CS 240 – Data Structures and Data Management

Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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Spring 2025

version 2025-06-16 17:40

Outline

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Dictionary ADT: Implementations thus far

Realizations we have seen so far:

- **Balanced Binary Search trees** (AVL trees):
  $\Theta(\log n)$ search, insert, and delete (worst-case)
- **Skip lists**:
  $\Theta(\log n)$ search, insert, and delete (expected)
- Various other realizations sometimes faster on insert,
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  but *search* always takes $\Omega(\log n)$ time.

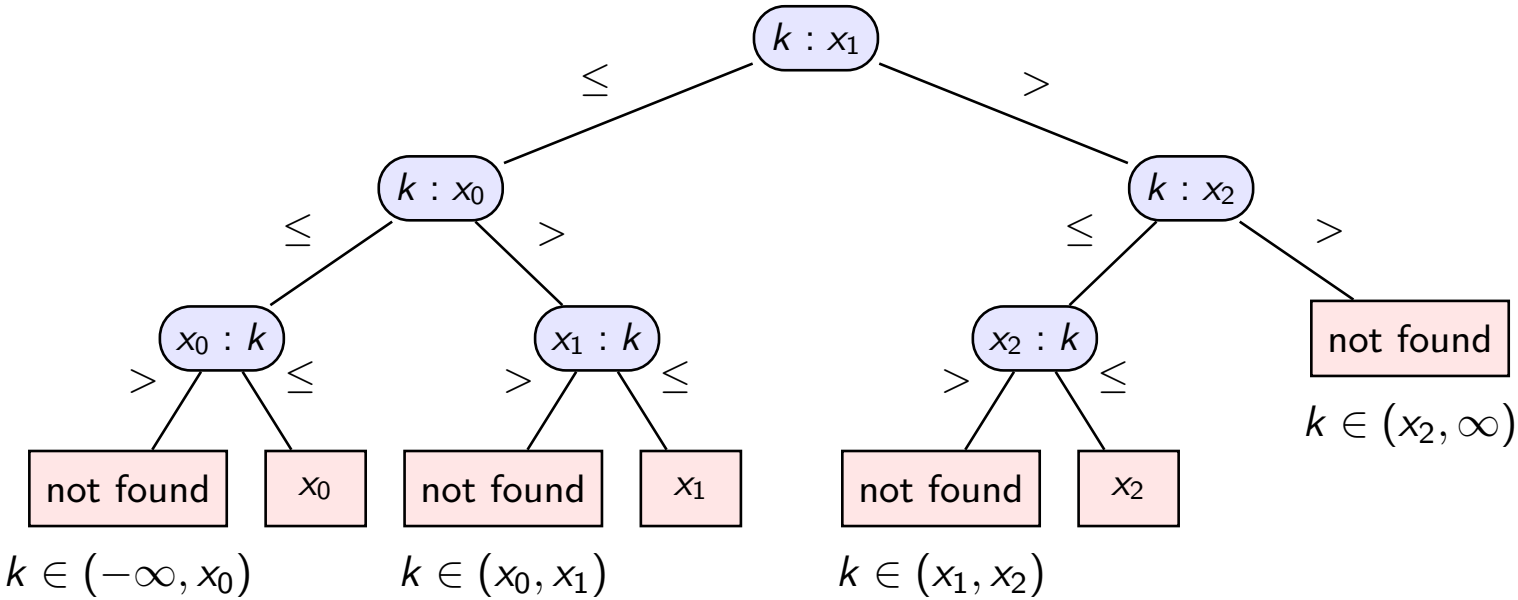
Question: Can one do better than $\Theta(\log n)$ time for *search*?
Answer: Yes and no! *It depends on what we allow*.

- No: Comparison-based searching lower bound is $\Omega(\log n)$.
- Yes: Non-comparison-based searching can achieve $o(\log n)$ (under restrictions!).

Lower bound for search

Theorem: Any *comparison-based* algorithm requires in the worst case $\Omega(\log n)$ comparisons to search among n distinct items.

Proof: Via decision tree for items $x_0, \ldots, x_{n-1}$ and search for k



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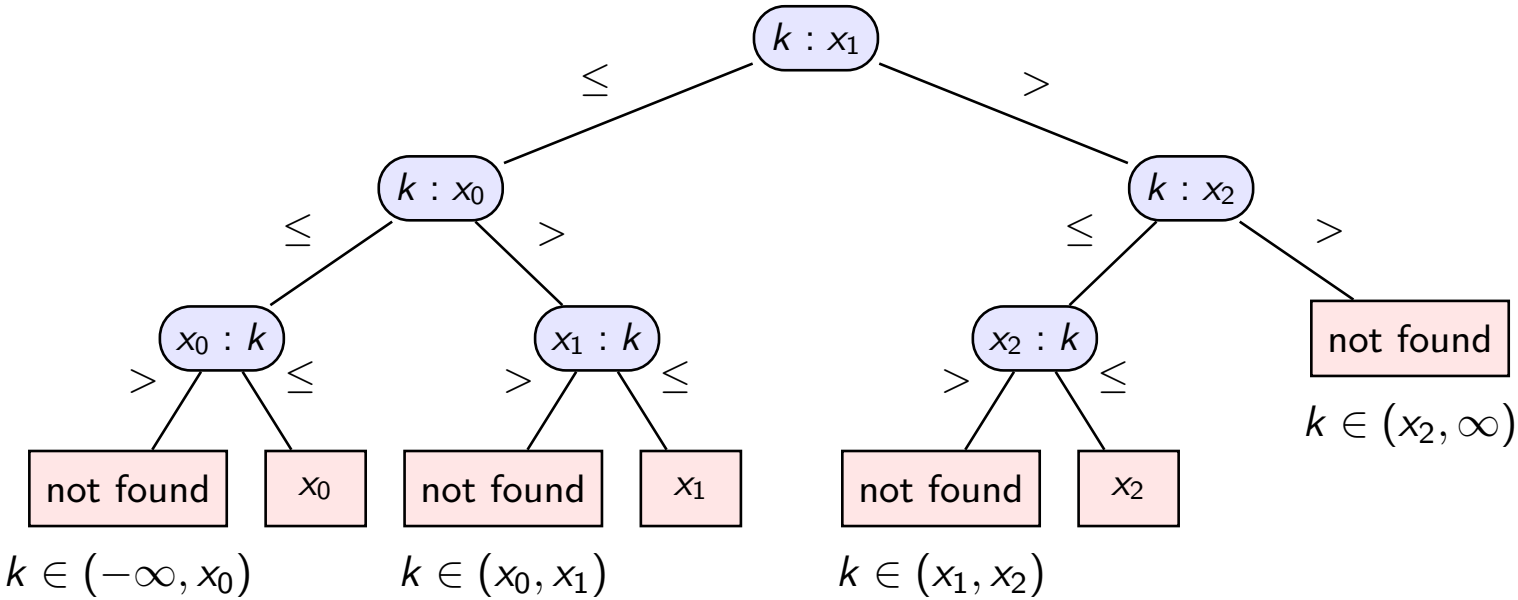


- How many possible outcomes are there?
- What does that tell us about the height of the decision tree?

Outline

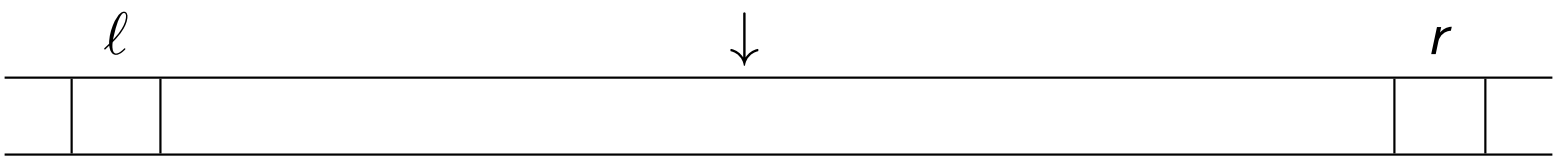
Towards interpolation search

We can match the lower bound asymptotically in a *sorted array*.

binary-search(A, n, k)

1. $\ell \leftarrow 0$, $r \leftarrow n-1$
2. **while** $(\ell \leq r)$
3. $\quad m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
4. $\quad$ **if** $(A[m]$ equals $k)$ **then return** "found at $A[m]$"
5. $\quad$ **else if** $(A[m] < k)$ **then** $\ell \leftarrow m+1$
6. $\quad$ **else** $r \leftarrow m-1$
7. **return** "not found, but would be between $A[\ell-1]$ and $A[\ell]$"

binary-search: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lceil \frac{1}{2}(r-\ell-1) \rceil$



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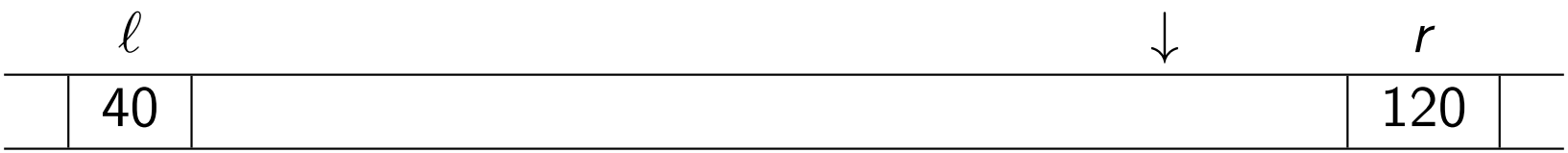


Question: If keys are *numbers*, where would you expect key $k = 100$?

