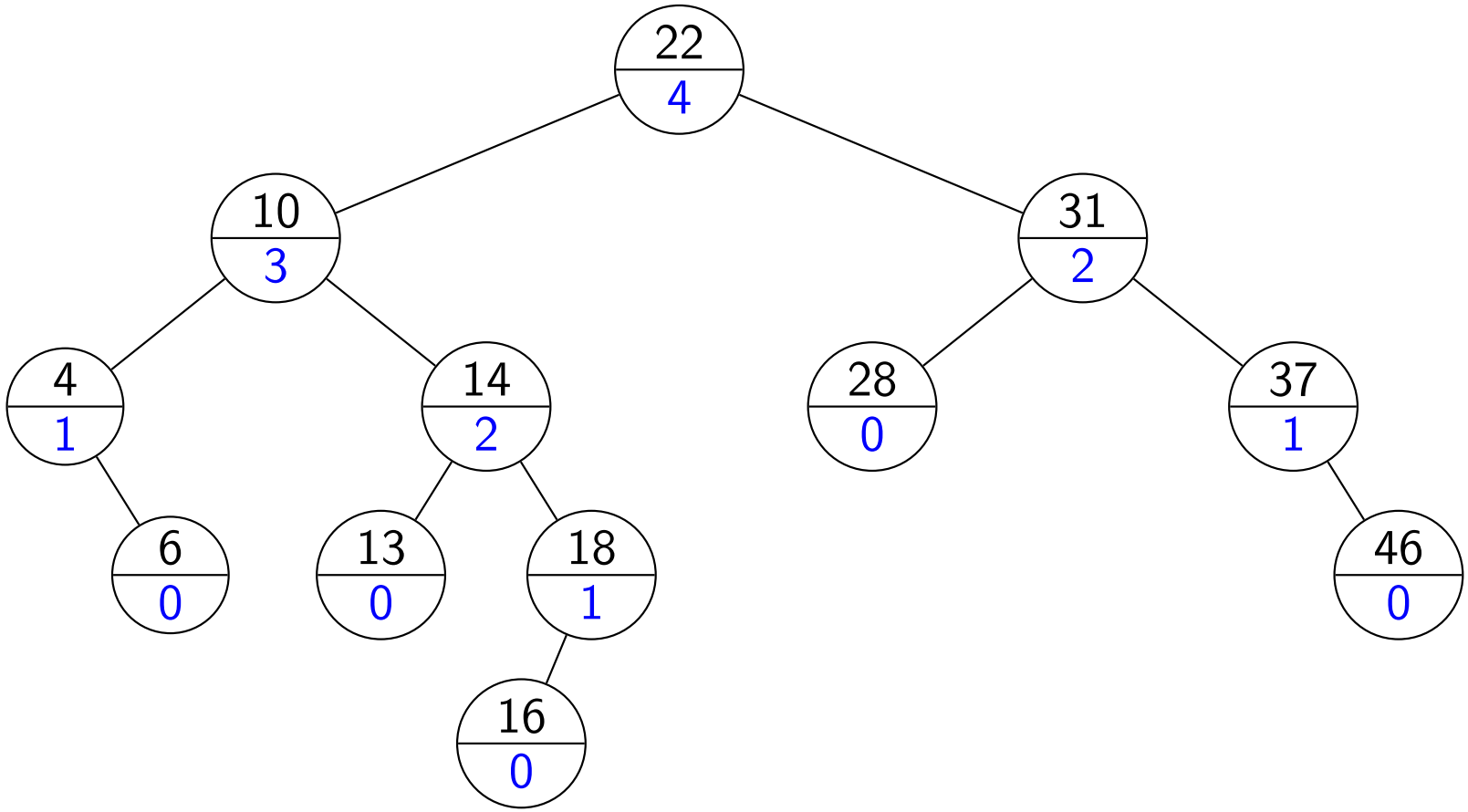
- Update height (easy to do in constant time):

*set-height-from-subtrees*$(u)$
1. $u.height \leftarrow 1 + \max\{u.left.height, u.right.height\}$

- If the height difference becomes $\pm 2$ at node $z$, then $z$ is **unbalanced**. Must re-structure the tree to rebalance.

# AVL tree insertion: Example

**Example**: *AVL::insert*(8)



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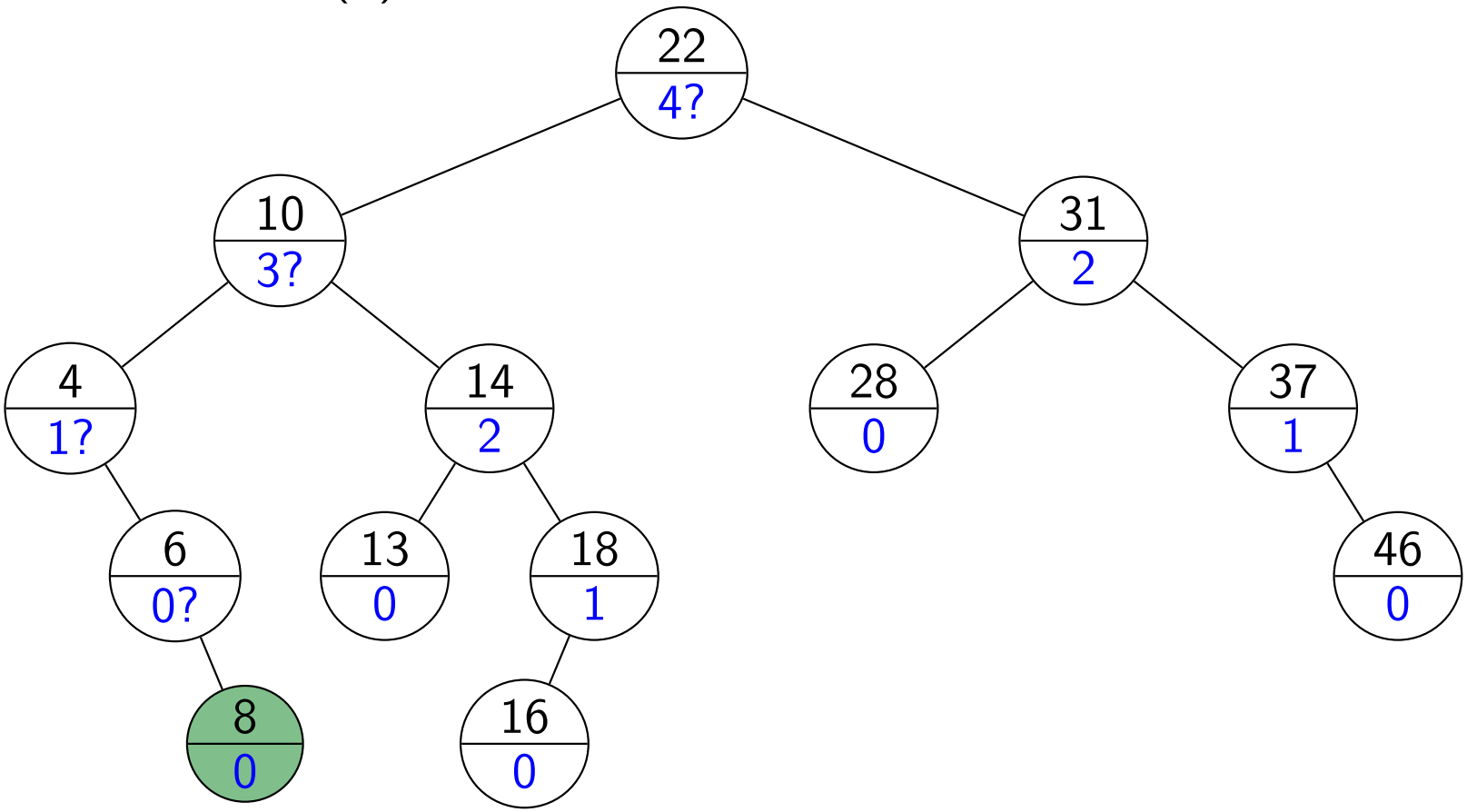

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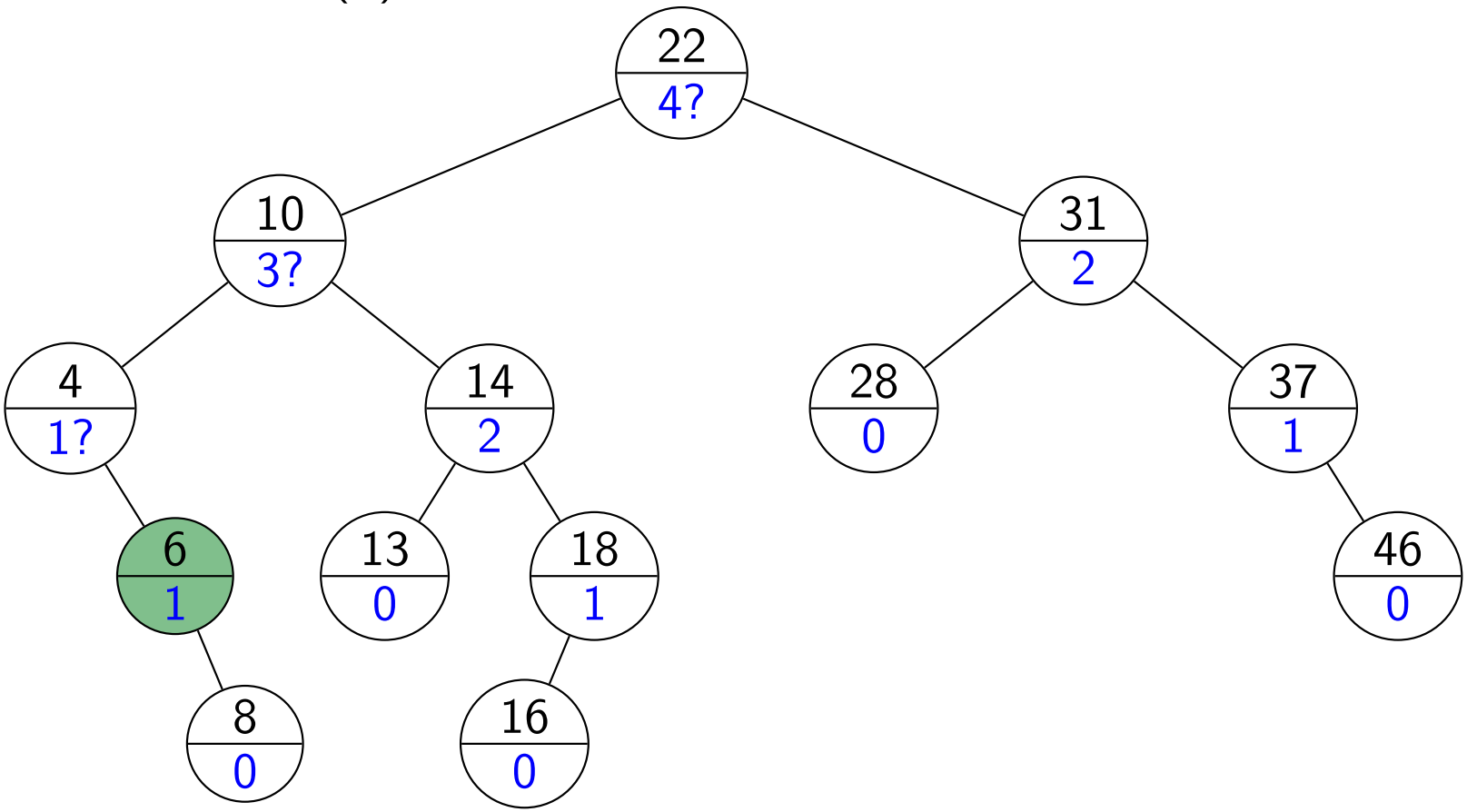

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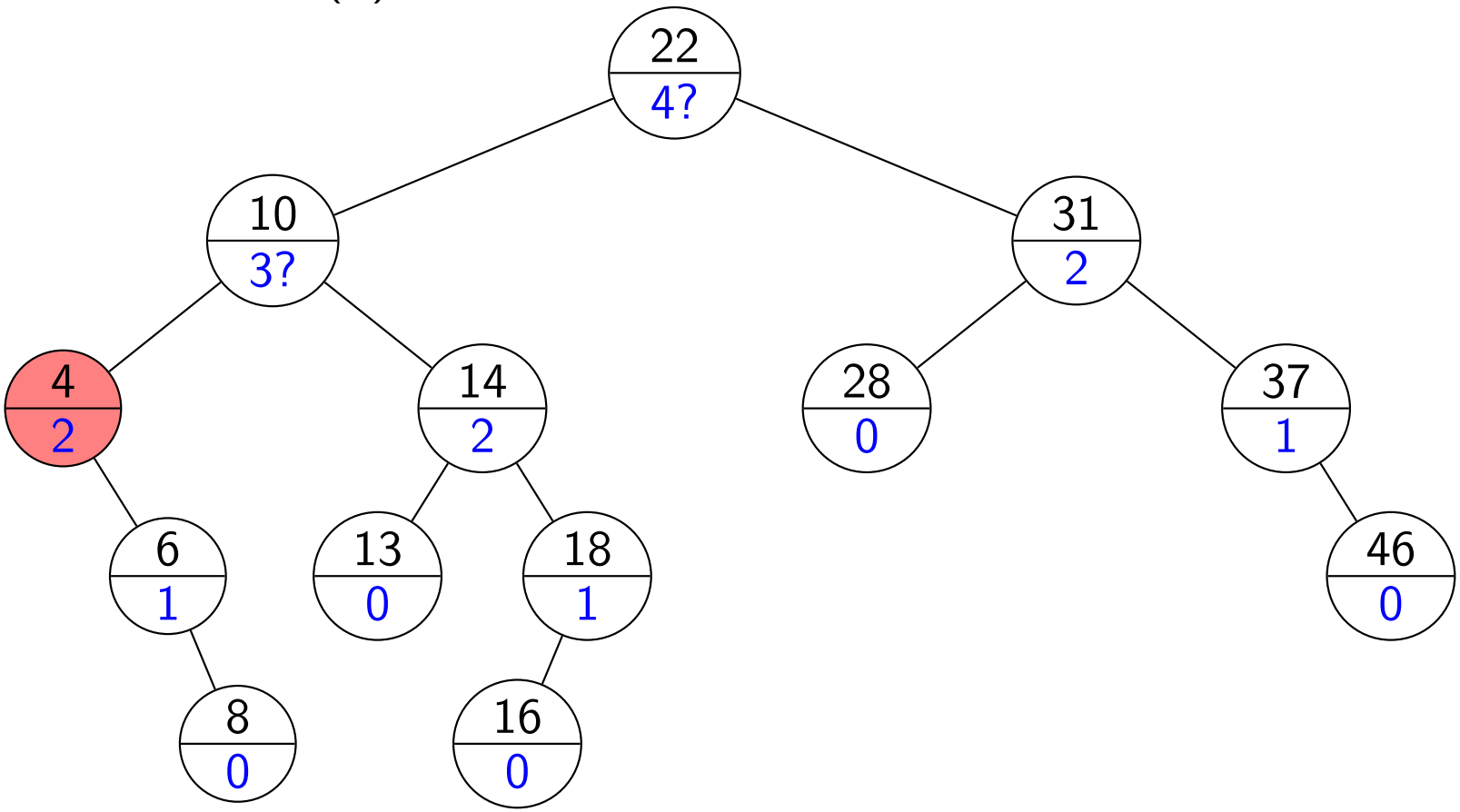
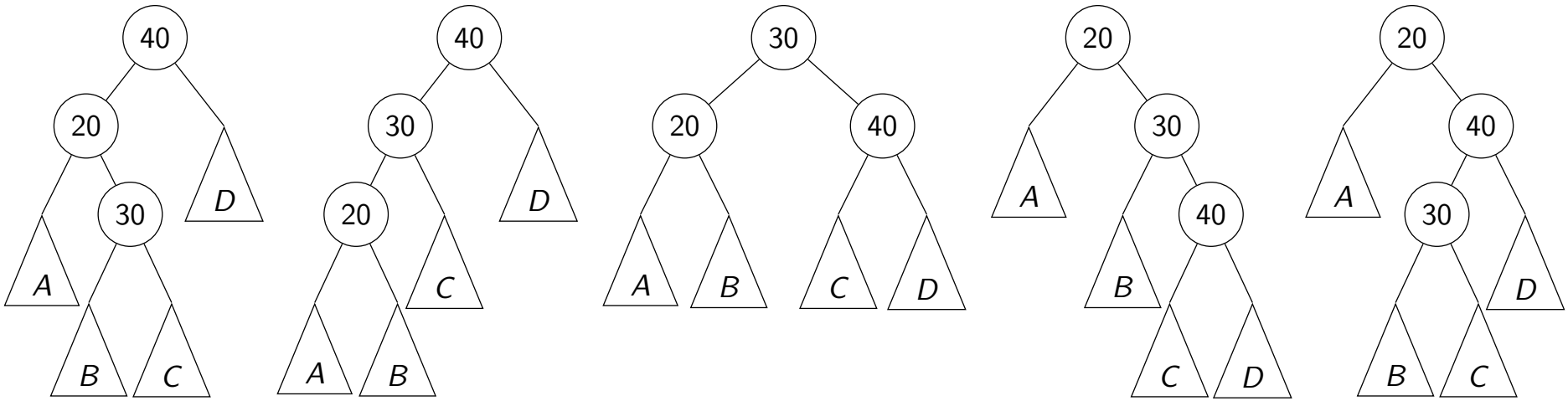