Interpolation search

- Code very similar to binary search, but compare at index

$$\ell + \left\lceil \frac{\overbrace{k - A[\ell]}^{\text{distance from left key}}}{\underbrace{A[r] - A[\ell]}_{\text{distance between left and right keys}}} \underbrace{(r - \ell - 1)}_{\#\text{ unknown keys in range}} \right\rceil$$

- Need a few extra tests to avoid crash during computation of m.

interpolation-search$(A, n \leftarrow A.\textit{size}, k)$

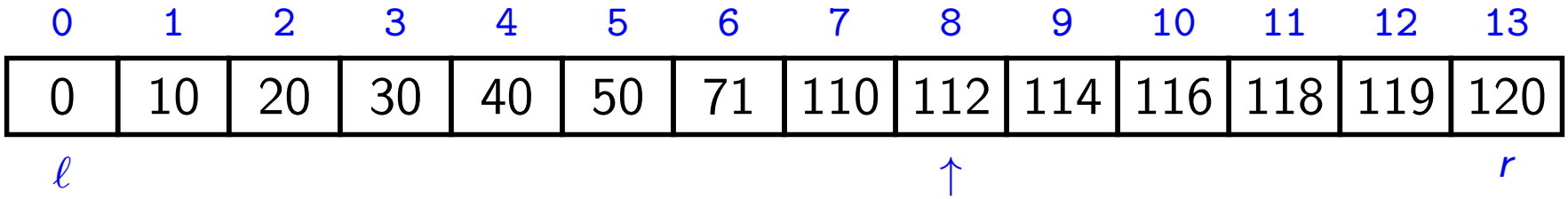
1. $\ell \leftarrow 0,\ r \leftarrow n - 1$
2. **while** $(\ell \leq r)$
3. **if** $(k < A[\ell]$ or $k > A[r])$ **return** "not found"
4. **if** $(k = A[r])$ **then return** "found at $A[r]$"
5. $m \leftarrow \ell + \left\lceil \frac{k - A[\ell]}{A[r] - A[\ell]} \cdot (r - \ell - 1) \right\rceil$
6. **if** ($A[m]$ equals k) **then return** "found at $A[m]$"
7. **else if** $(A[m] < k)$ **then** $\ell \leftarrow m + 1$
8. **else** $r \leftarrow m - 1$

Interpolation search: Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	10	20	30	40	50	71	110	112	114	116	118	119	120

interpolation-search(A[0..13],14,71):

Interpolation search: Example



interpolation-search(A[0..13],14,71):

- $\ell = 0,\ r = n - 1 = 13,\ m = \ell + \lceil \frac{71-0}{120-0}(13-0-1) \rceil = \ell + 8 = 8$

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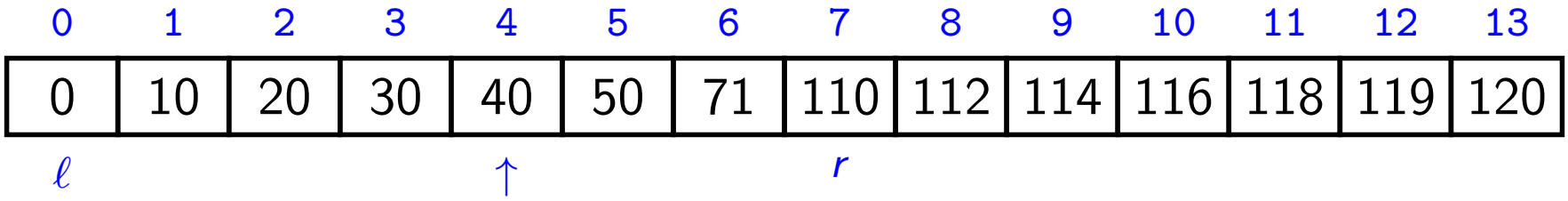

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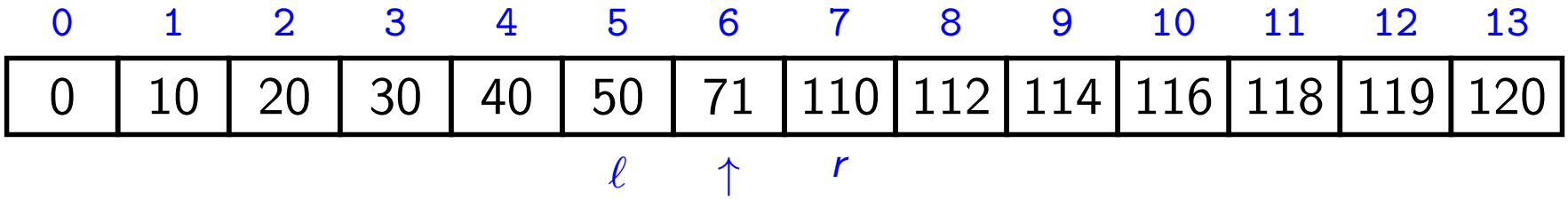

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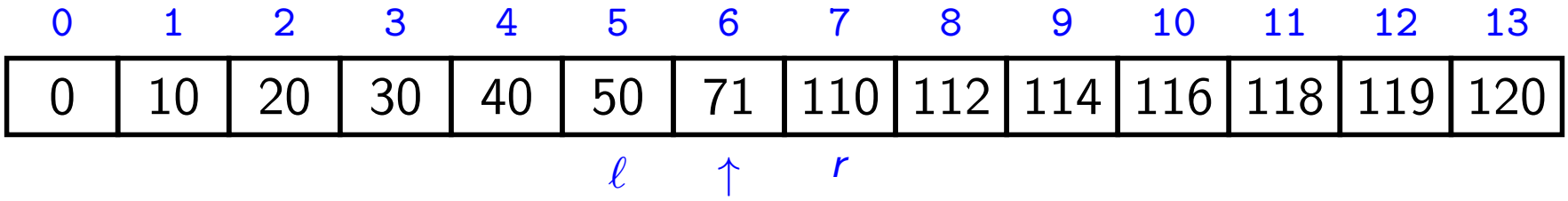
- $\ell = 5,\ r = 7,\ m = \ell + \lceil \frac{71-50}{110-50}(7-5-1) \rceil = \ell + 1 = 6$, found at $A[6]$

Interpolation search: Example



interpolation-search(A[0..13],14,71):

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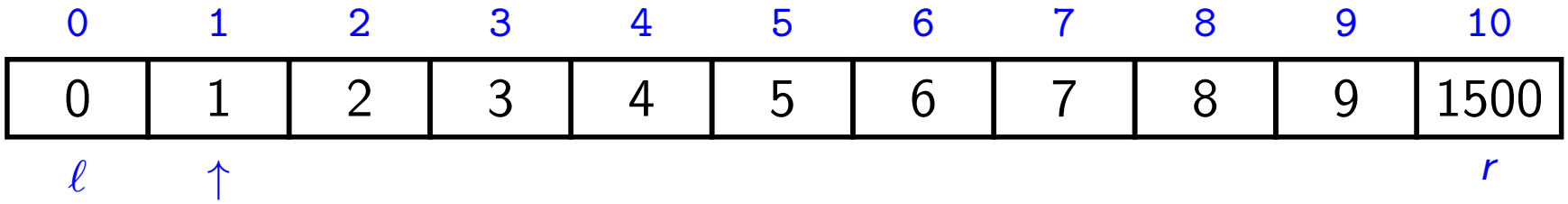
If instead we had $A[6] = 72$:

- $\ell = 5 = r$, exit at line 3 with "not found"

Interpolation search: Example 2



interpolation-search(A[0..10],10):

- $\ell = 0,\ r = n - 1 = 10,\ m = \ell + \lceil \frac{10-0}{1500-0}(10-0-1) \rceil = \ell + 1 = 1$

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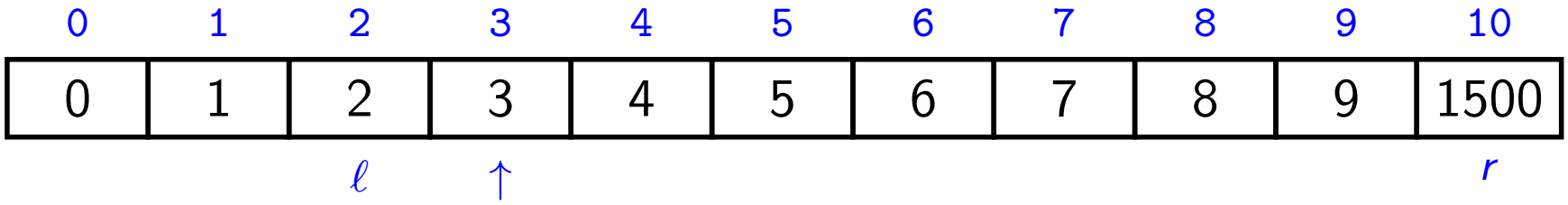

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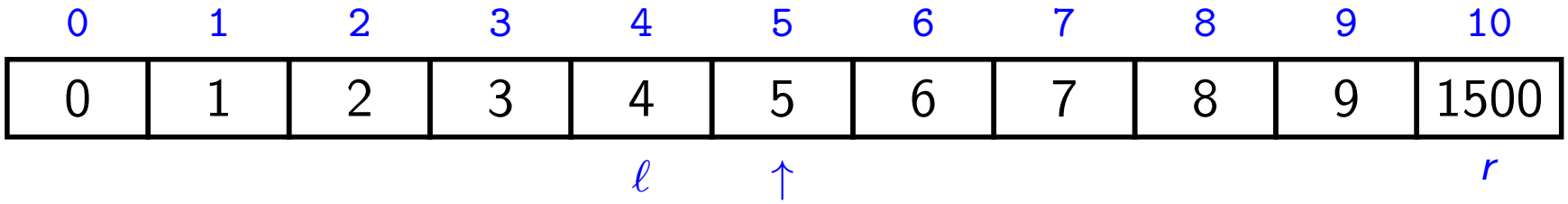

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- ... in the worst case this can be very slow ($\Theta(n)$ time)

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- ... in the worst case this can be very slow ($\Theta(n)$ time)

But it works well on average:

- Can show (difficult): $\boxed{T^{\text{avg}}(n) \leq T^{\text{avg}}(\sqrt{n}) + \Theta(1).}$
- This resolves to $T^{\text{avg}}(n) \in O(\log \log n)$.

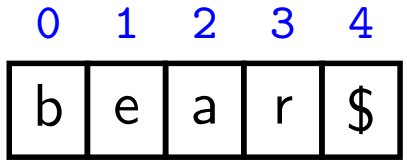
Outline

Review: Words

Scenario: Keys in dictionary are *words*. Need brief review.

Words (= strings): sequences of characters over alphabet Σ

- Typical alphabets: $\{0, 1\}$ ($\rightarrow$bitstrings), ASCII, $\{C, G, T, A\}$
- Stored in an array: $w[i]$ gets ith character (for $i = 0, 1, \ldots$)



- Words have end-sentinel \$ (sometimes not shown)
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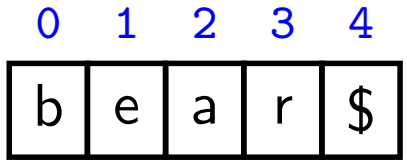

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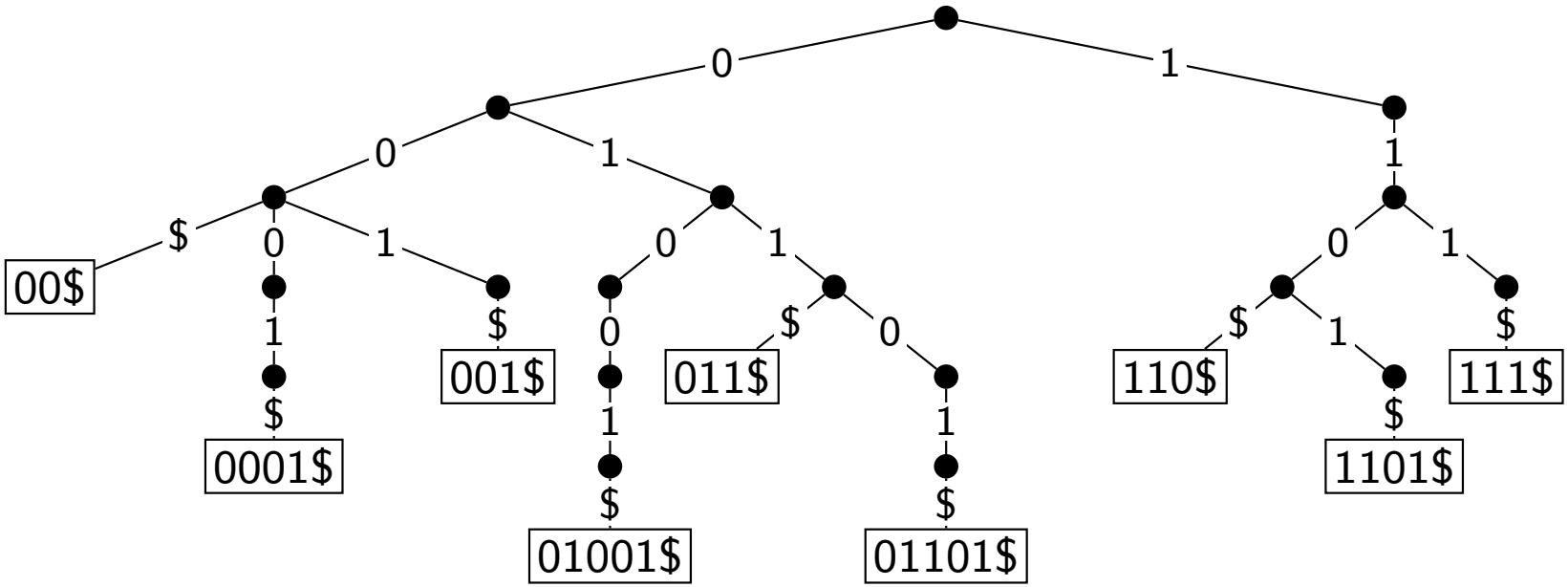
Should know:

- prefix, suffix, substring
- Sort words **lexicographically**: $\texttt{be\$} <_{\text{lex}} \texttt{bear\$} <_{\text{lex}} \texttt{beer\$}$
  This is different from sorting numbers: $\texttt{001\$} <_{\text{lex}} \texttt{010\$} <_{\text{lex}} \texttt{1\$}$

Tries: Introduction

Trie (also know as **radix tree**): A dictionary for bitstrings.

- Comes from retrieval, but pronounced "try"
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key
- Due to end-sentinels, all key-value pairs are at leaves.



Tries: Search

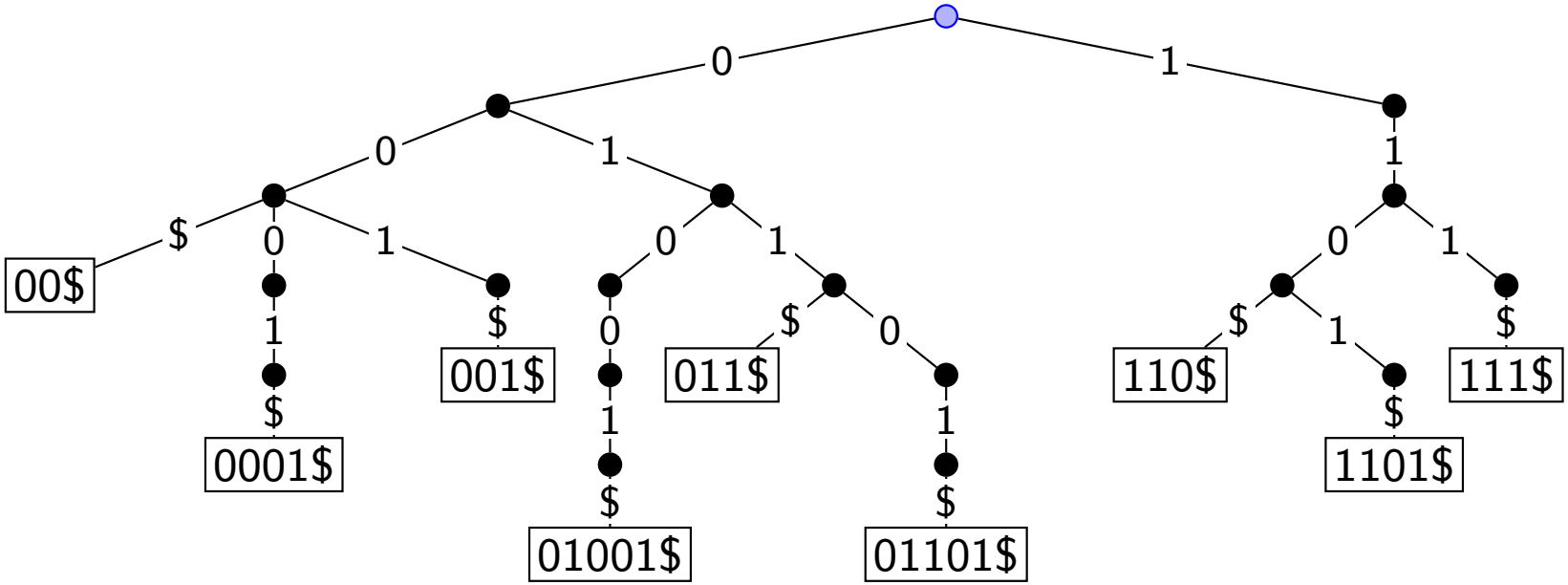
- Follow links that correspond to current bits in w, keeping track of current depth d
- Repeat until no such link or w found at a leaf

Similar as for skip lists, we find search-path P separately first.