# Outline

# Changing structure without changing order

**Note**: There are many different BSTs with the same keys.

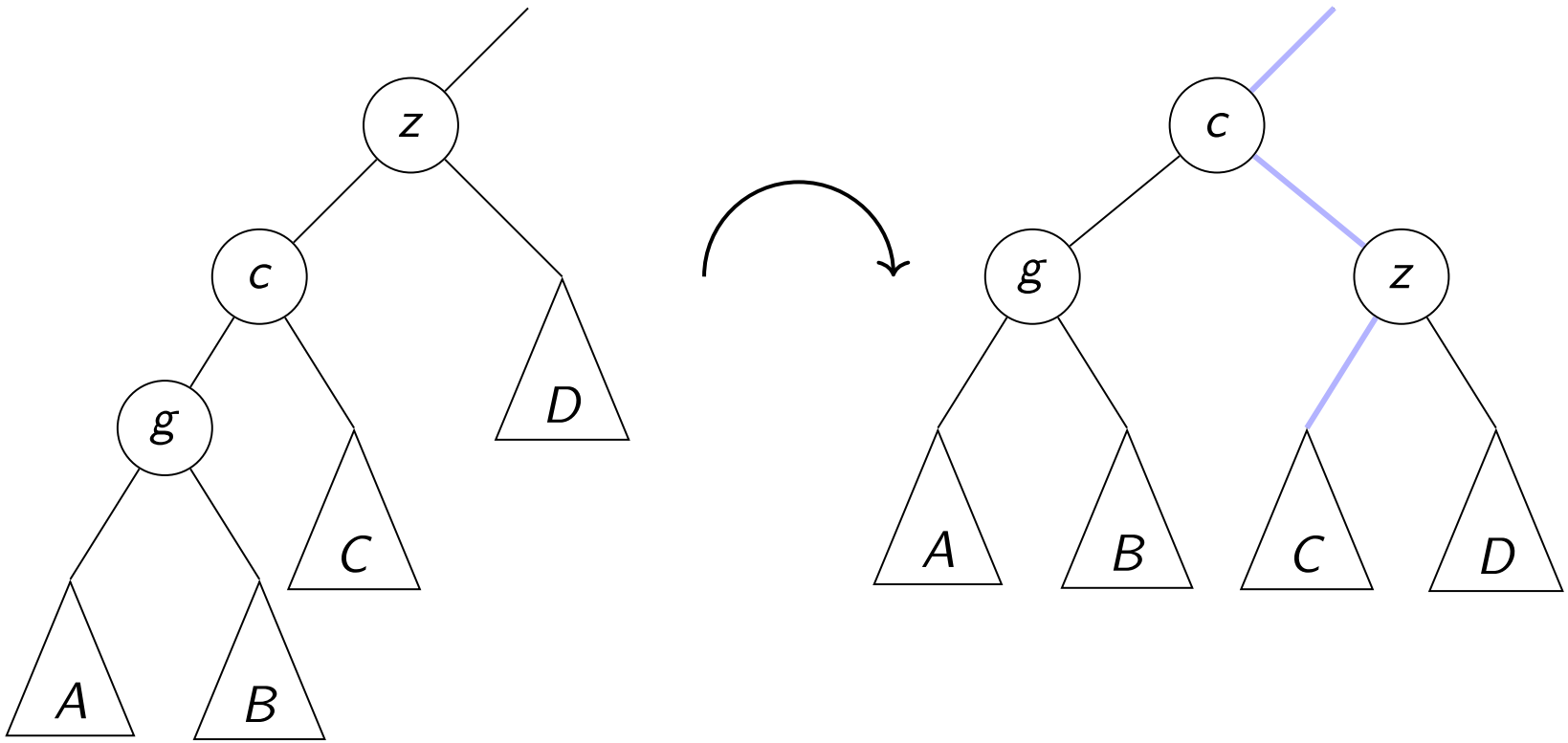



**Goal**: Change the *structure* locally without changing the *order*.

**Longterm goal**: Restructure such that the subtree becomes balanced.

# Right Rotation

This is a **right rotation** on node $z$:



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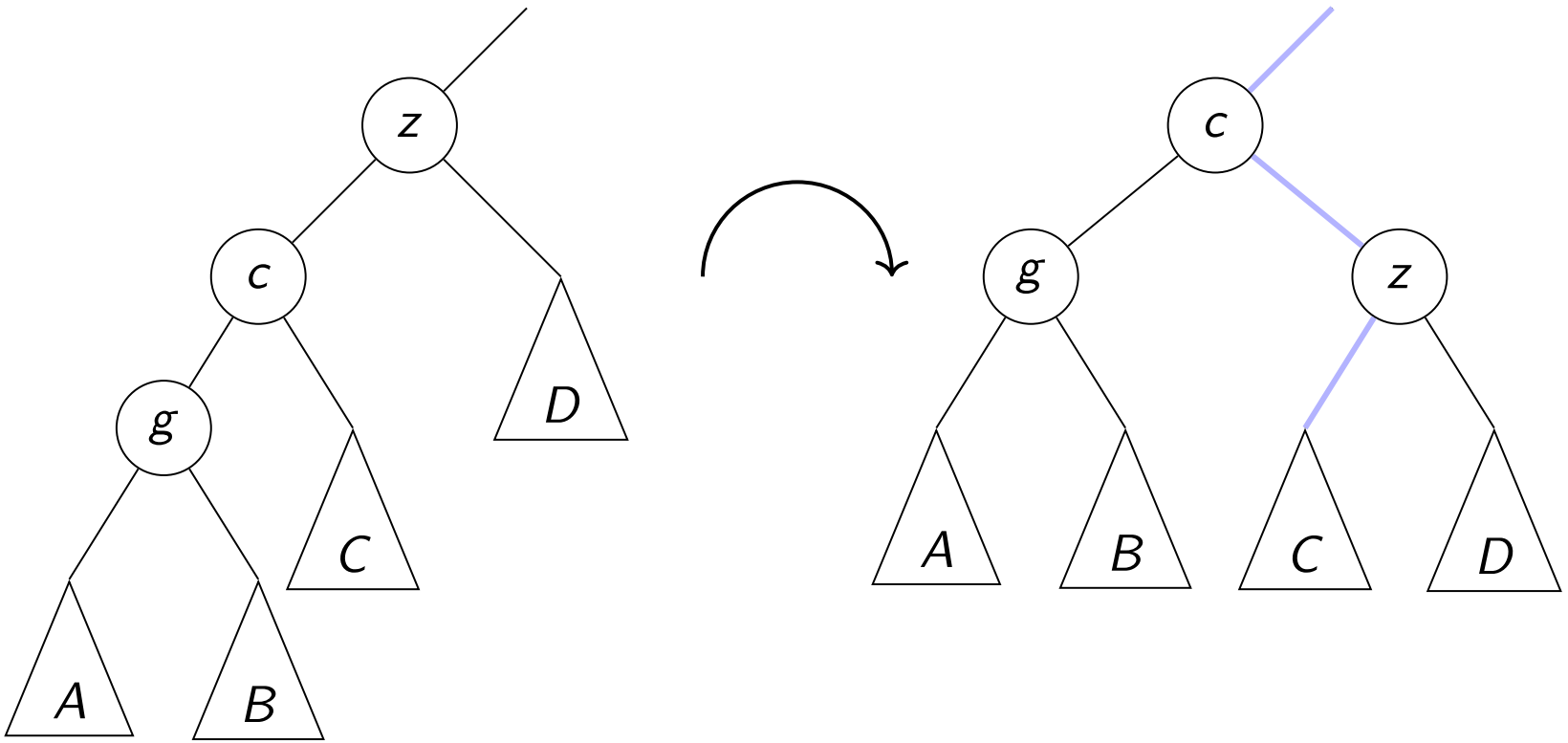
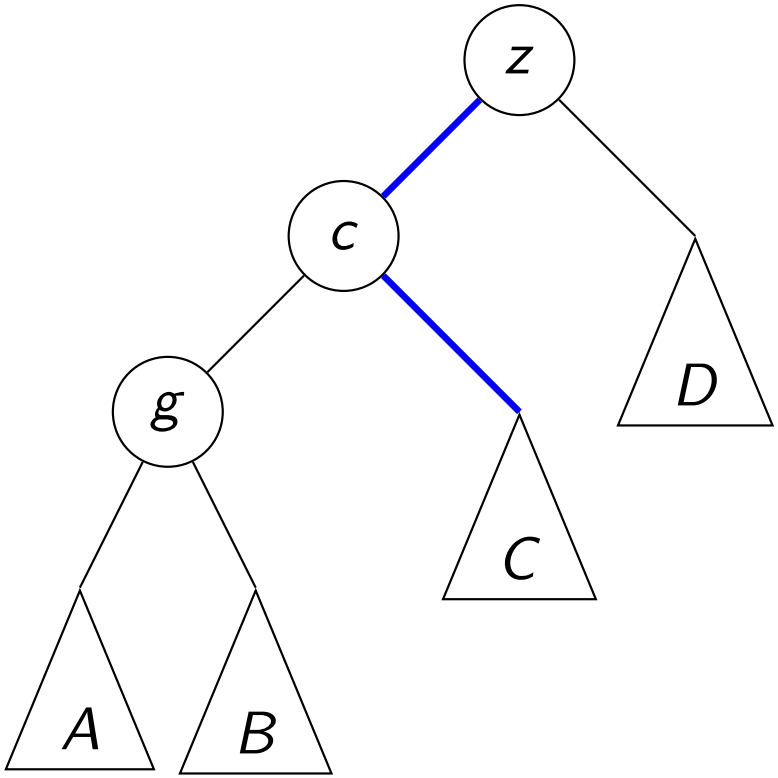



**Note**: Only $O(1)$ links are changed. Useful to fix left-left imbalance.

# Why do we call this a rotation?



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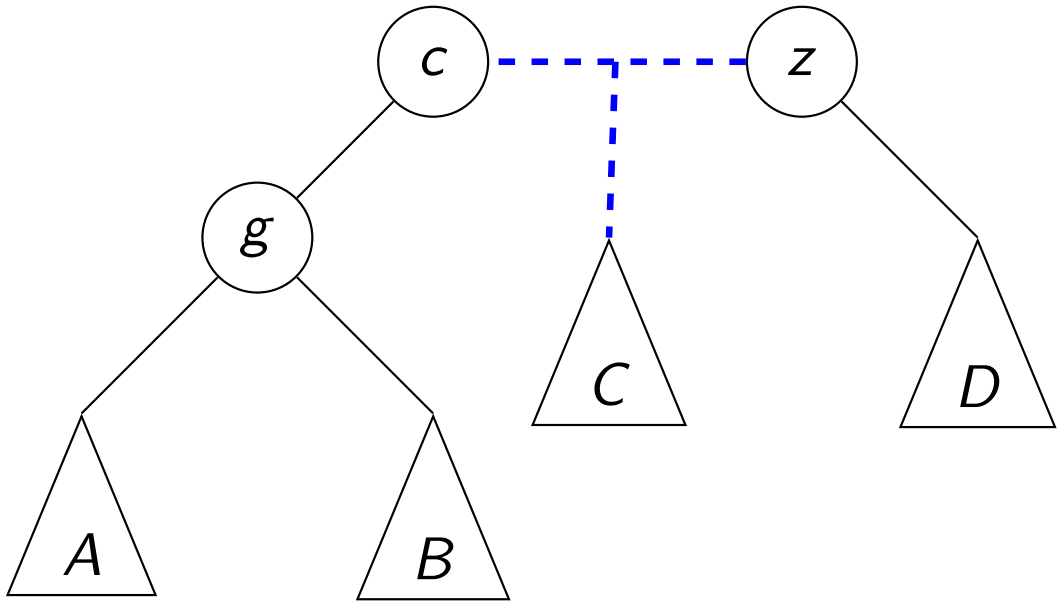