Trie::get-path-to(w)
Output: Stack with all ancestors of where w would be stored
1. $P \leftarrow$ empty stack; $z \leftarrow root$; $d \leftarrow 0$; P.*push*(z)
2. **while** $d \leq |w|$
3. **if** z has a child-link labelled with $w[d]$
4. $z \leftarrow$ child at this link; d++; P.*push*(z)
5. **else break**
6. **return** P

Tries search: Example

Example: Trie::search(011$)



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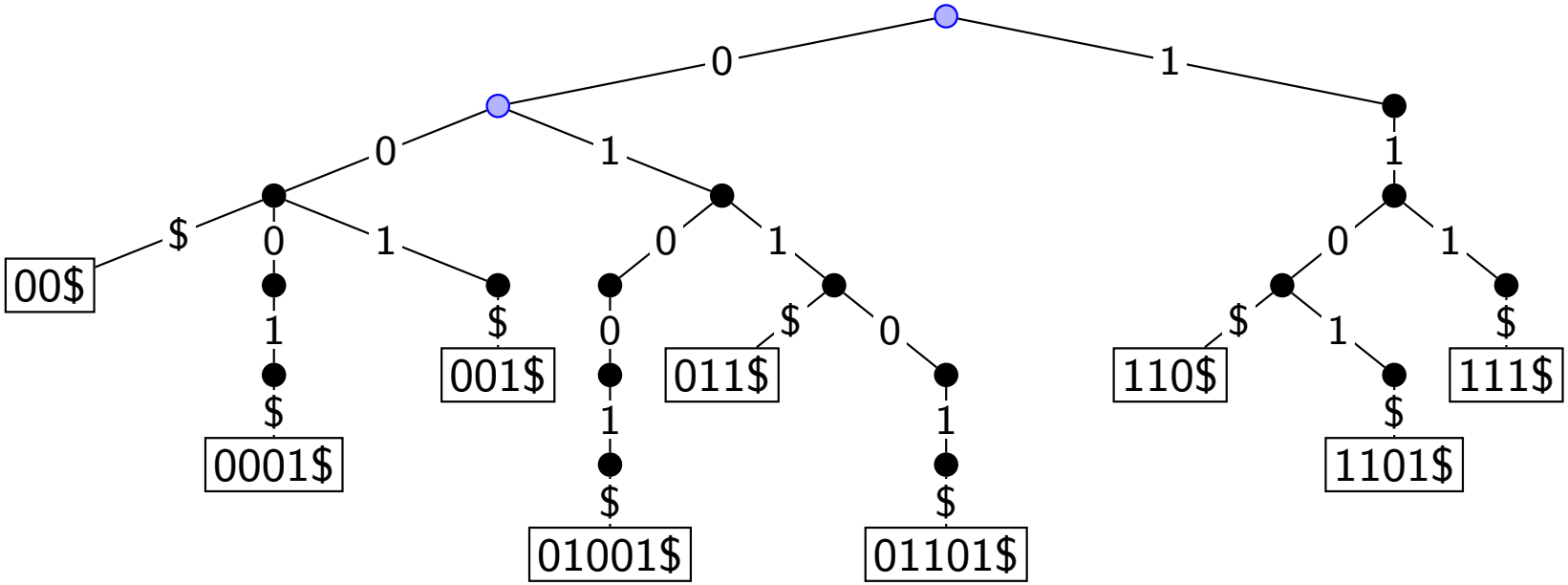

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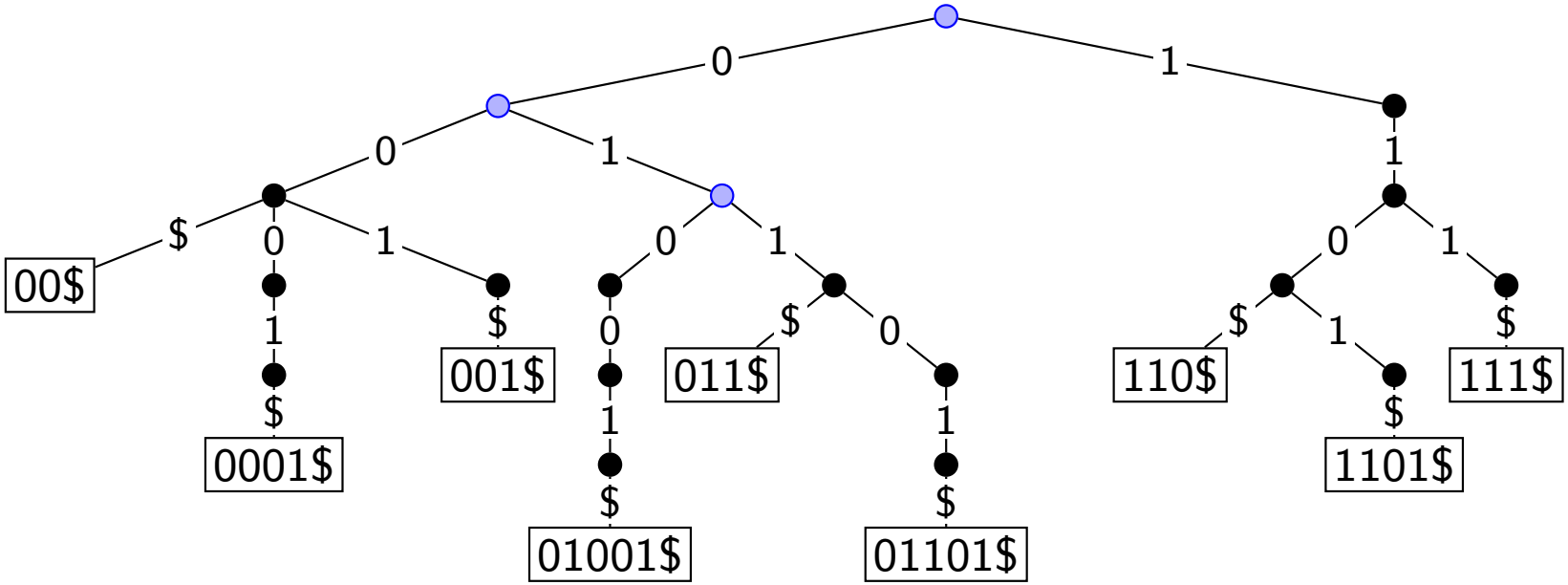

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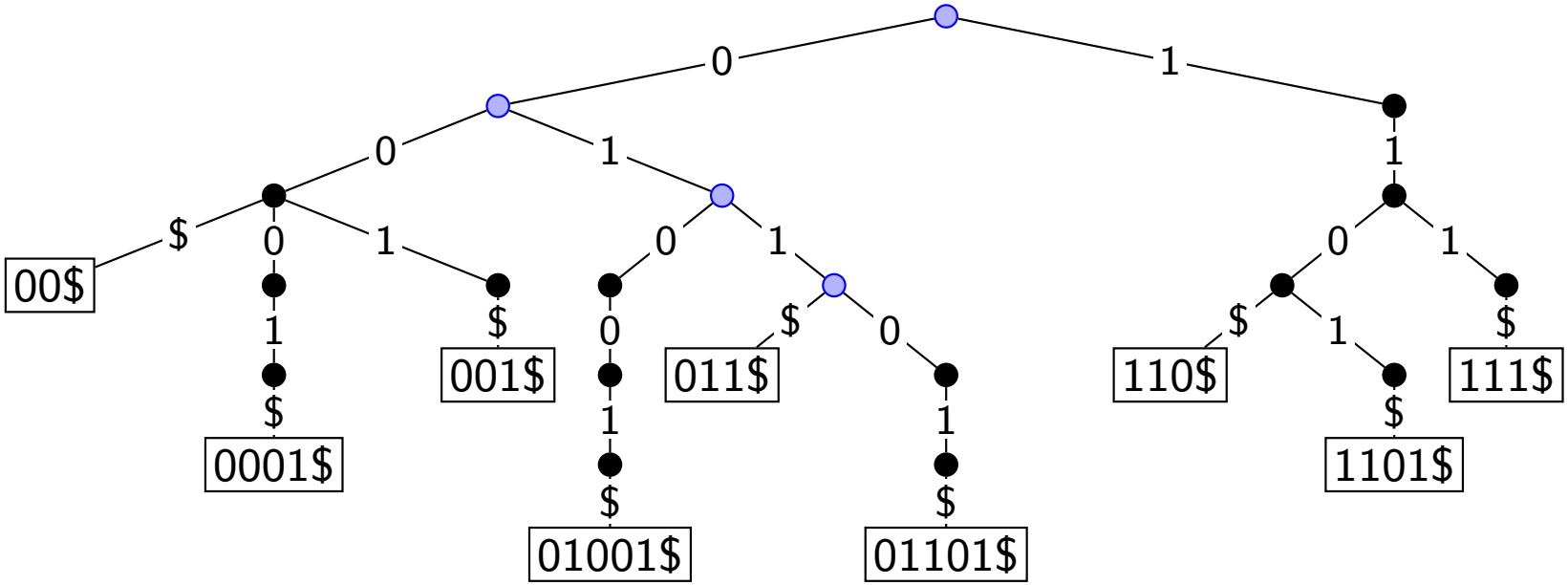

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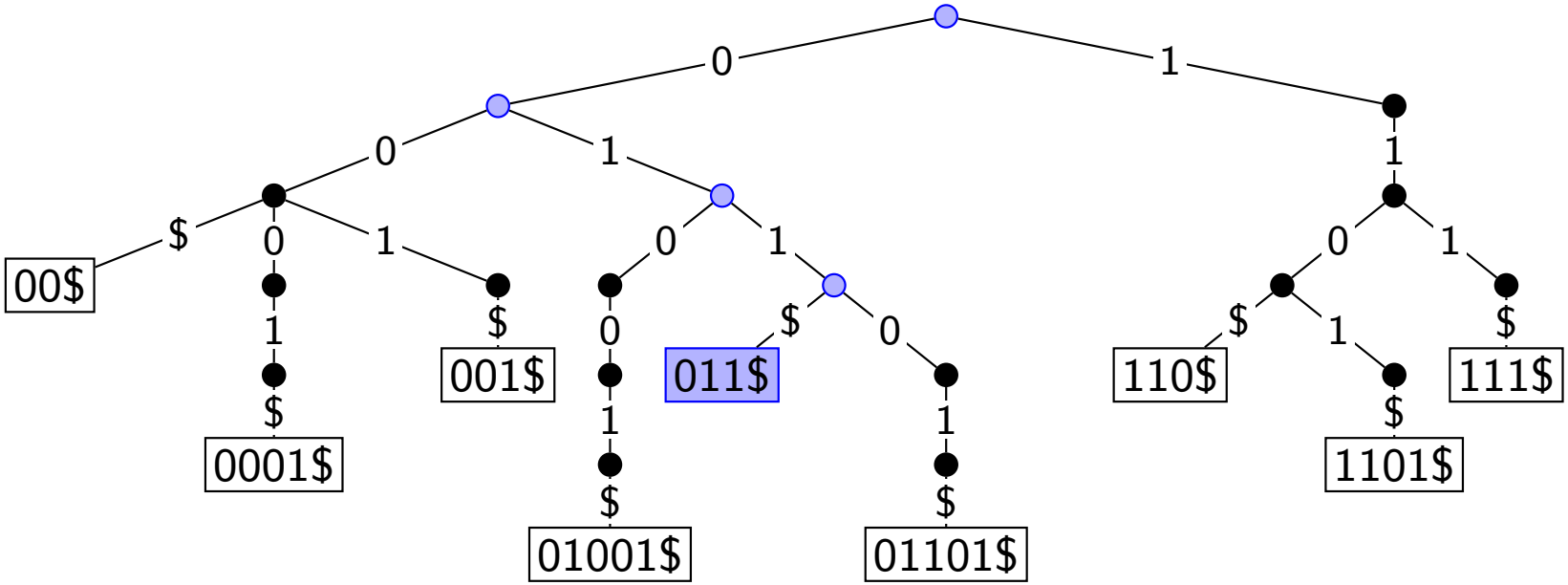