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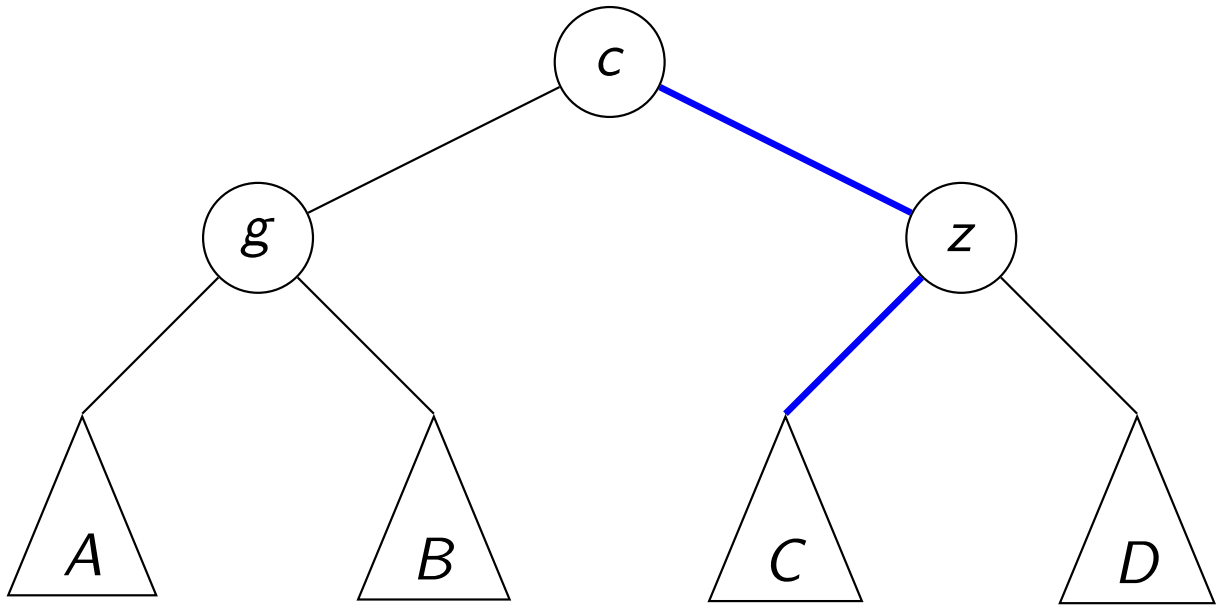


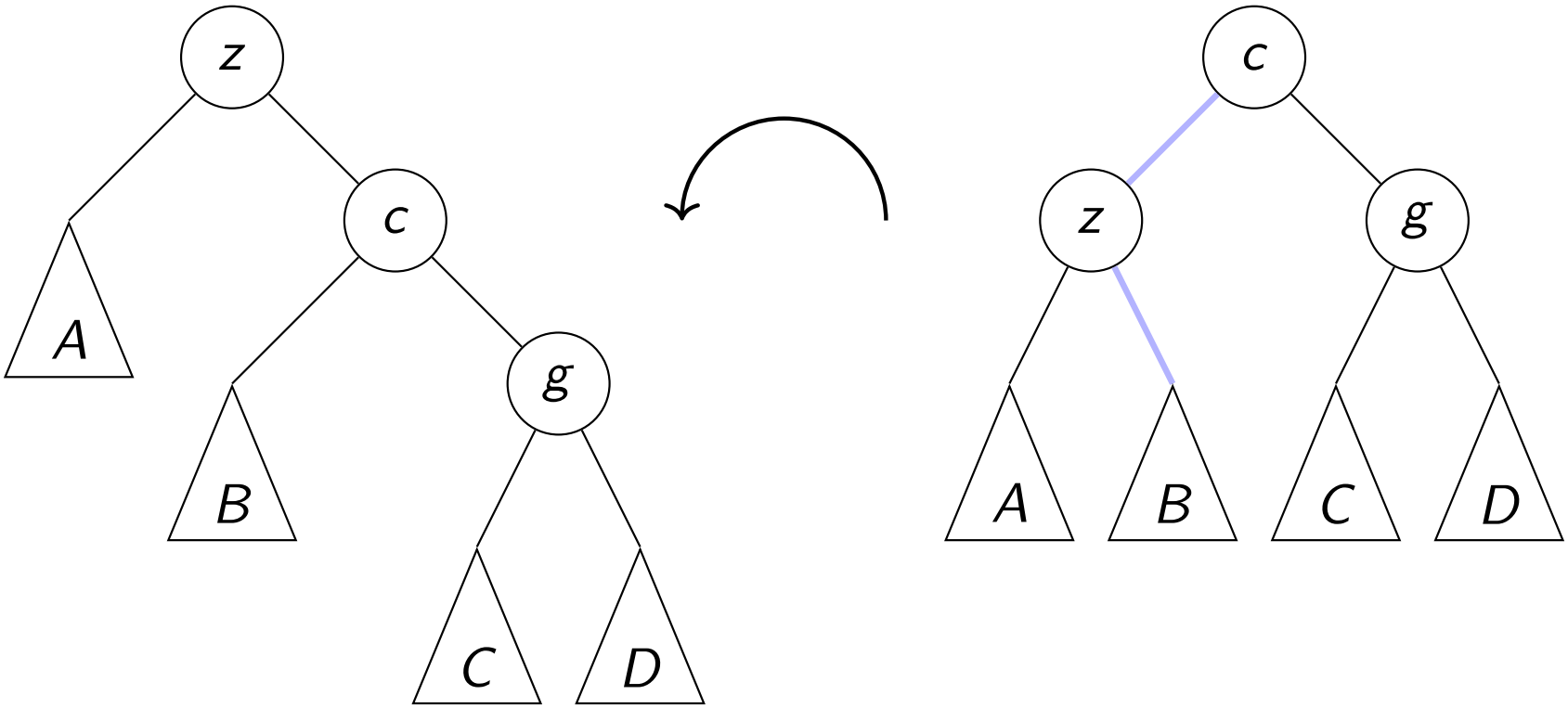
# Right Rotation: Pseudocode

```
rotate-right(z)
1.  c ← z.left

2.  // fix links connecting to above
3.  c.parent ← (p ← z.parent)
4.  if p = NULL then root ← c else
5.        if p.left = z then p.left ← c else p.right ← c

6.  // actual rotation
7.  z.left ← c.right, c.right.parent ← z
8.  c.right ← z, z.parent ← c

9.  set-height-from-subtrees(z), set-height-from-subtrees(c)
10. return c      // returns new root of subtree
```

Run-time: $O(1)$

# Left Rotation

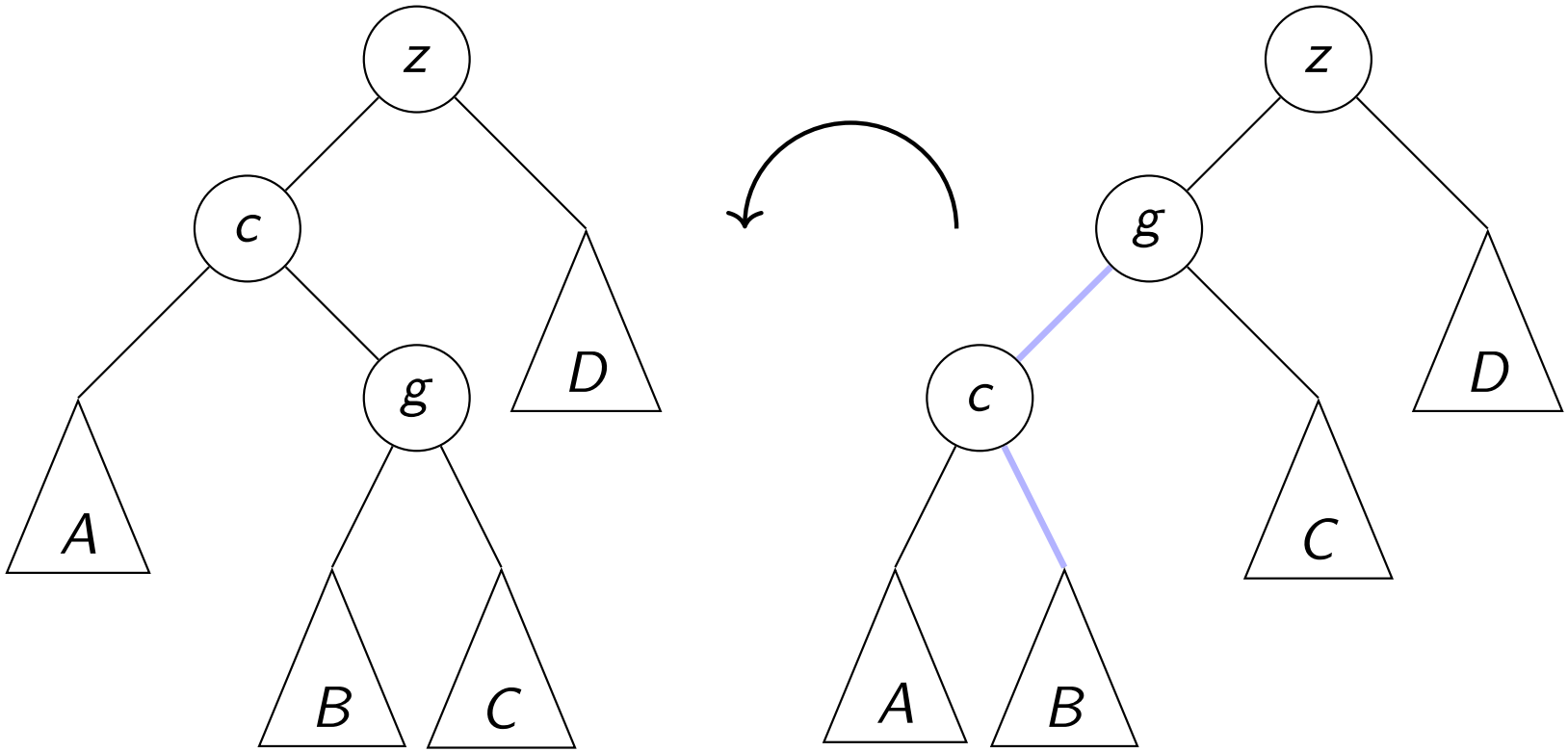
Symmetrically, this is a **left rotation** on node $z$:




Again, only $O(1)$ links need to be changed. Useful to fix right-right imbalance.

# Double Right Rotation

This is a **double right rotation** on node $z$:




First, a left rotation at $c$.

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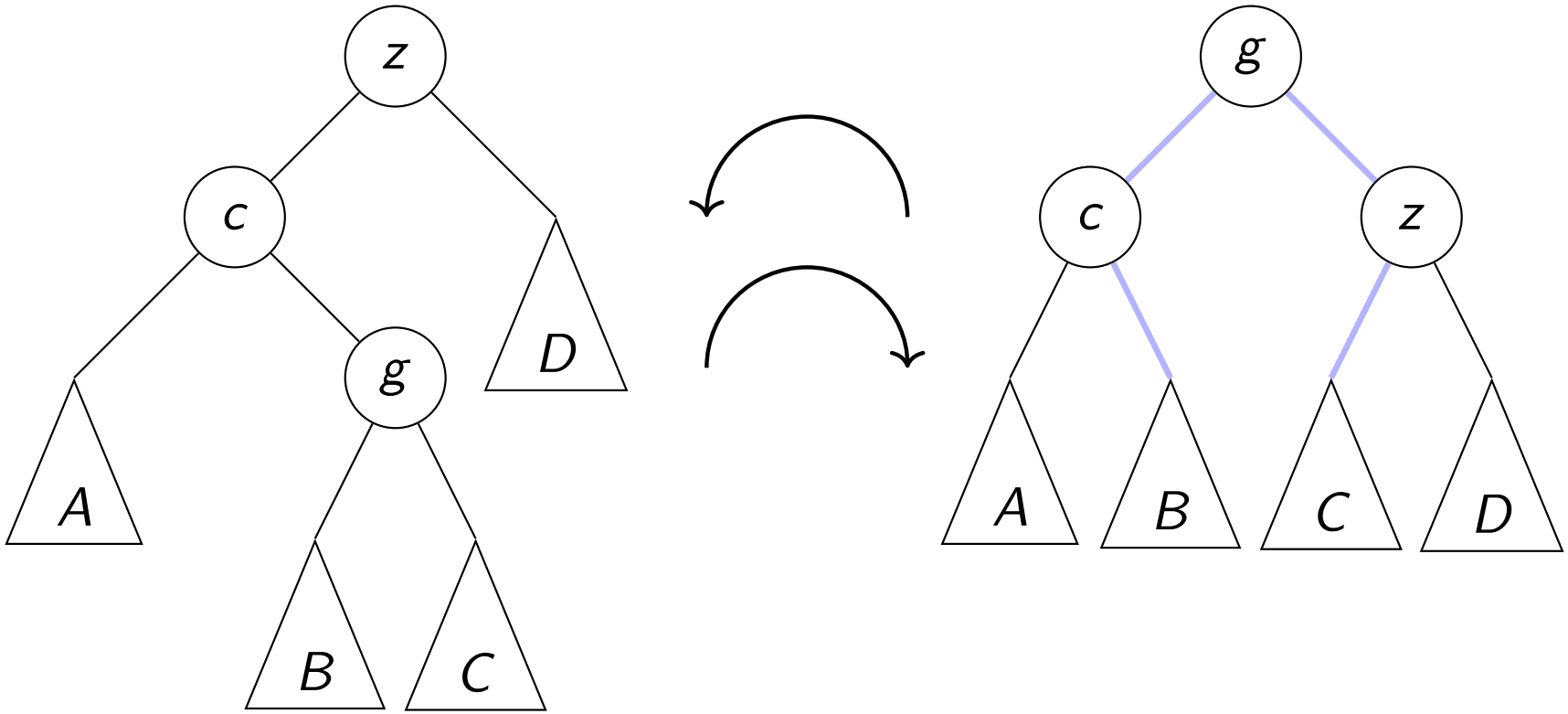


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Second, a right rotation at $z$.