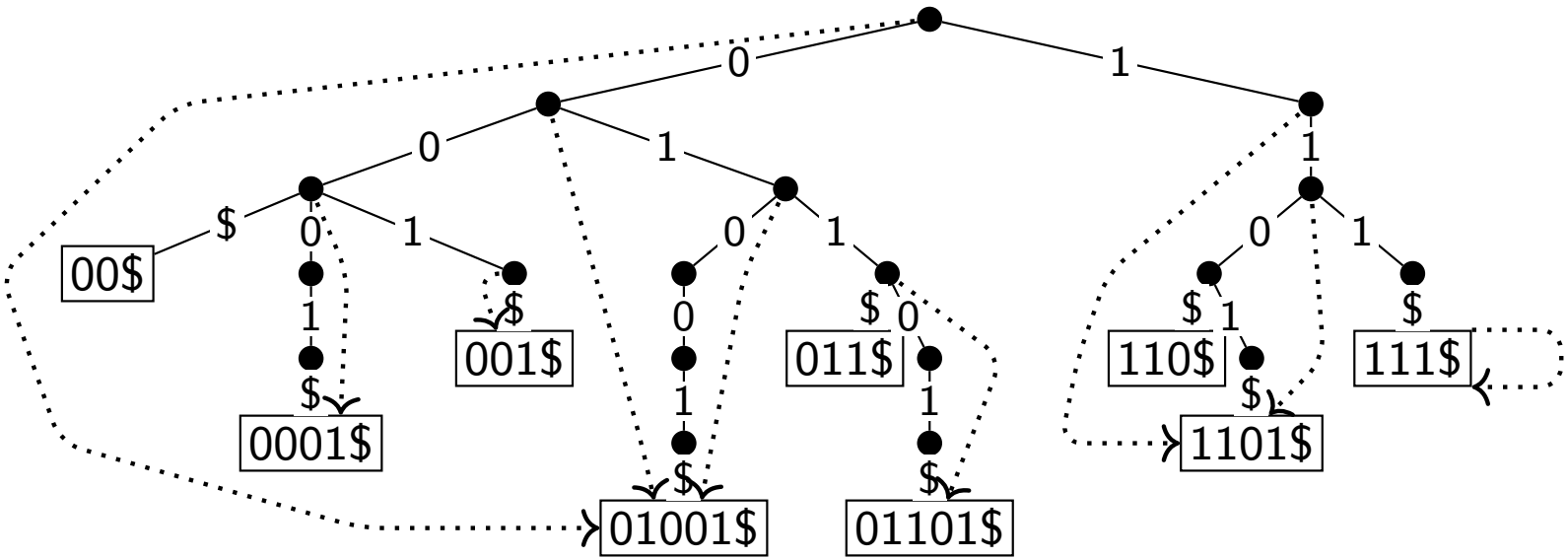
Trie::search(w)

1. $P \leftarrow$ *get-path-to*(w), $z \leftarrow P$.*top*
2. **if** (z is not a leaf) **then**
3. **return** "not found, would be in sub-trie of z"
4. **return** key-value pair at z

Tries: Prefix-search

For later applications of tries, we want another search-operation:

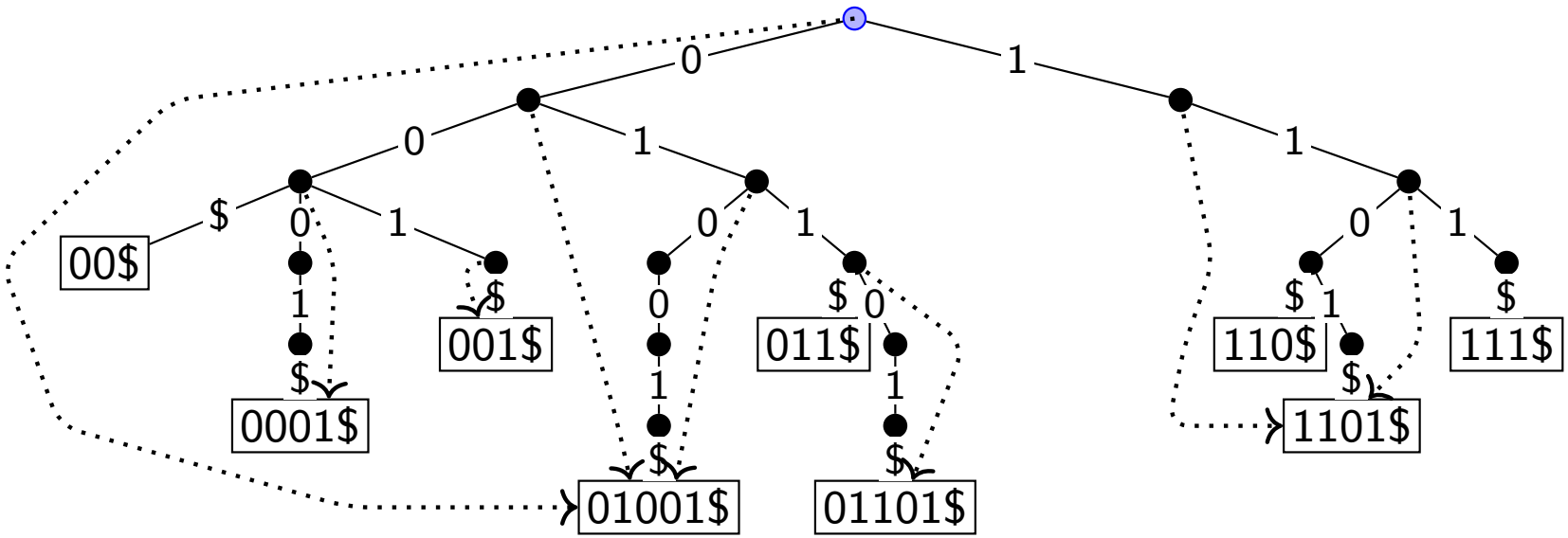
- *prefix-search*(w): Find **extension**(word for which w is a prefix).
- Testing whether extension exists is easy (how?)
- To find extension quickly, we need **leaf-references**
  - Every node z stores reference $z.\mathit{leaf}$ to a leaf in subtree
  - **Convention:** store leaf with longest word



(not all leaf-references are shown)

Tries prefix-search: Example

Example: Trie::prefix-search(11$)



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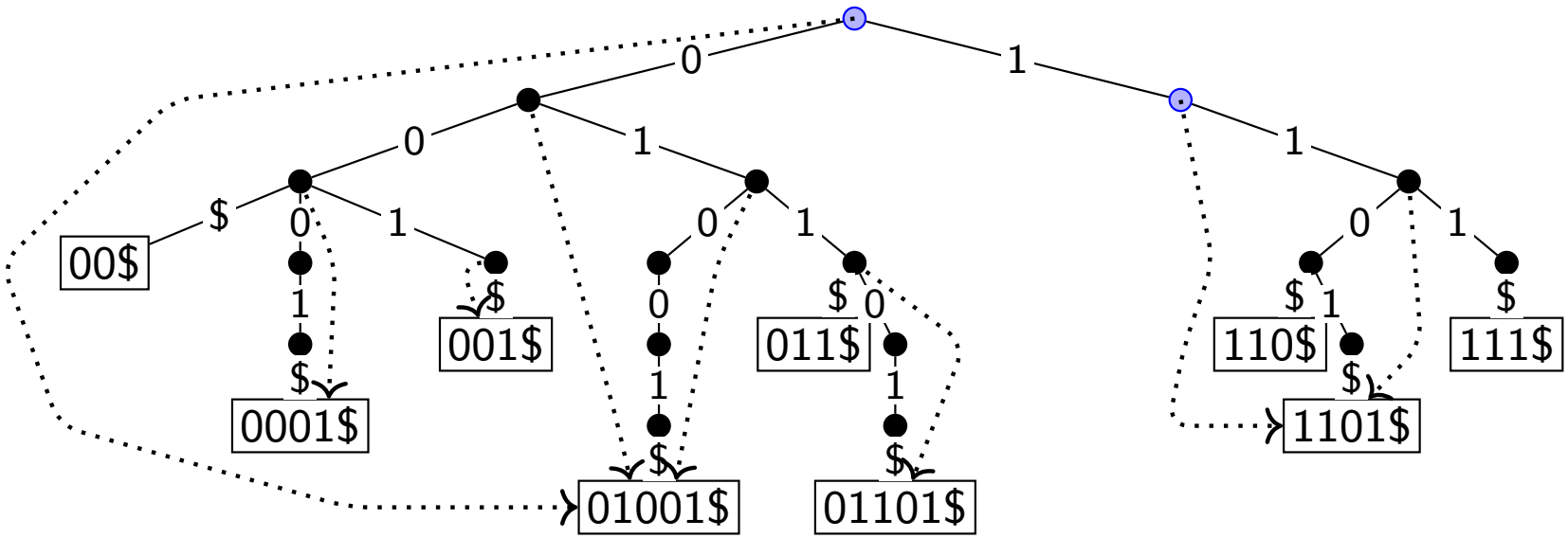

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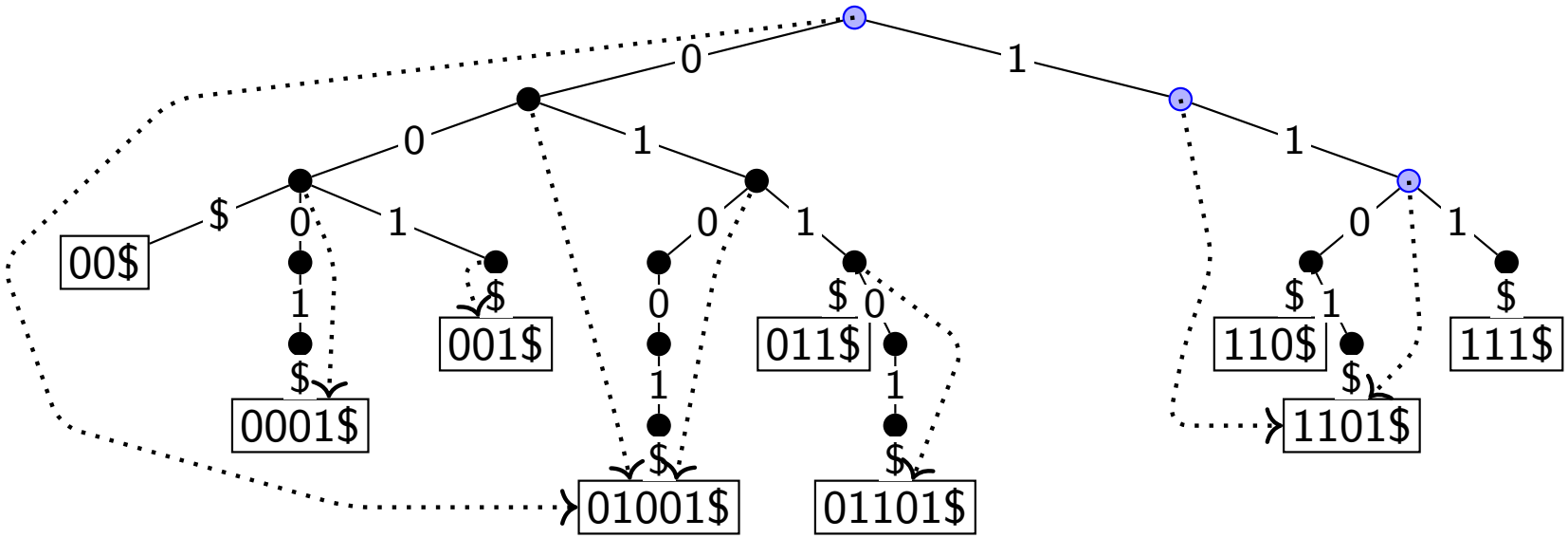

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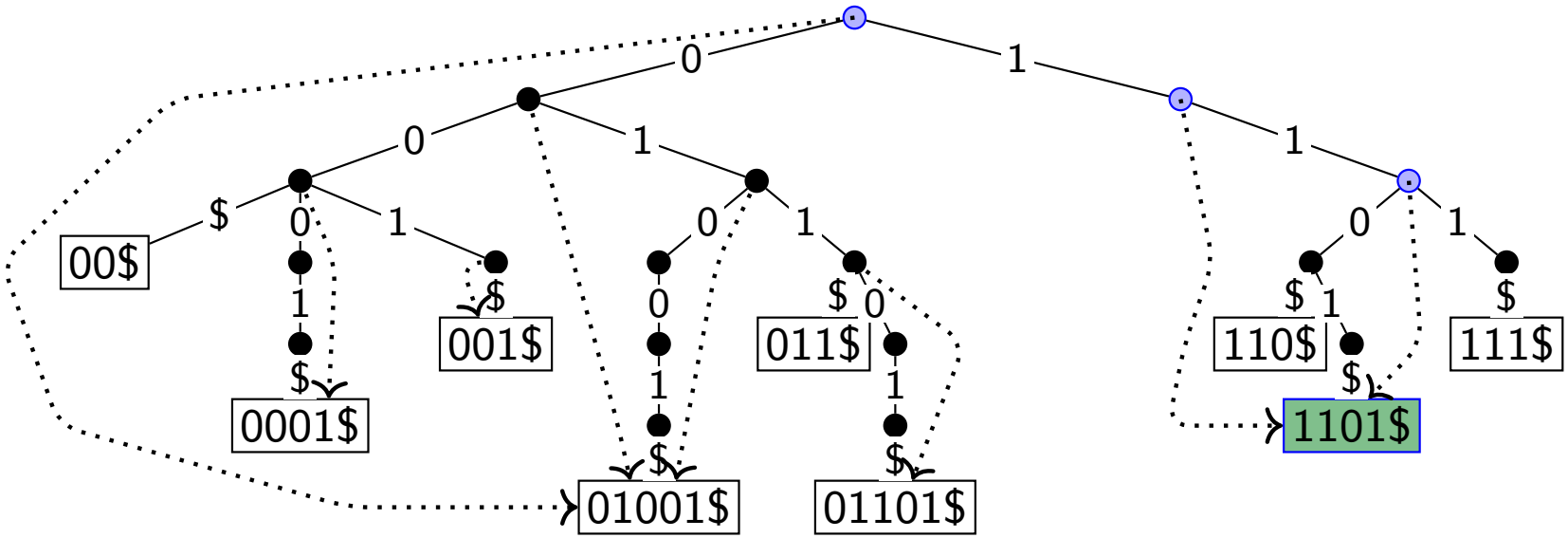
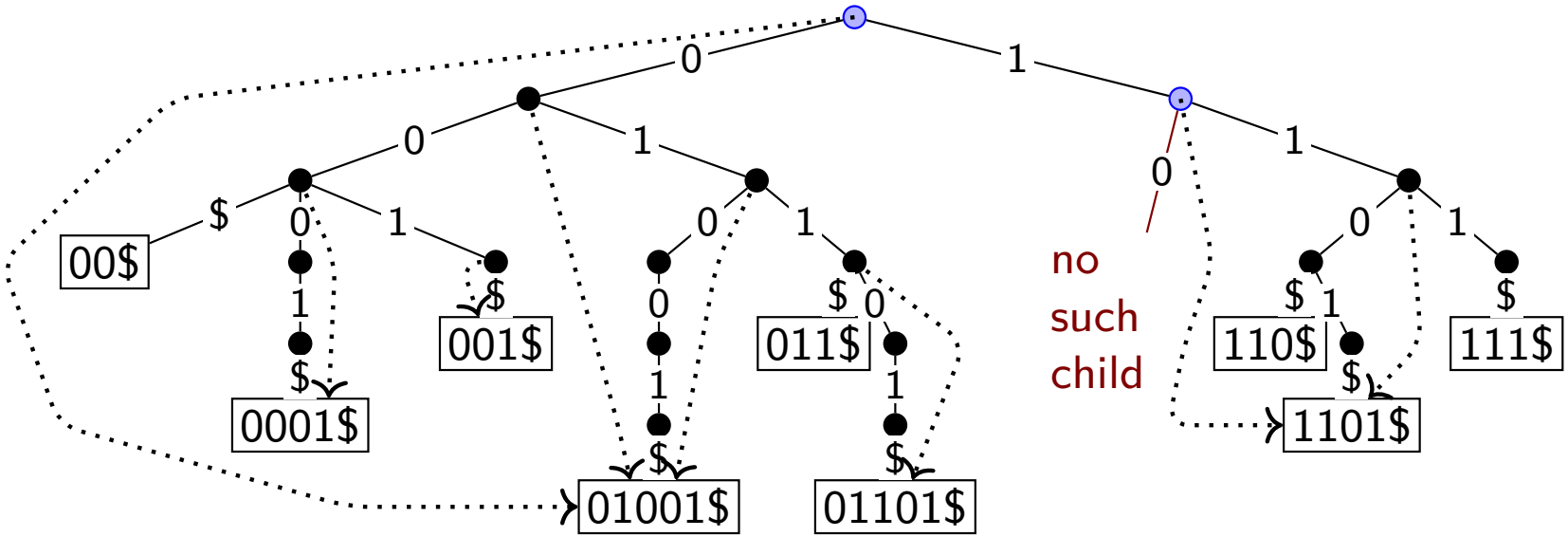


Trie::prefix-search(w)

1. $P \leftarrow$ *get-path-to*$(w[0..|w|-1])$ `// ignore end-sentinel`
2. **if** number of nodes on P is at most $|w|$
3. **return** "no extension of w found"
4. **return** P.*top*().*leaf*

Tries prefix-search: Example

Example: Trie::prefix-search(10$)