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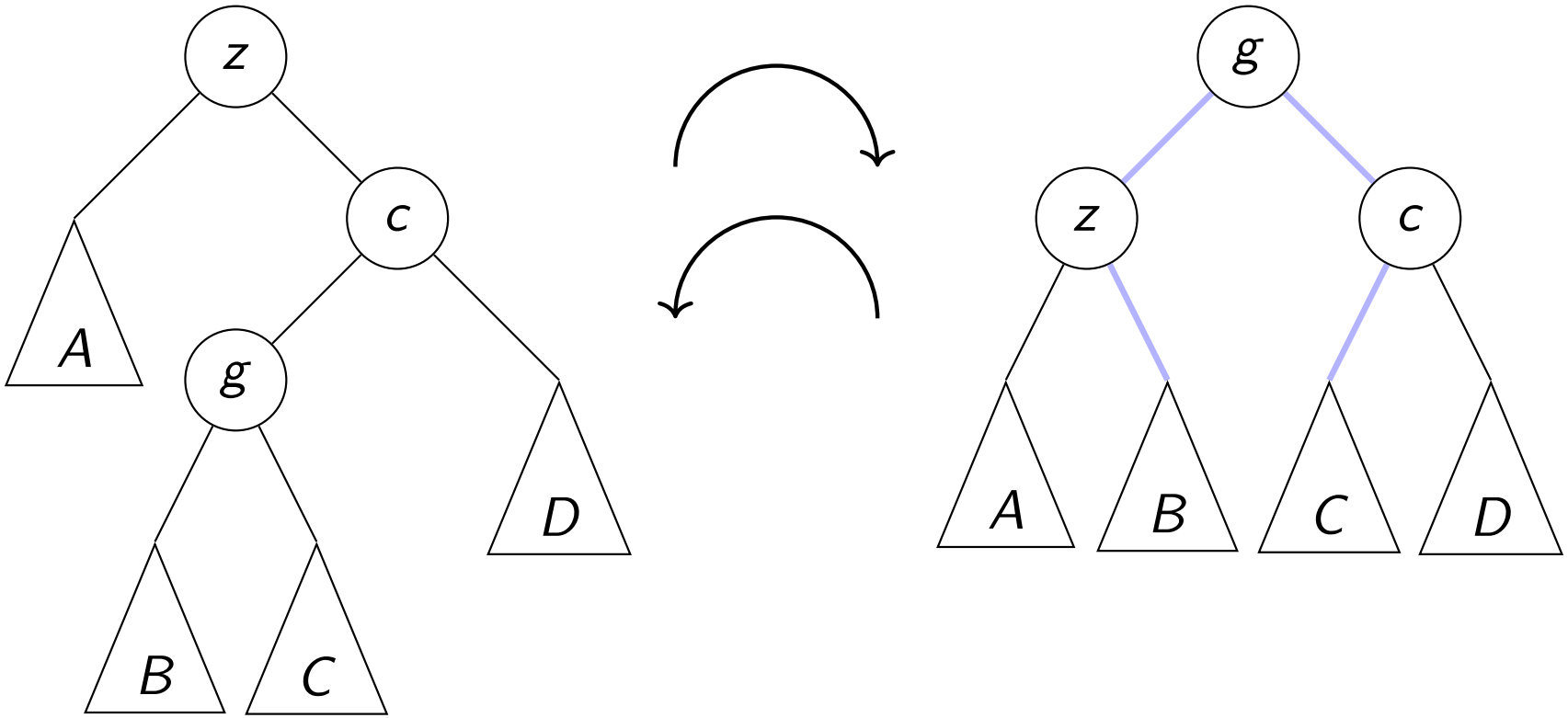
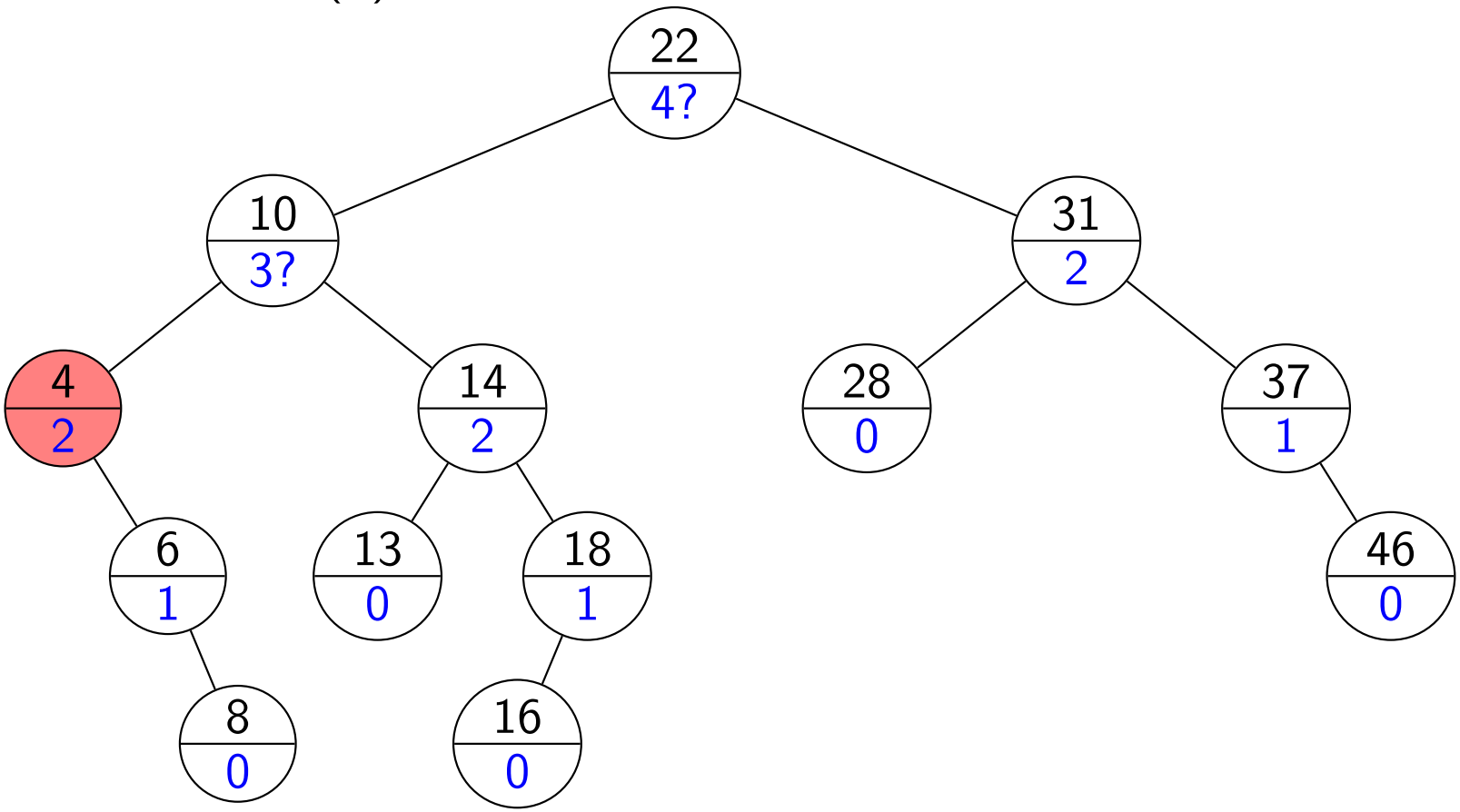



First, a right rotation at $c$.
Second, a left rotation at $z$.

# Outline

# AVL tree insertion: Example revisited

**Example**: *AVL::insert*(8)



# AVL tree insertion revisited

- Imbalance at $z$: do (single or double) rotation
- Choose $c$ as child where subtree has bigger height.

```
AVL::insert(k, v)
1.  z ← BST::insert(k, v)              // new leaf with k
2.  while (z is not NULL)
3.      if (|z.left.height − z.right.height| > 1) then
4.          Let c be taller child of z
5.          Let g be taller child of c (so grandchild of z)
6.          z ← restructure(g, c, z) // see later
7.          break          // can argue that we are done
8.      set-height-from-subtrees(z)
9.      z ← z.parent
```

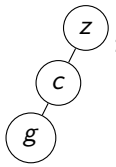
Can argue: For insertion *one* rotation restores all heights of subtrees.

$\Rightarrow$ No further imbalances, can stop checking.