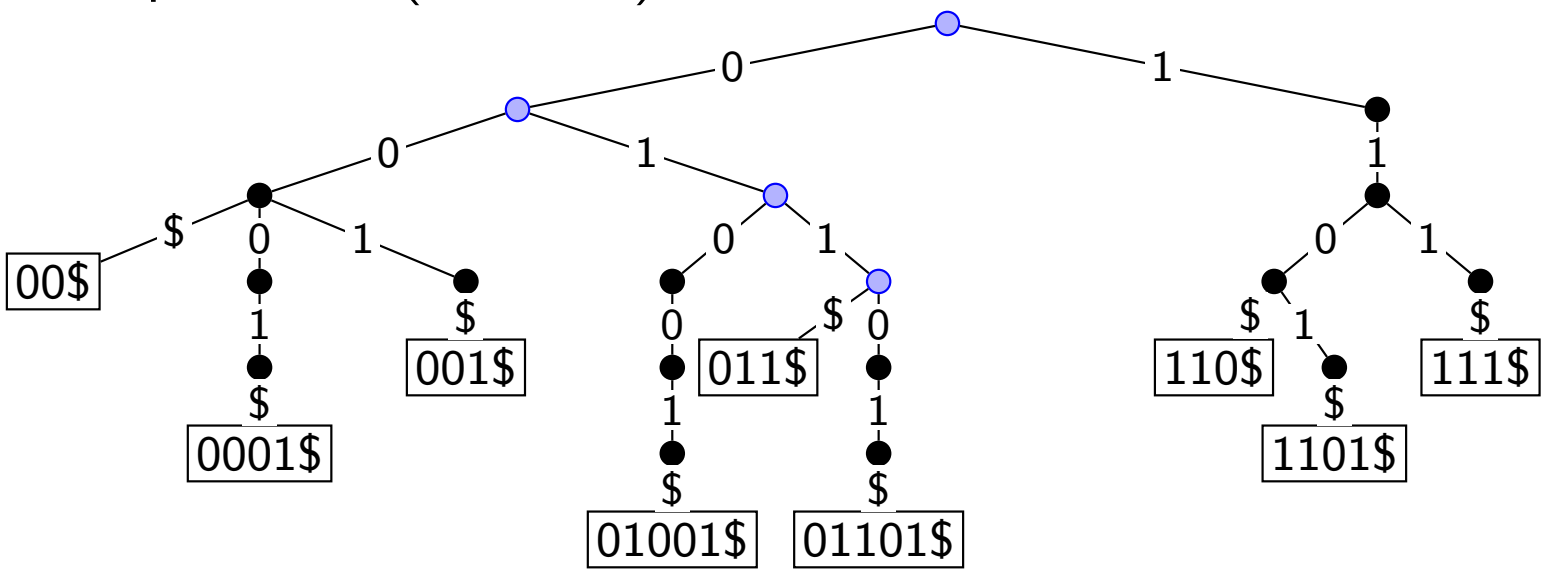
- Word 10$ has size 2.
- *get-path-to*(10) returns stack with two nodes.
- We need more than $|w|$ nodes on P to have an extension.

Tries: Insert

Trie::insert(w)

- $P \leftarrow$ *get-path-to*(w) gives ancestors that exist already,
- Expand the trie from P.*top*() by adding necessary nodes that correspond to extra bits of w.
- Update leaf-references (also at P if w is longer than previous leaves)

Example: *insert*(011101$)



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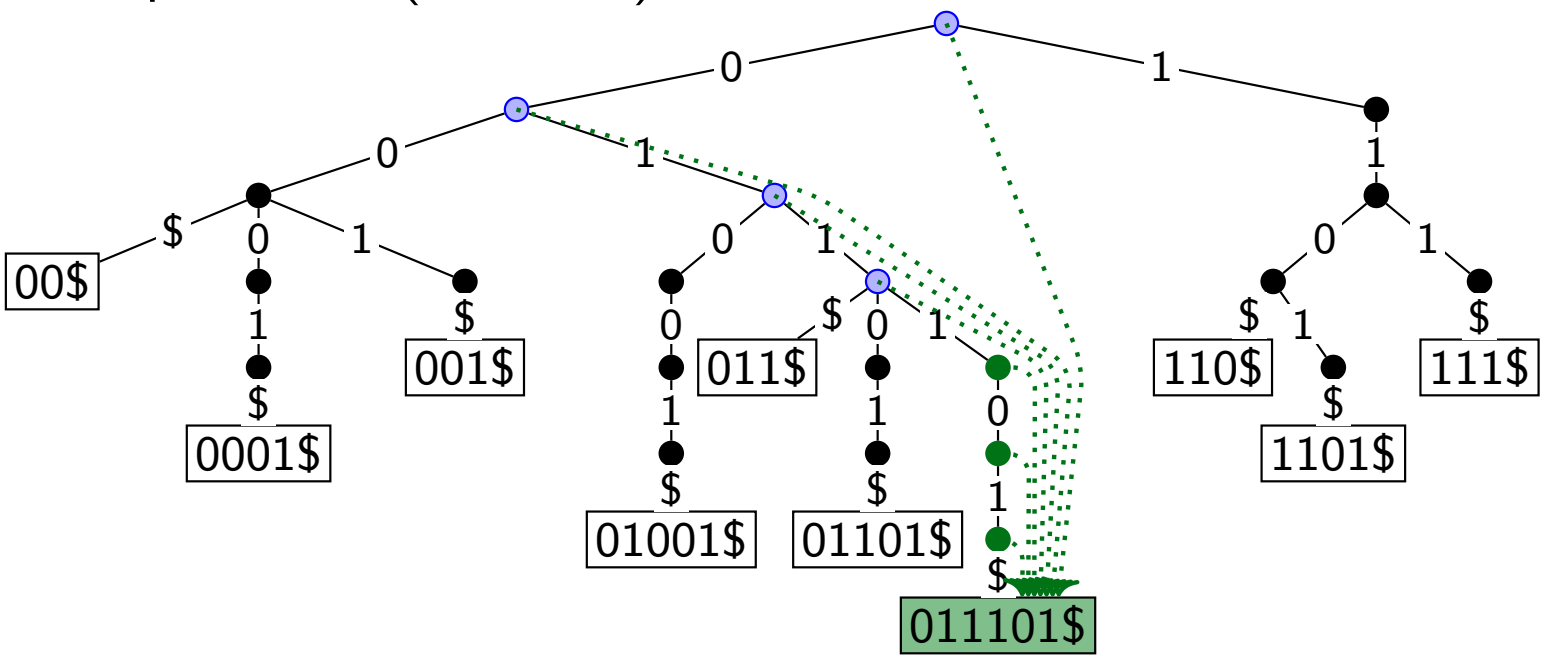


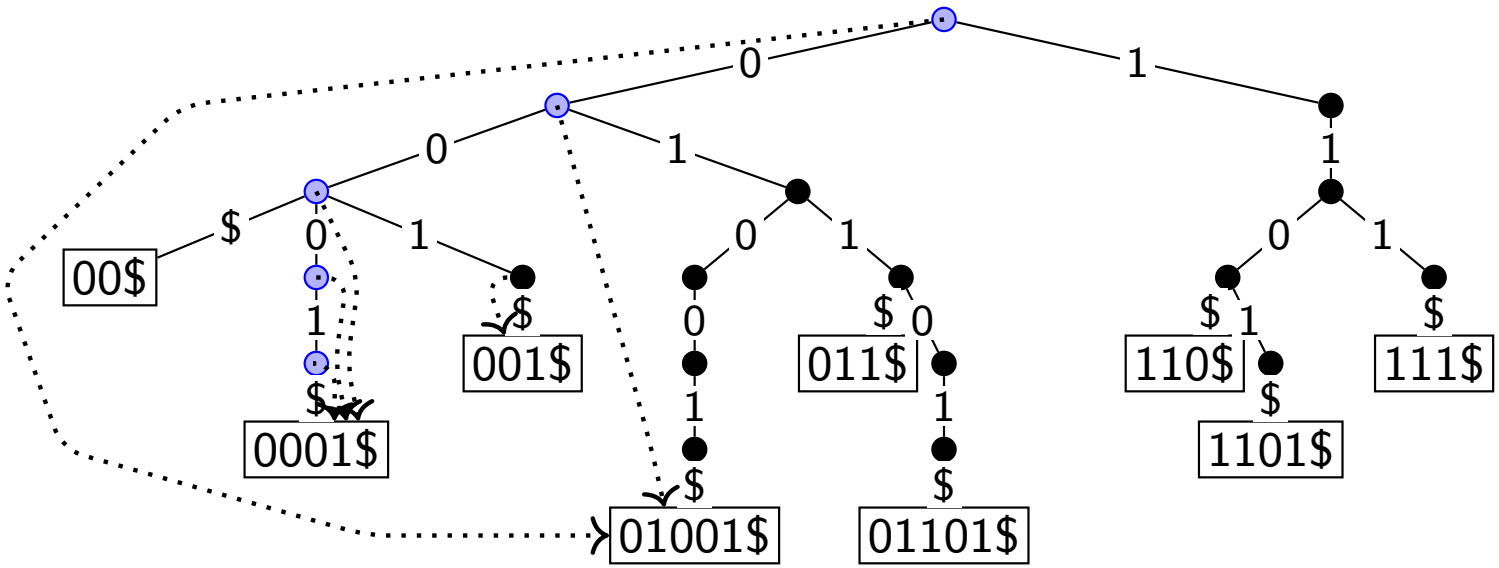
(only updated leaf-references are shown)

Tries: Delete

Trie::delete(w)

- $P \leftarrow$ *get-path-to*(w) gives all ancestors.
- Let ℓ be the leaf where w is stored
- Delete ℓ and nodes on P until ancestor that had two or more children.
- Update leaf-references on rest of P.
  (If $z \in P$ referred to ℓ, find new z.*leaf* from other children.)