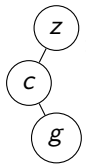
# Fixing a slightly-unbalanced AVL tree

*restructure*$(g, c, z)$
node $g$ is child of $c$ which is child of $z$

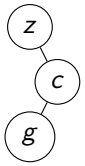
1. **case**



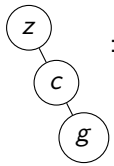

: // Right rotation
$u \leftarrow$ *rotate-right*$(z)$




: // Double-right rotation
*rotate-left*$(c)$
$u \leftarrow$ *rotate-right*$(z)$




: // Double-left rotation
*rotate-right*$(c)$
$u \leftarrow$ *rotate-left*$(z)$



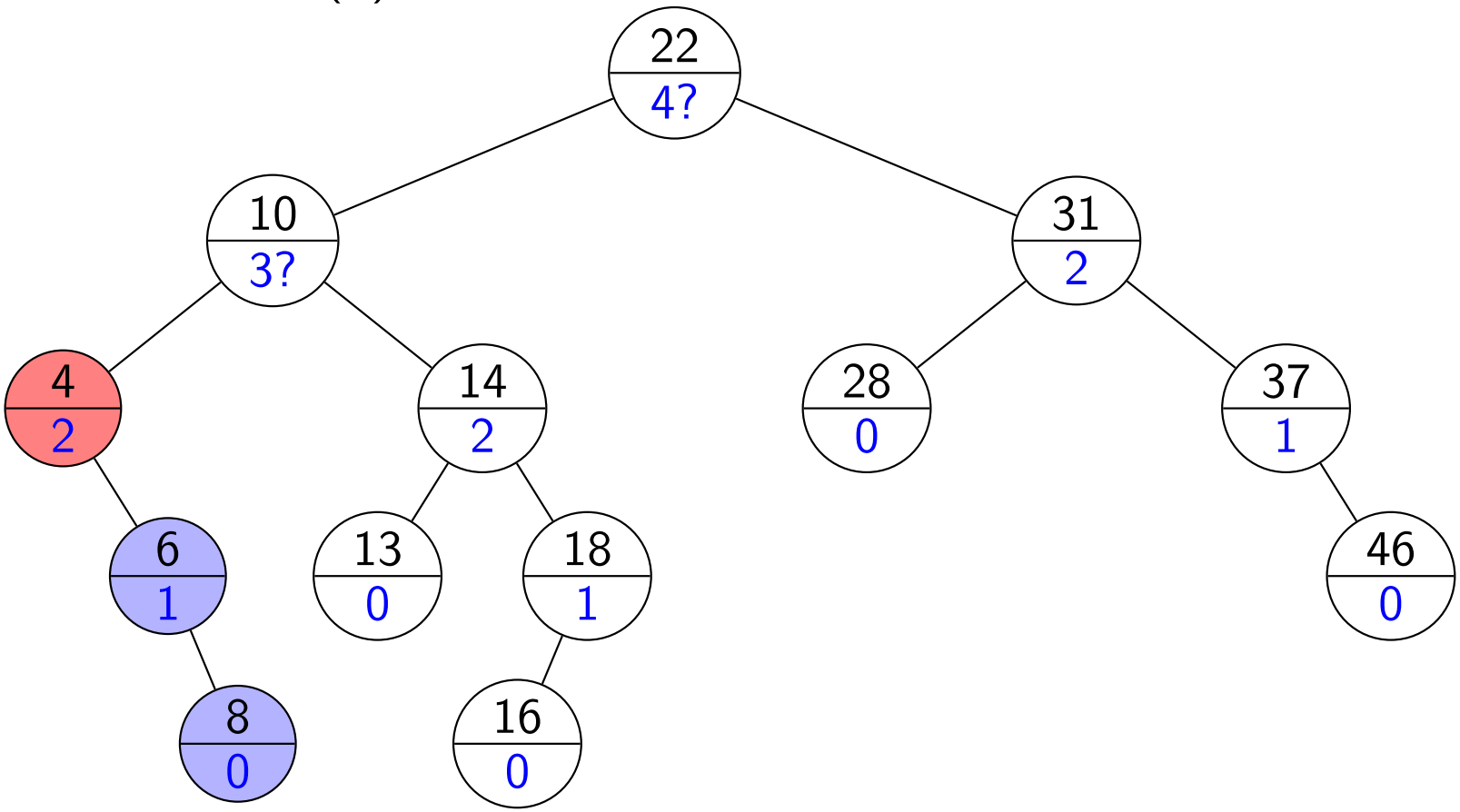

: // Left rotation
$u \leftarrow$ *rotate-left*$(z)$

2. **return** $u$

**Rule**: The middle key of $g, c, z$ becomes the new root.

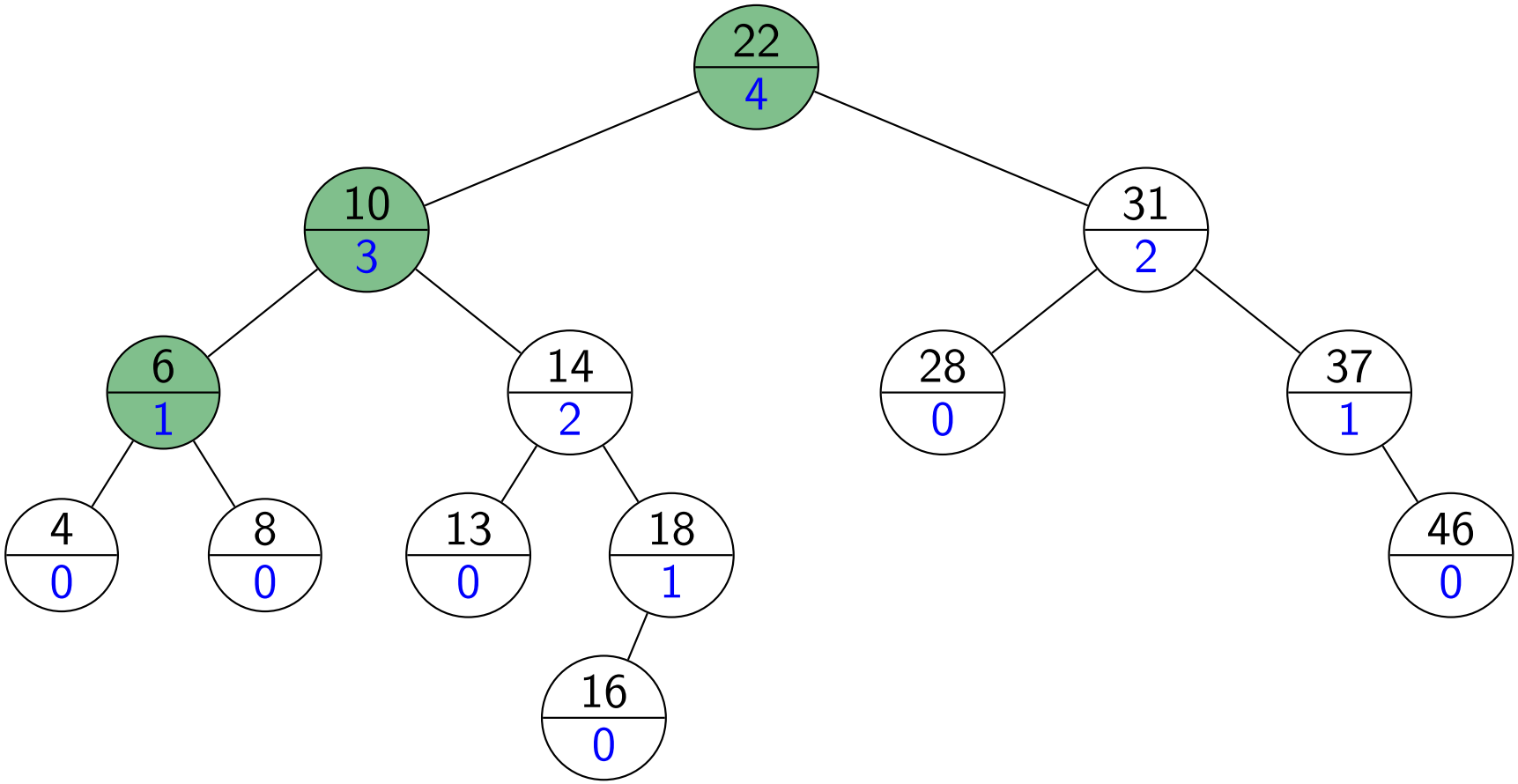
# AVL tree insertion: Example revisited

**Example**: *AVL::insert*(8)



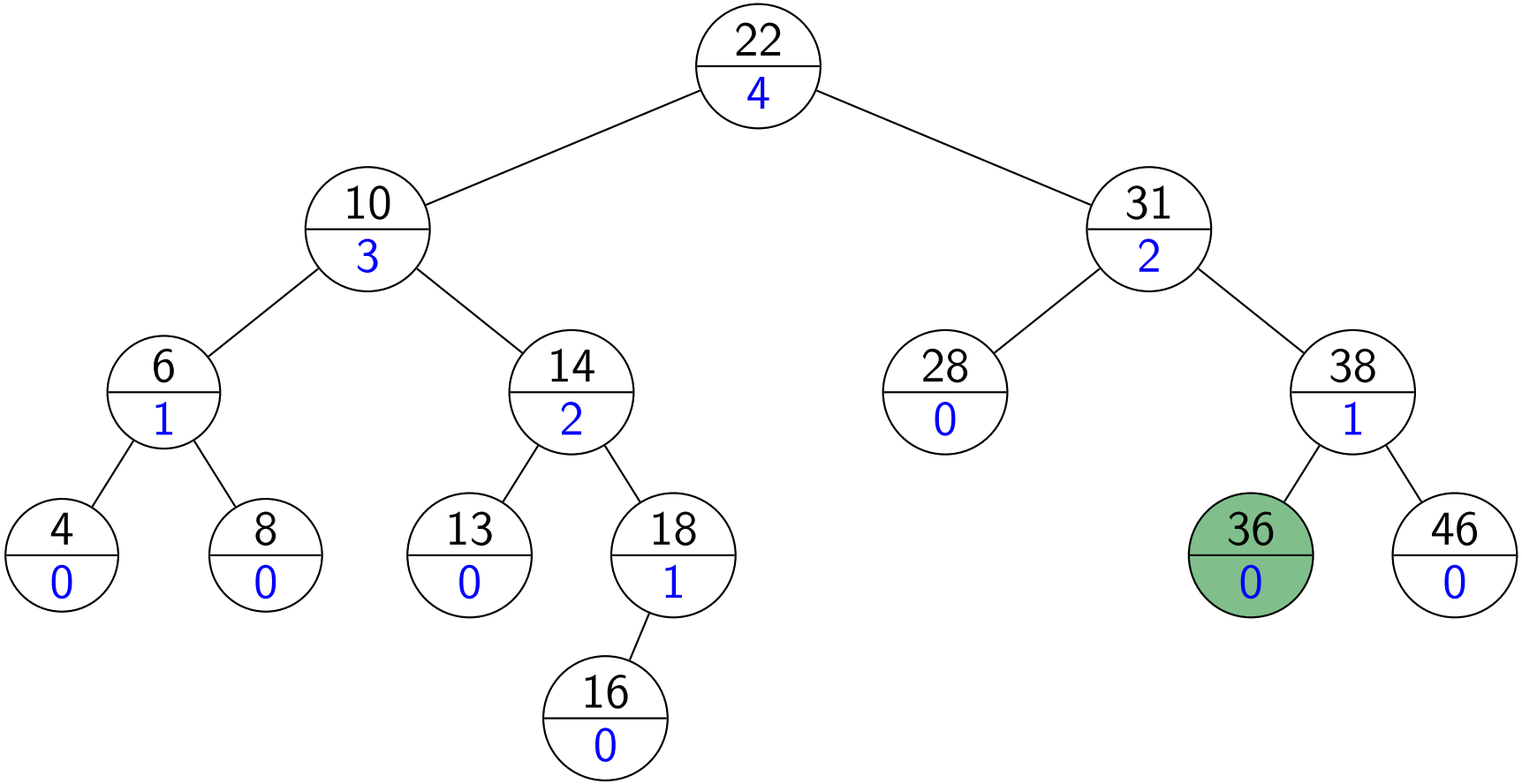
# AVL tree insertion: Example revisited

**Example**: *AVL::insert*(8)



# AVL tree insertion: Second example

**Example**: *AVL::insert*(36), *AVL::insert*(35)



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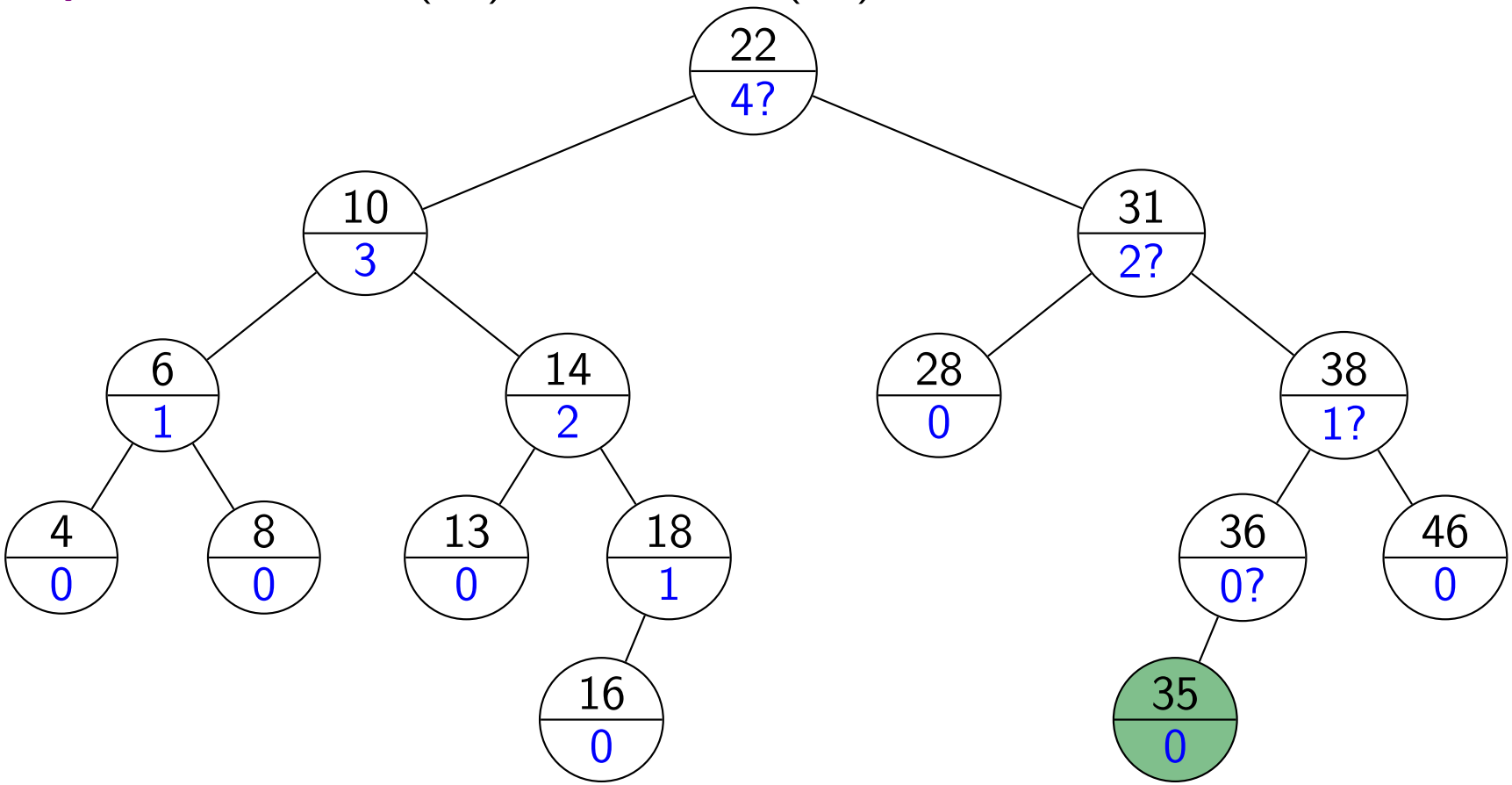

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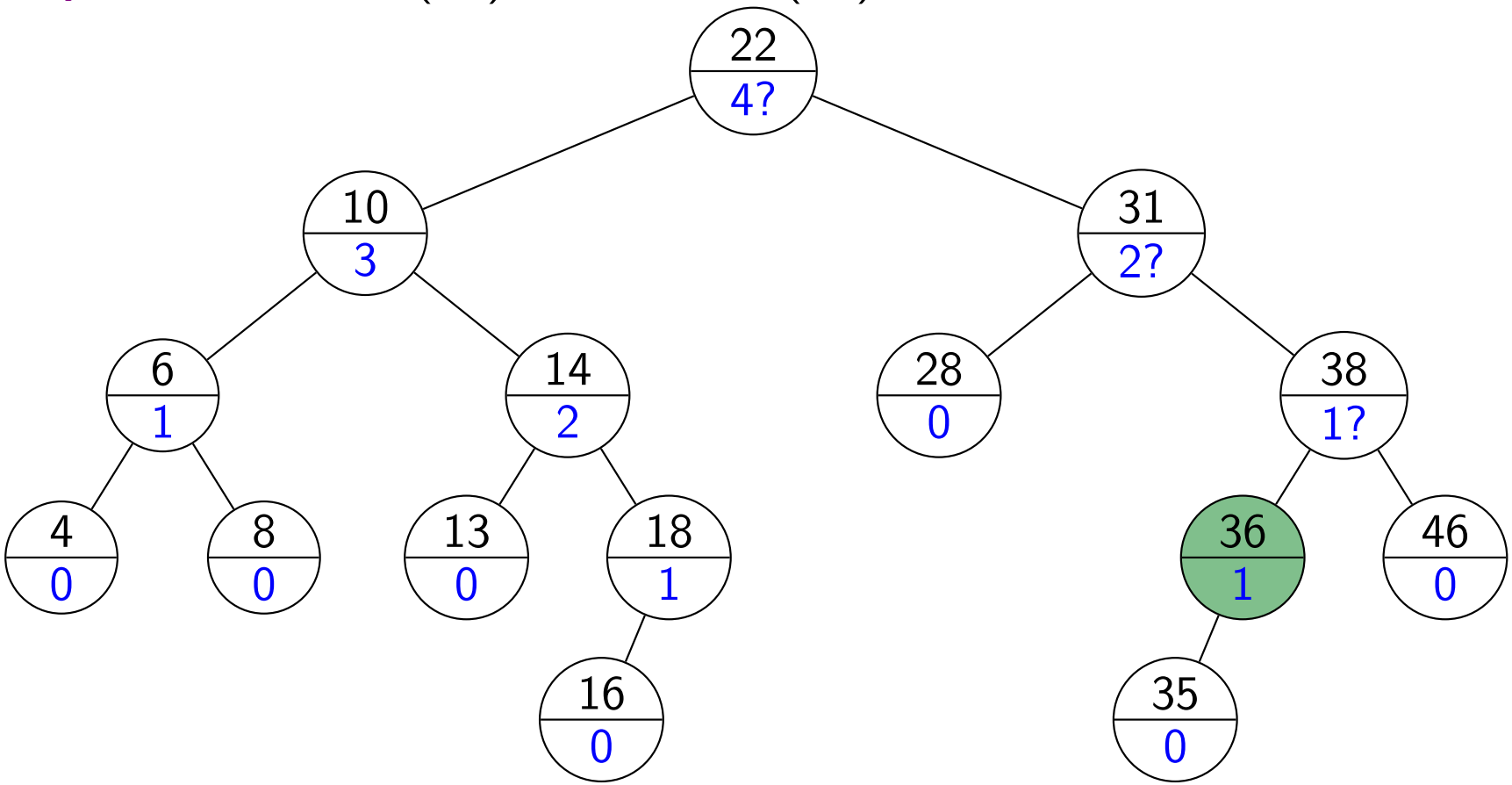

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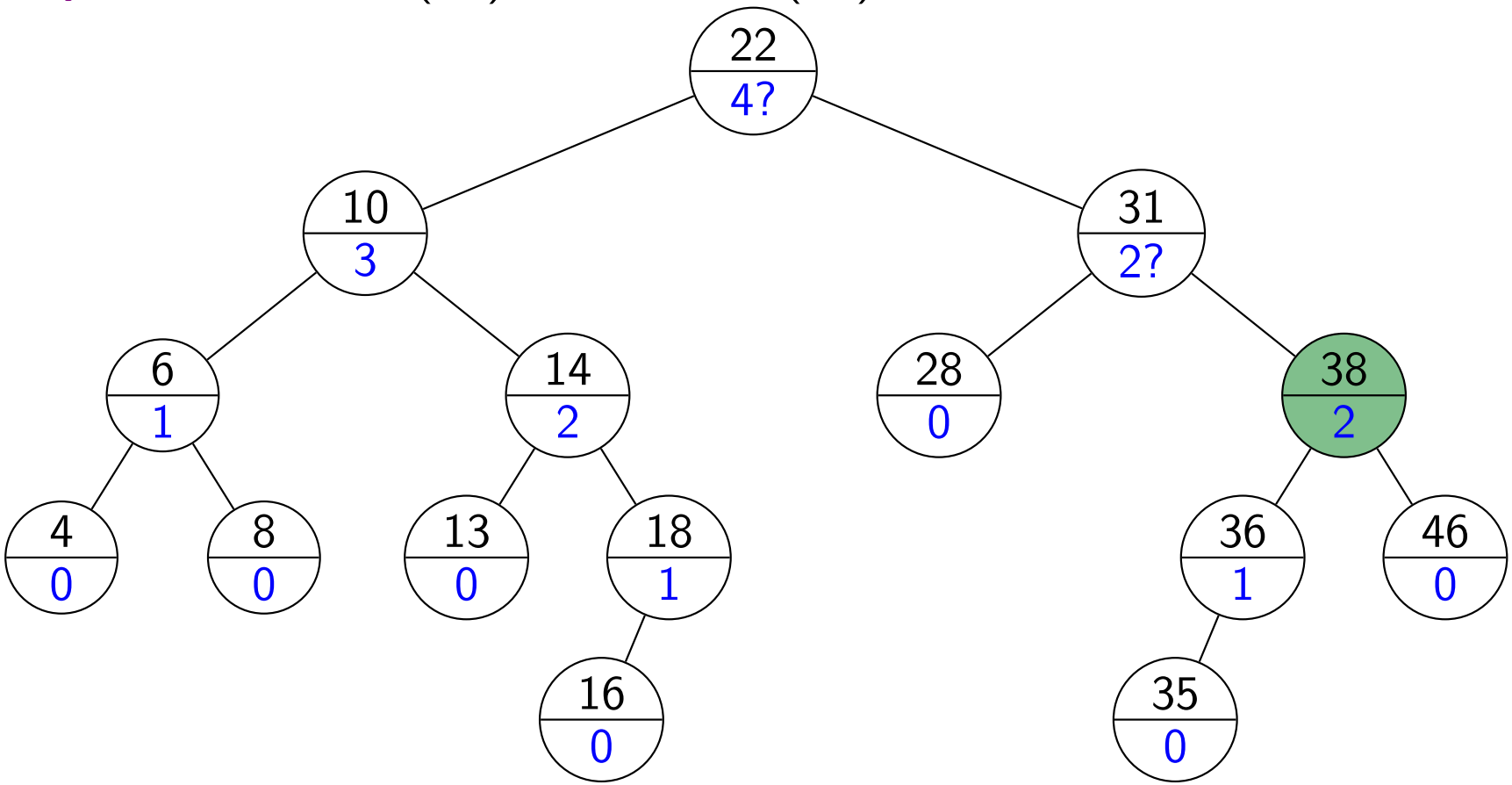

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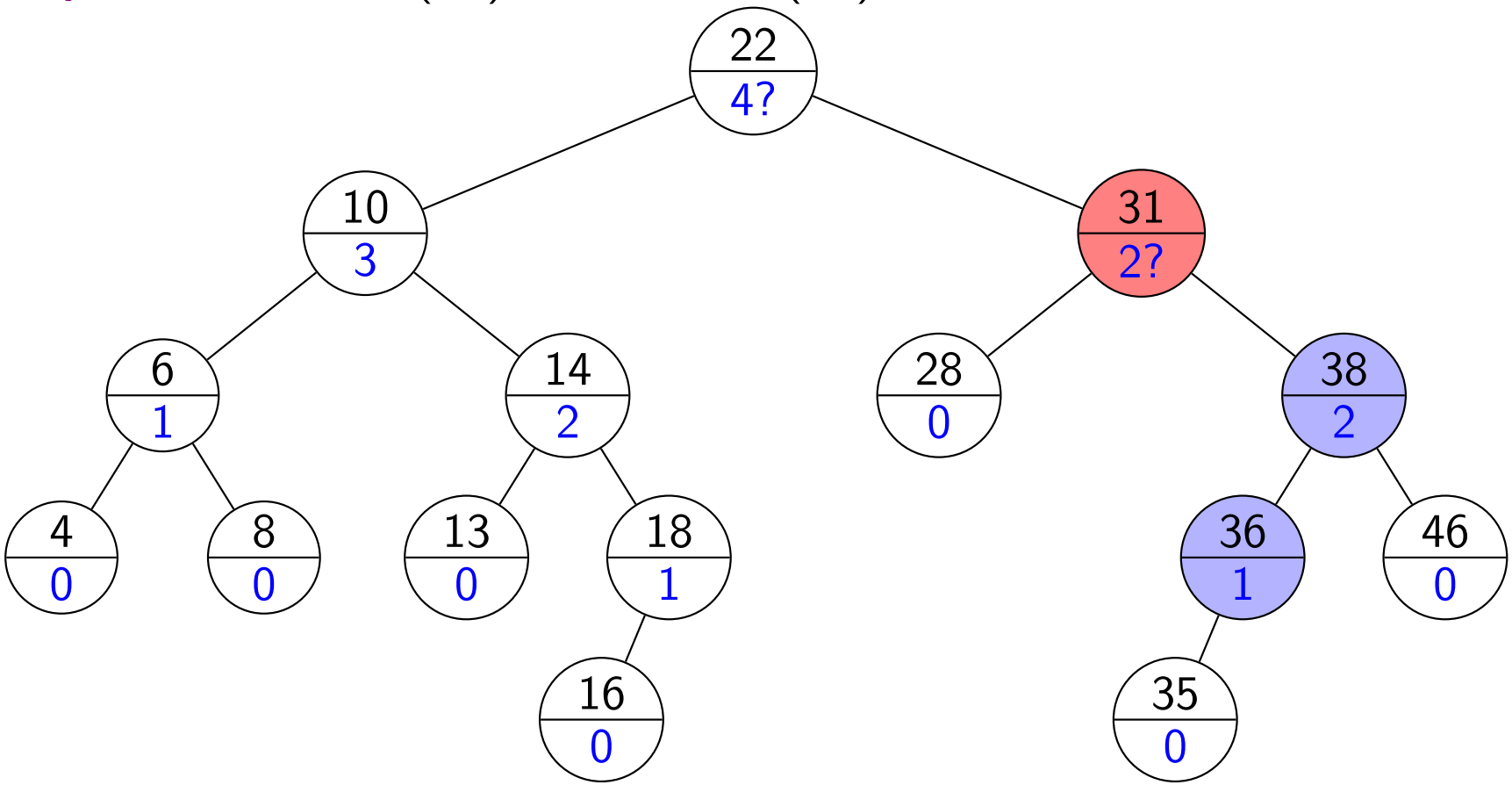

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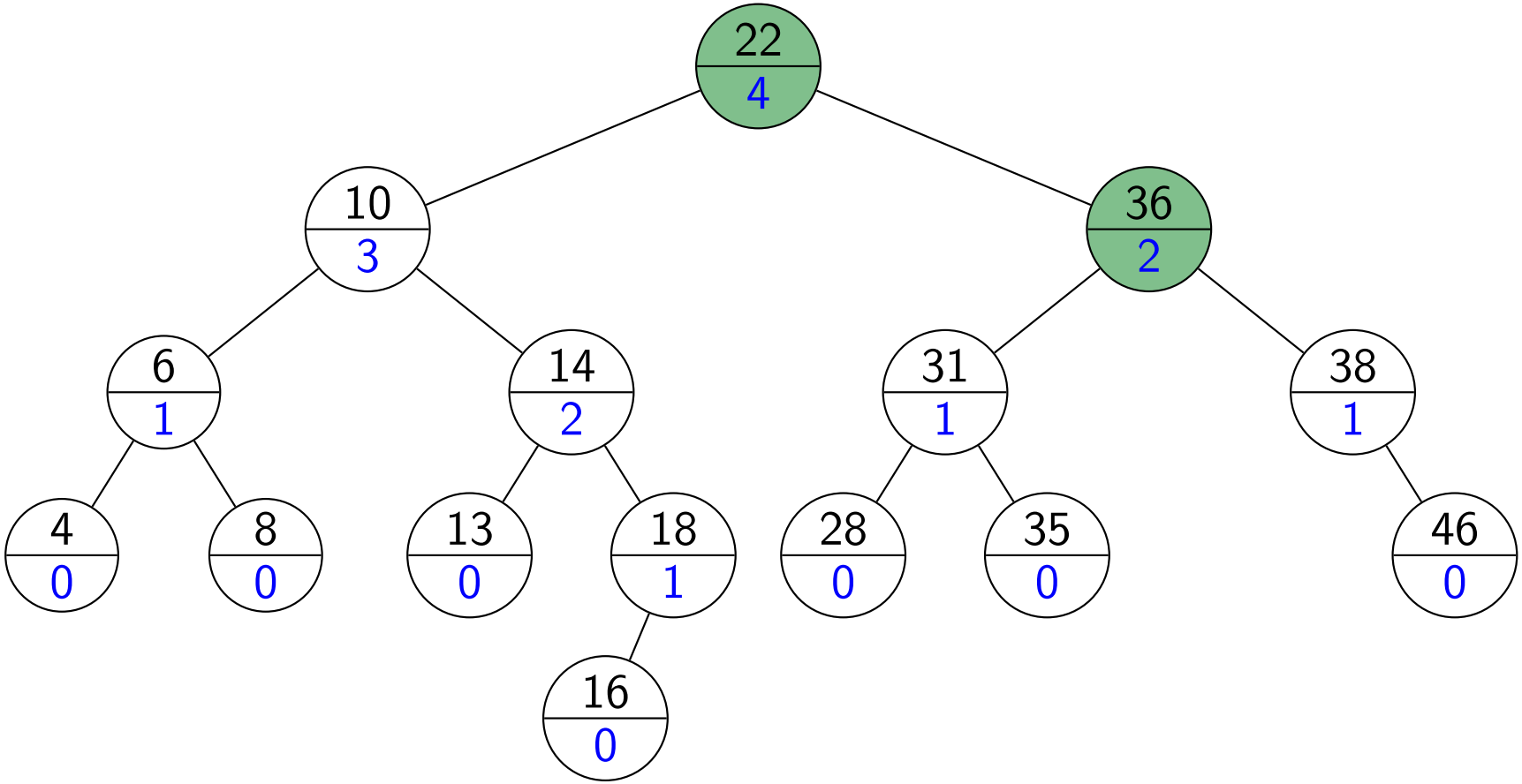


# Outline

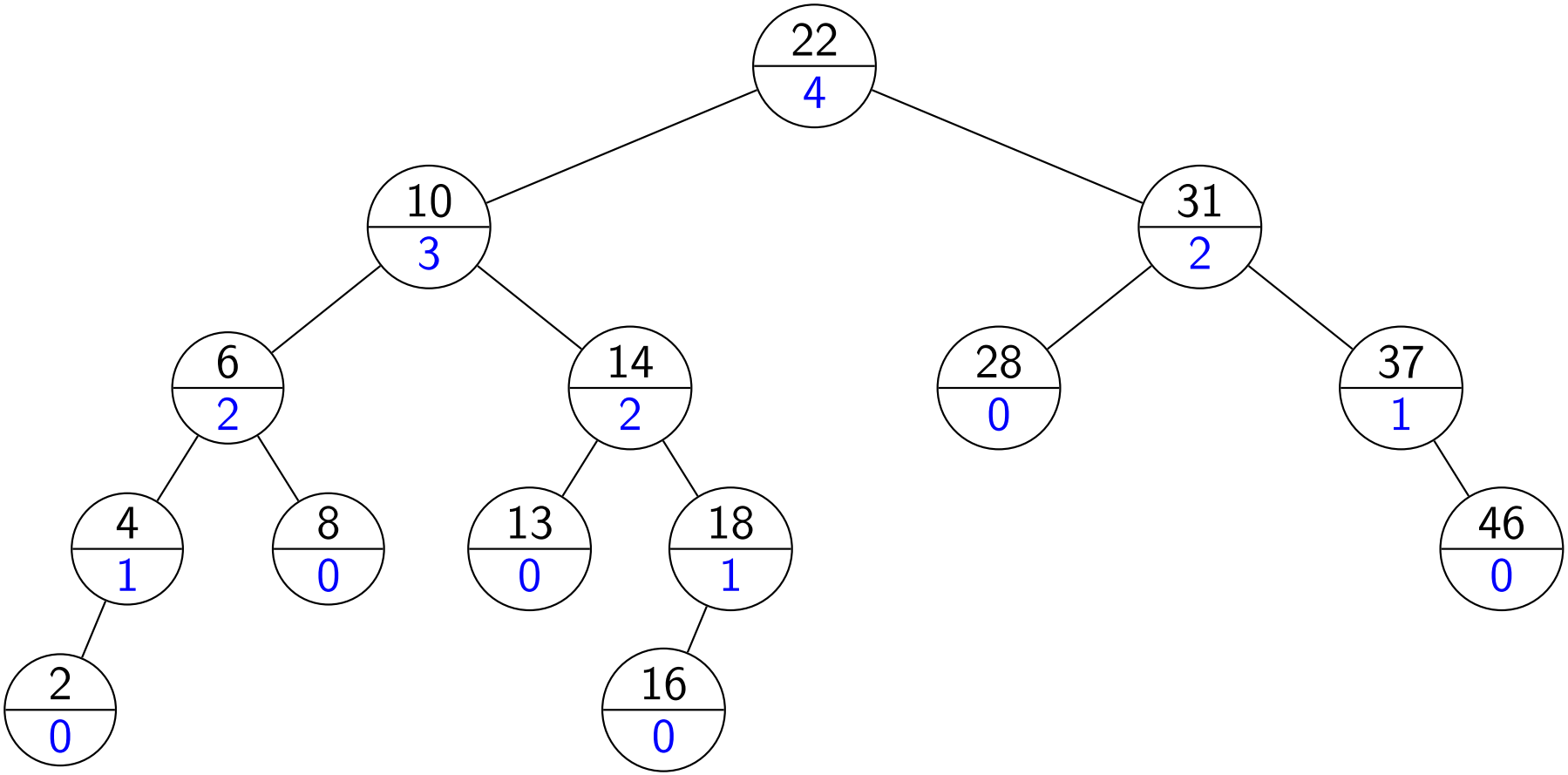
# AVL trees: Deletion

Remove the key $k$ with *BST::delete*.
Find node where *structural* change happened.
(This is not necessarily near the node that had $k$.)
Go back up to root, update heights, and rotate if needed.

```
AVL::delete(k)
1.  z ← BST::delete(k)
2.  // Assume z is the parent of the BST node that was removed
3.  while (z is not NULL)
4.      if (|z.left.height − z.right.height| > 1) then
5.          Let c be taller child of z
6.          Let g be taller child of c (break ties to avoid double rotation)
7.          z ← restructure(g, c, z)
8.      // Always continue up the path
9.      set-height-from-subtrees(z)
10.     z ← z.parent
```

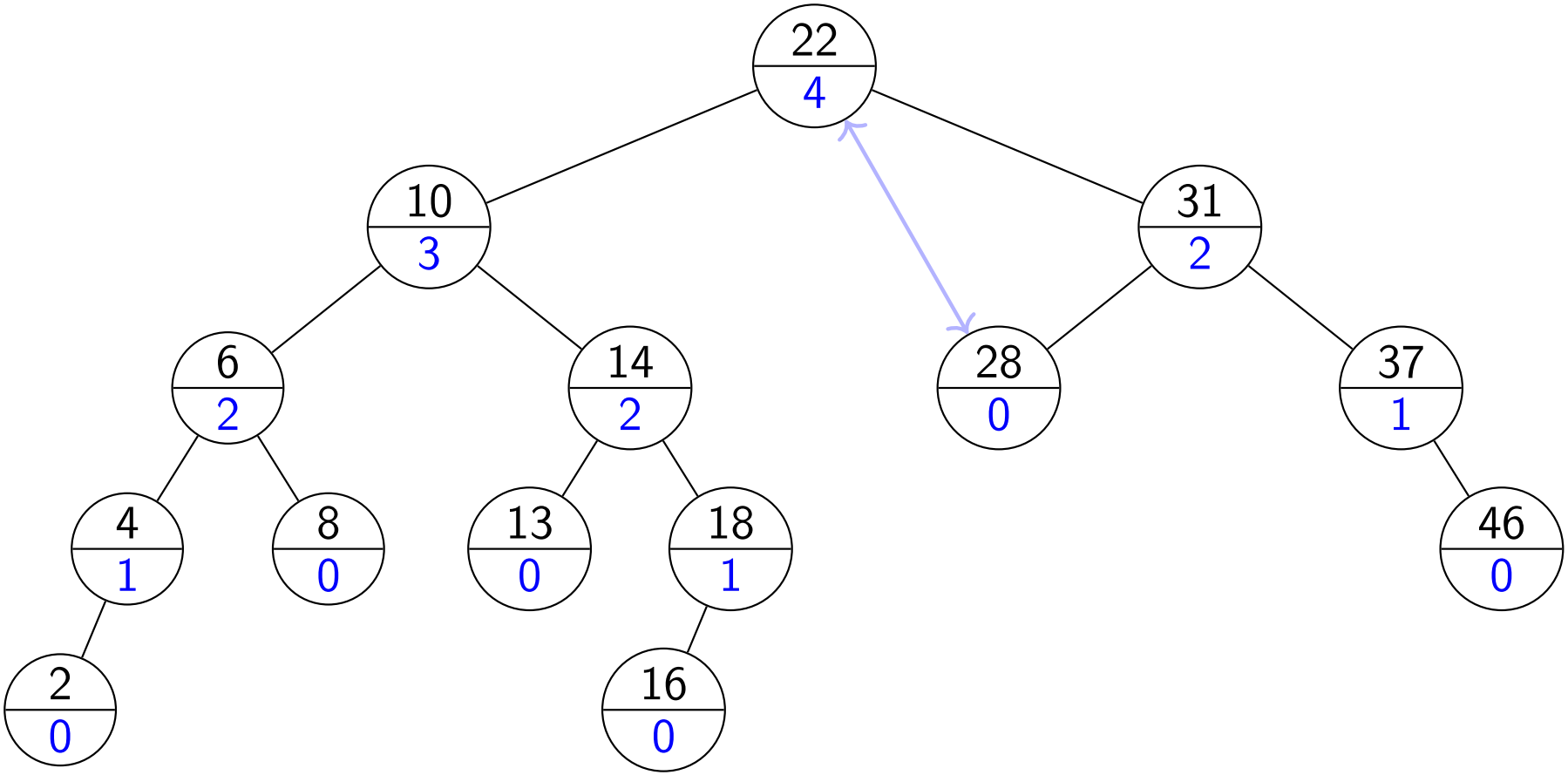
# AVL trees deletion: Example

**Example**: *AVL::delete*(22)



# AVL trees deletion: Example

**Example**: *AVL::delete*(22)



# AVL trees deletion: Example

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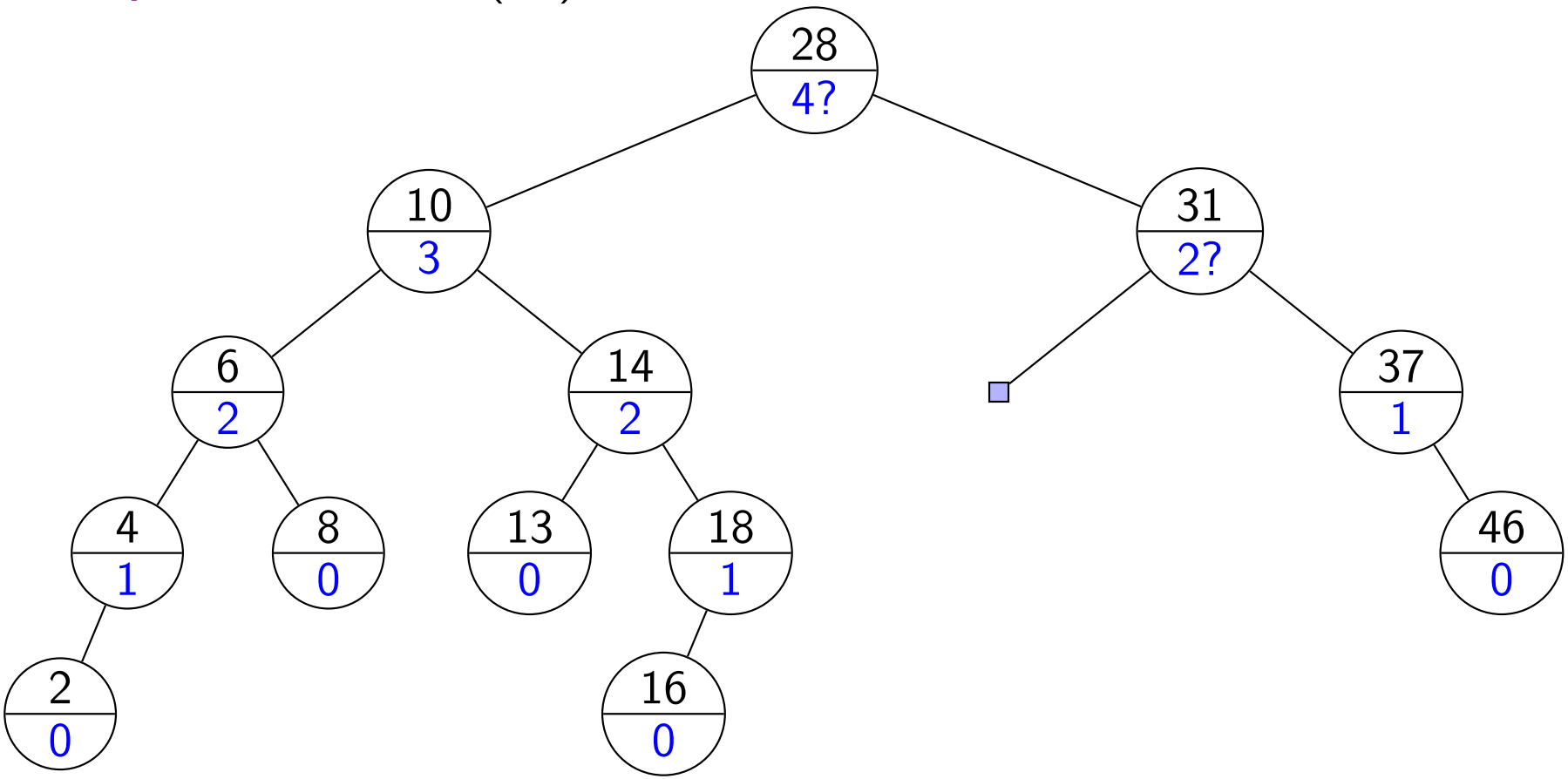

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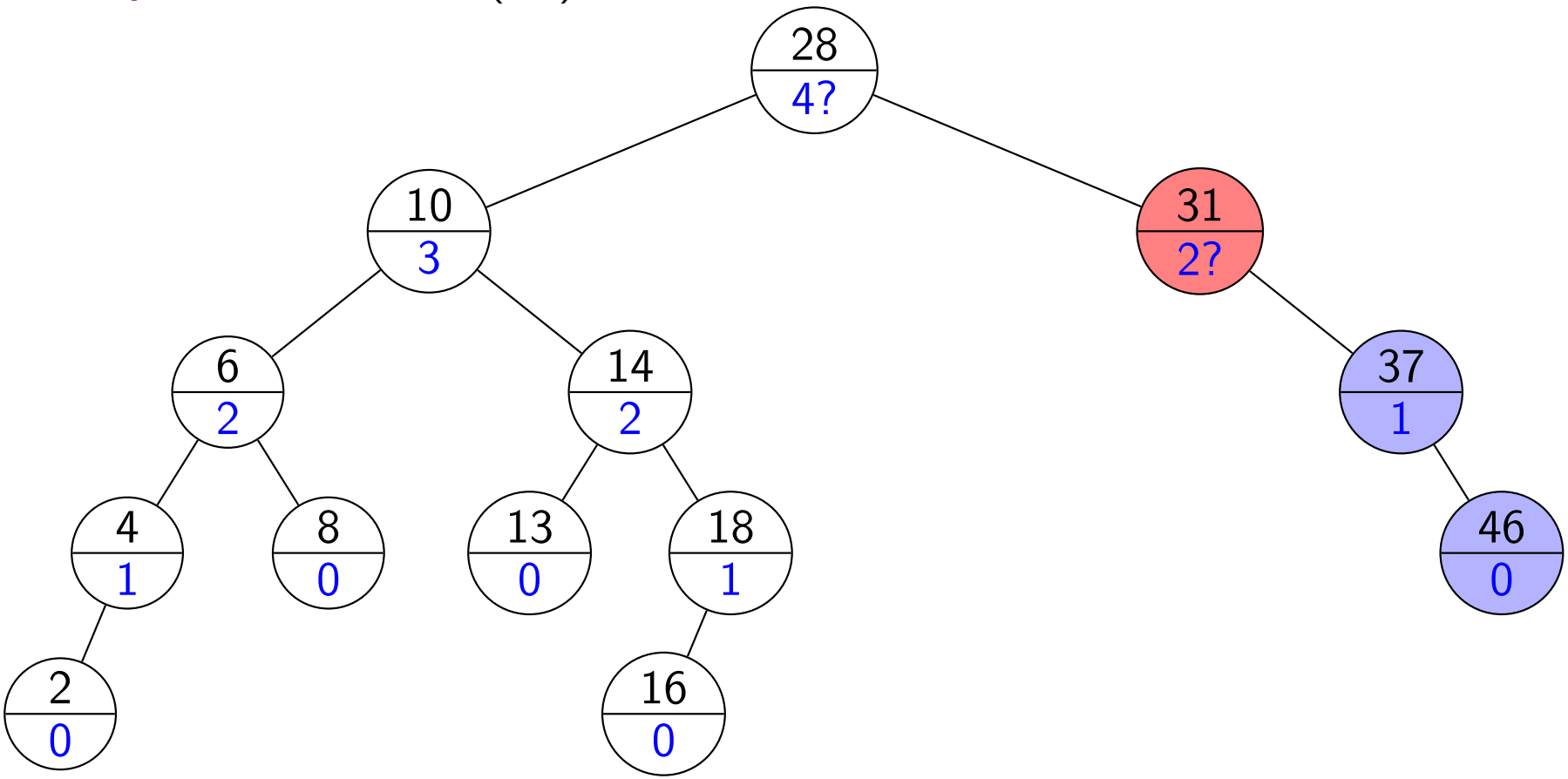
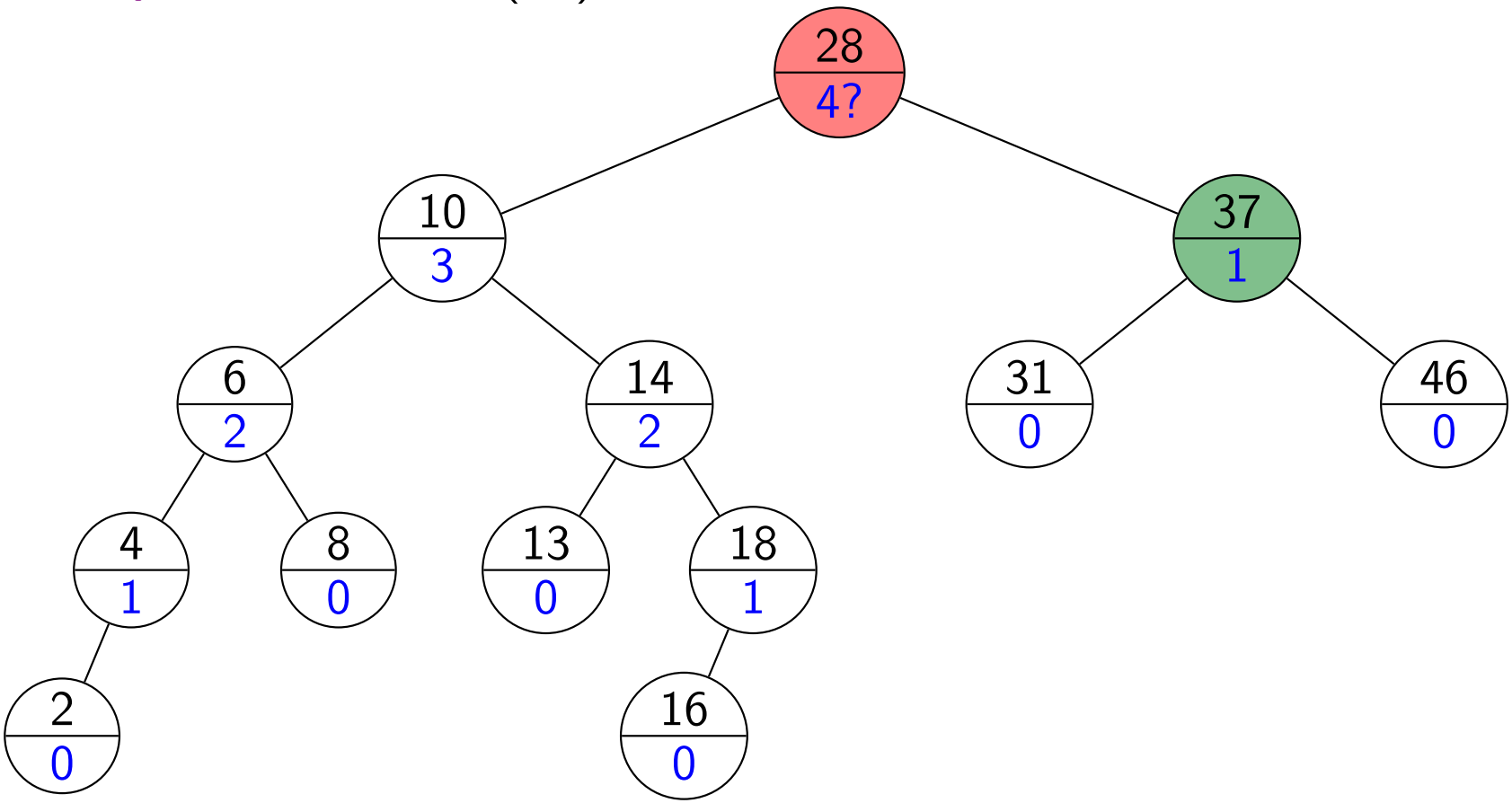


# AVL trees deletion: Example

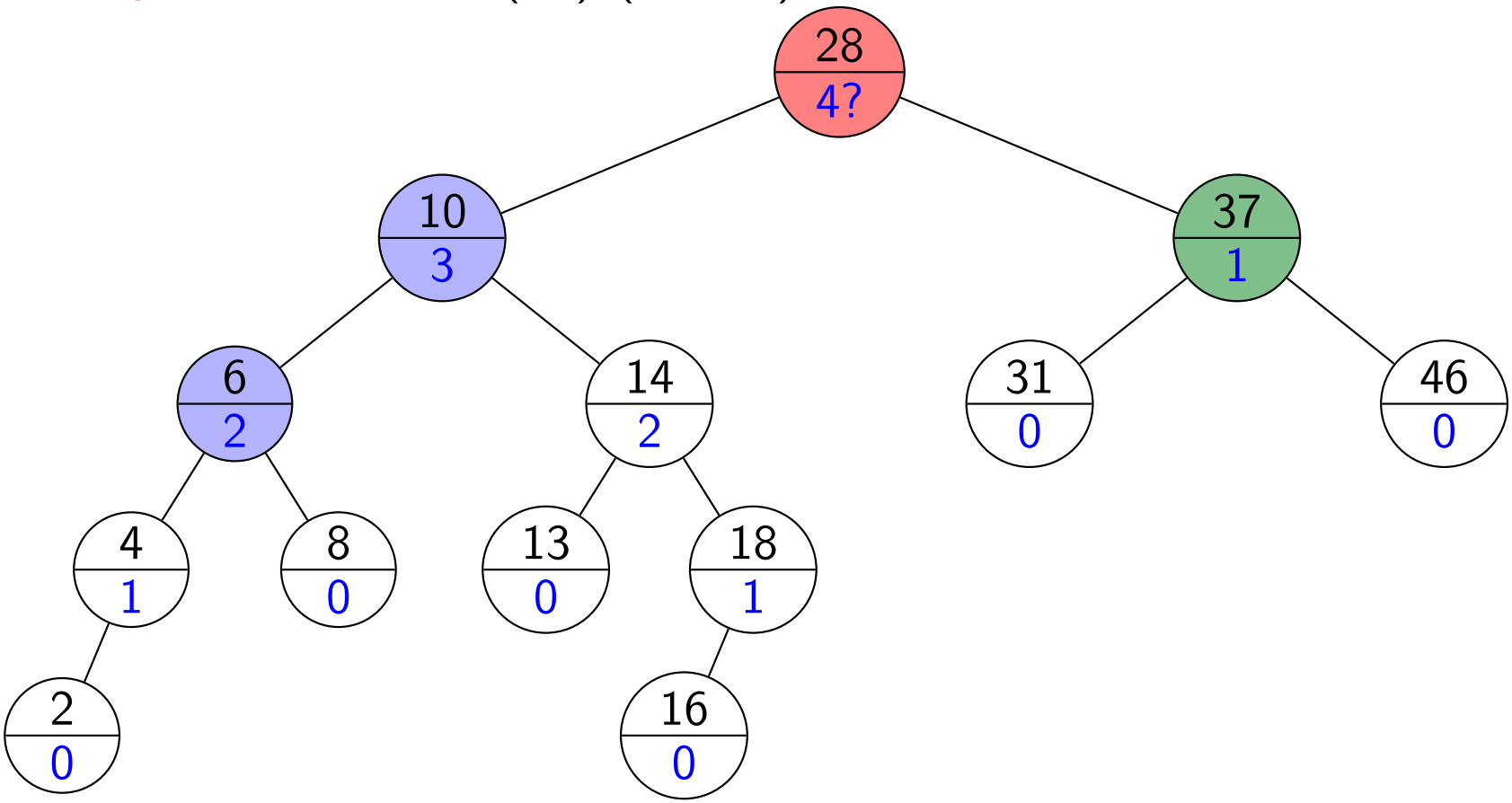
**Example**: *AVL::delete*(22)




A single *restructure* is not enough to restore all balances.

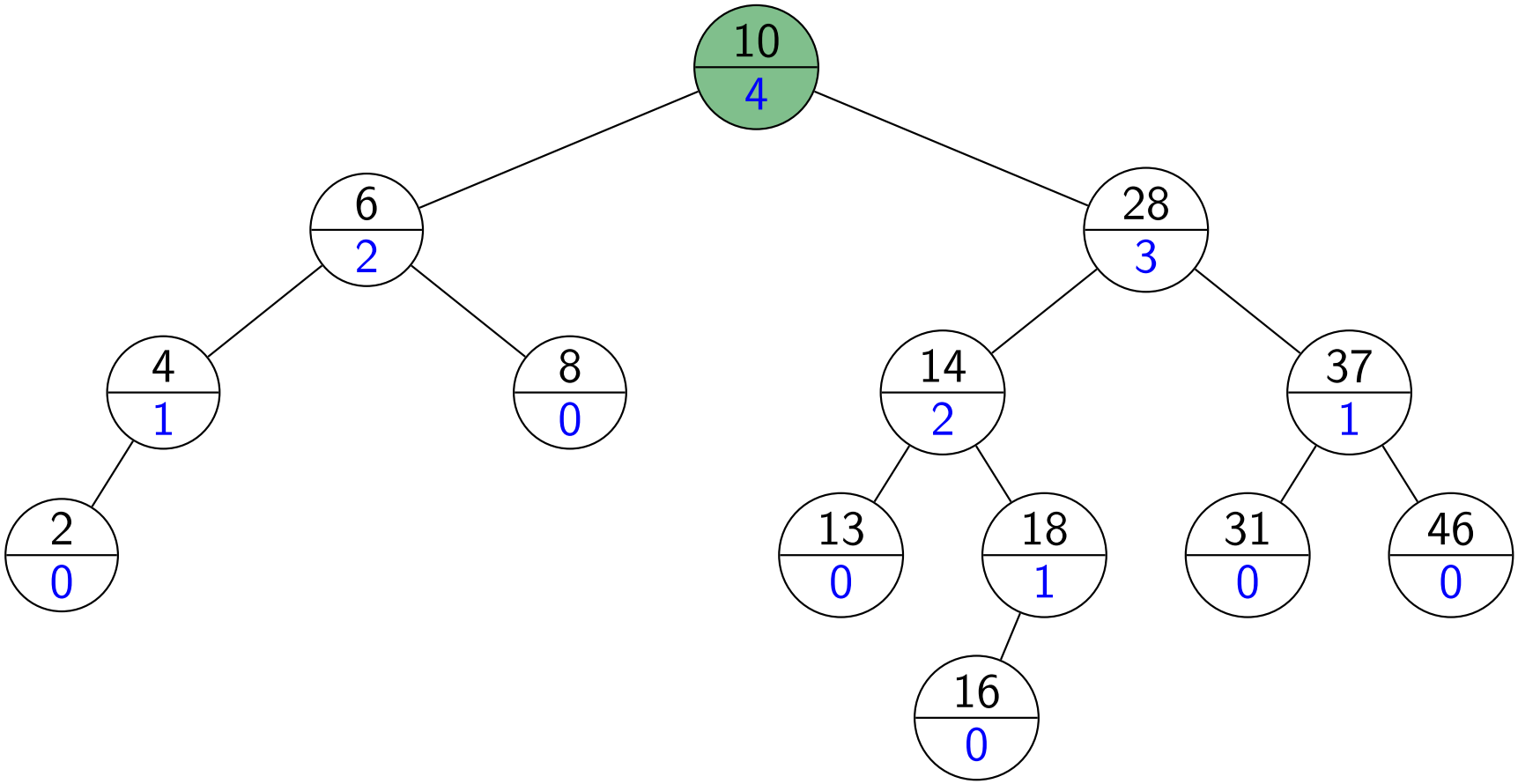
# AVL trees deletion: Example

**Example**: *AVL::delete*(22) (cont'd)



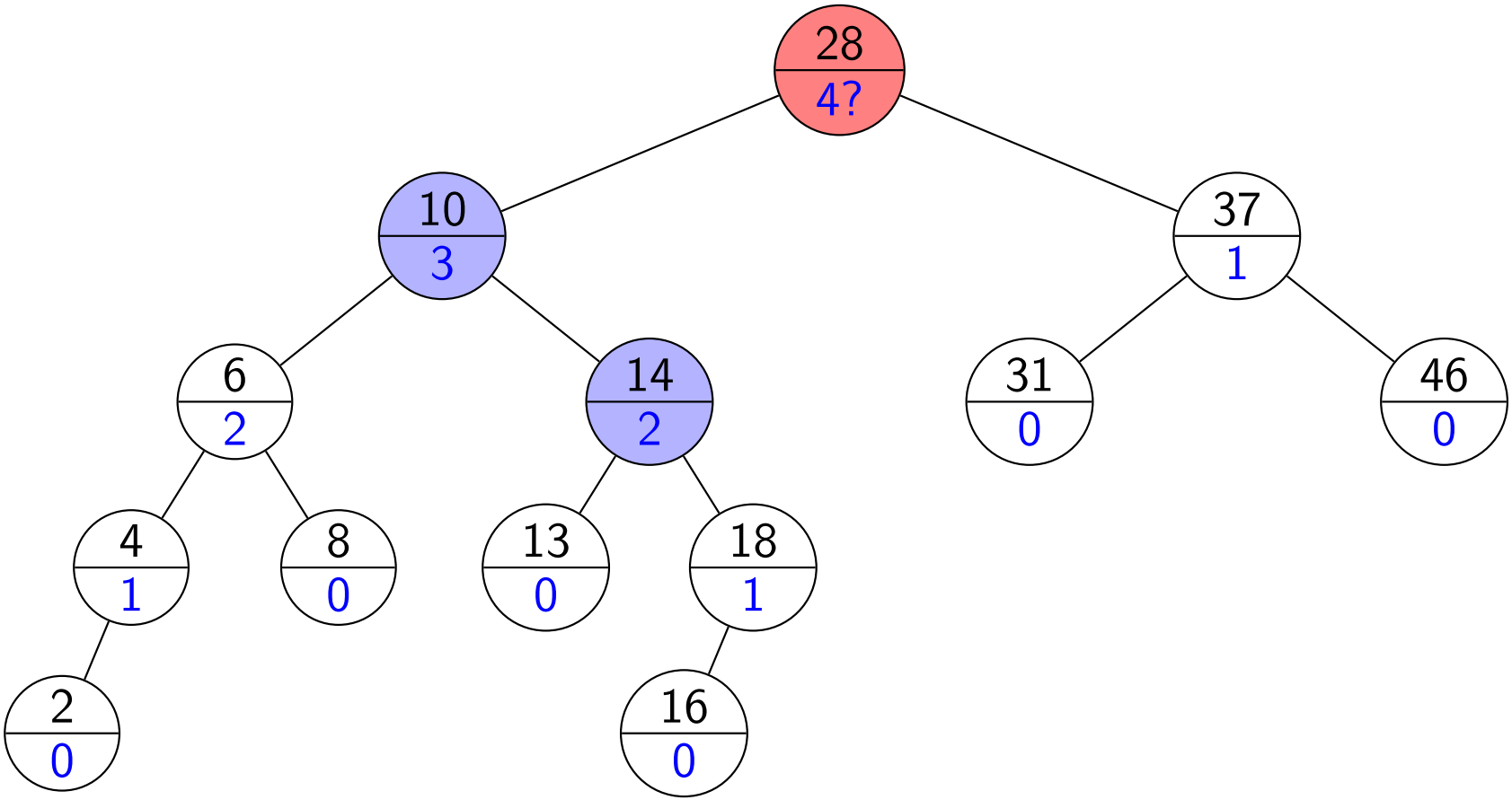
# AVL trees deletion: Example

**Example**: *AVL::delete*(22) (cont'd)



# AVL trees deletion: Example

**Important**: Ties *must* be broken to avoid double rotation.
Consider again the above example. If we applied double-rotation:



# AVL trees deletion: Example

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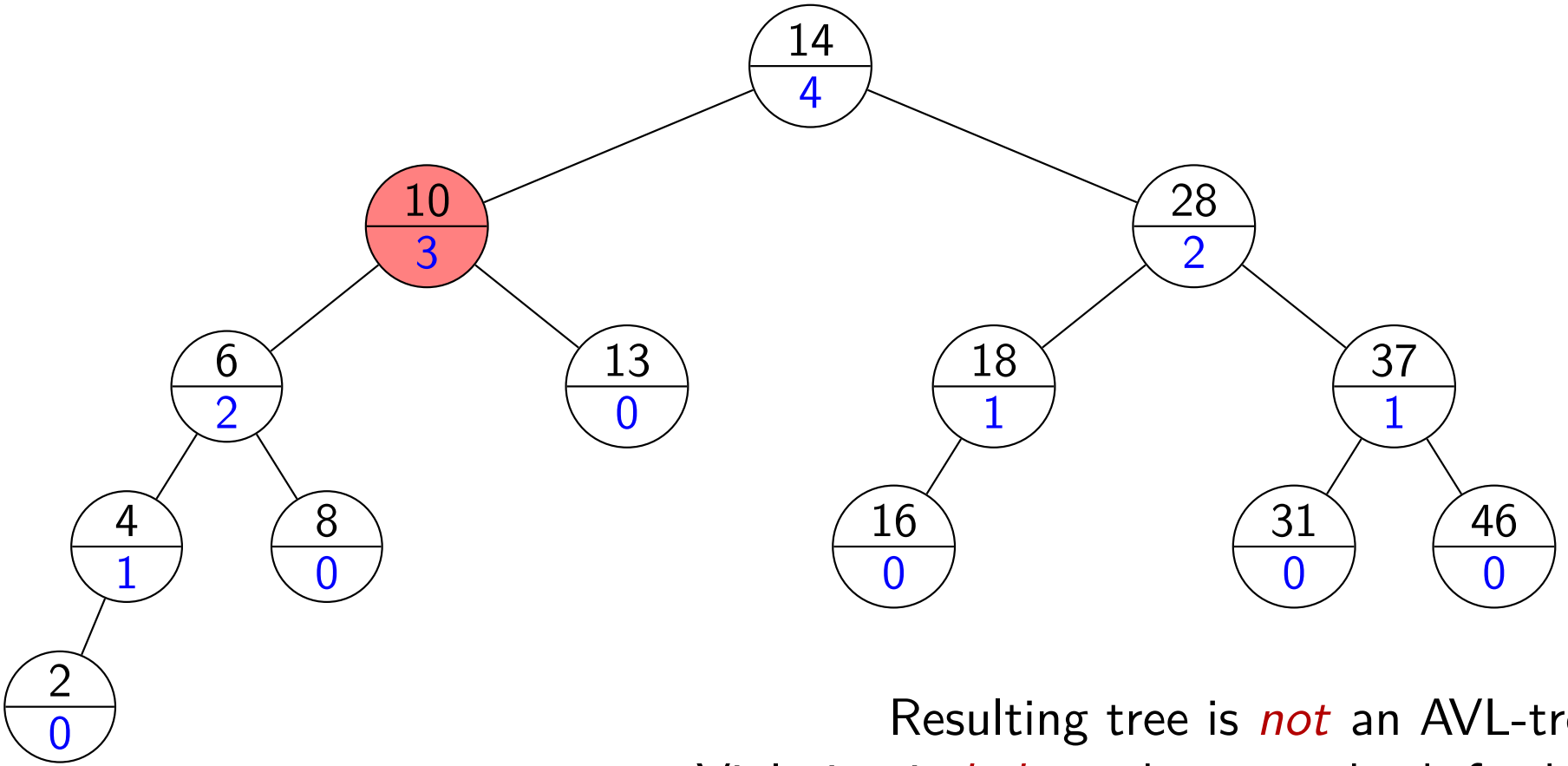


Resulting tree is *not* an AVL-tree.
Violation is *below* where we check further.

# AVL trees: Summary

**search**: Just like in BSTs, costs $\Theta(\mathit{height})$

**insert**: *BST::insert*, then check & update along path to new leaf

- total cost $\Theta(\mathit{height})$
- *restructure* will be called *at most once*.

**delete**: *BST::delete*, then check & update along path to deleted node

- total cost $\Theta(\mathit{height})$
- *restructure* may be called $\Theta(\mathit{height})$ times.

*Worst-case* cost for all operations is $\Theta(\mathit{height}) = \Theta(\log n)$.

- In practice, the constant is quite large.
- Other realizations of ADT Dictionary are better in practice ($\rightarrow$ later)