Example: *trie::delete*(0001$)



(only some leaf-references are shown)

Tries: Delete

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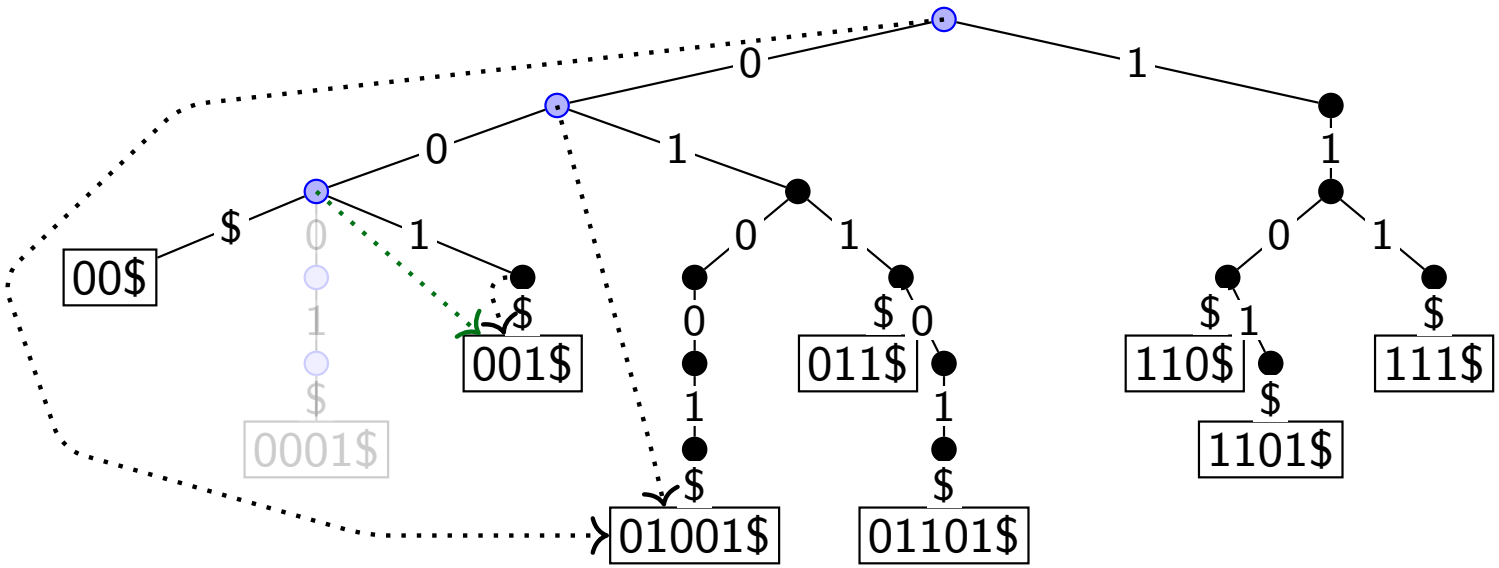
- $P \leftarrow$ *get-path-to*(w) gives all ancestors.
- Let ℓ be the leaf where w is stored
- Delete ℓ and nodes on P until ancestor that had two or more children.
- Update leaf-references on rest of P.
  (If $z \in P$ referred to ℓ, find new z.*leaf* from other children.)

Example: *trie::delete*(0001$)



(only some leaf-references are shown)

Binary tries: Summary

search(w), *prefix-search*(w), *insert*(w), *delete*(w) all take time $\Theta(|w|)$.

- Search-time is *independent* of number n of words stored in the trie!
- Search-time is small for short words.

The trie for a given set of words is unique

(except for order of children and ties among leaf-references)

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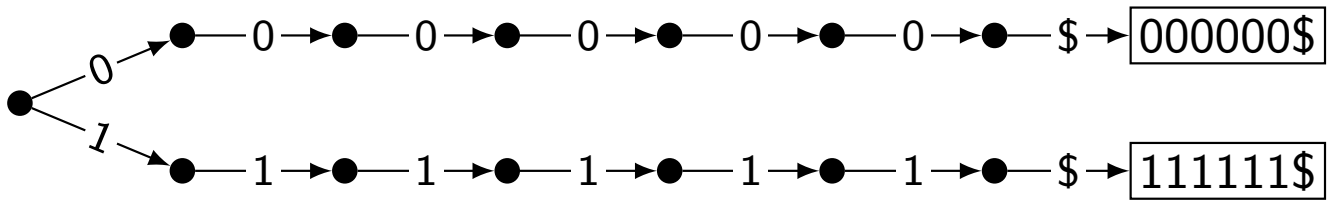
Disadvantages:

- Tries can be wasteful with respect to space.

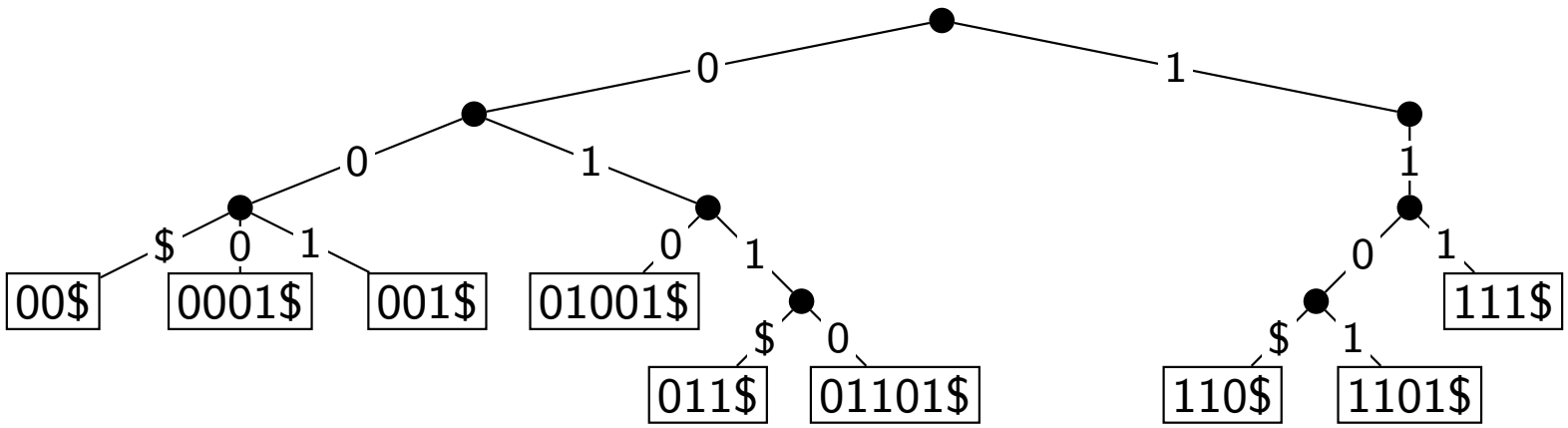


- Worst-case space is $\Theta(n \cdot (\text{maximum length of a word}))$
- What can we do to save space?

Variations of tries: Pruned tries

Pruned Trie: Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Saves space if there are only few bitstrings that are long.



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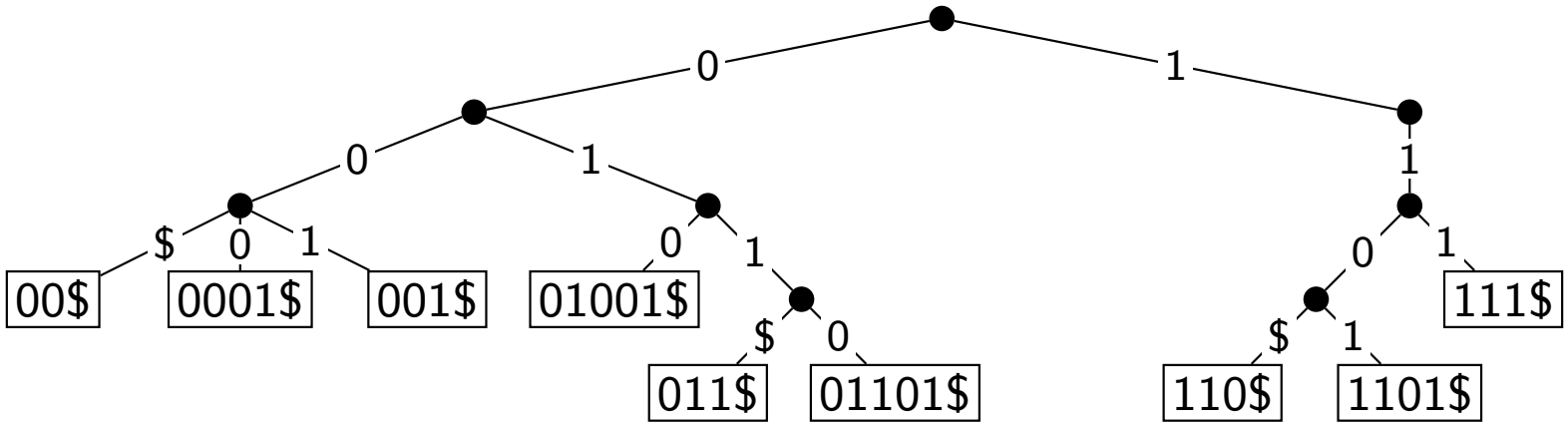
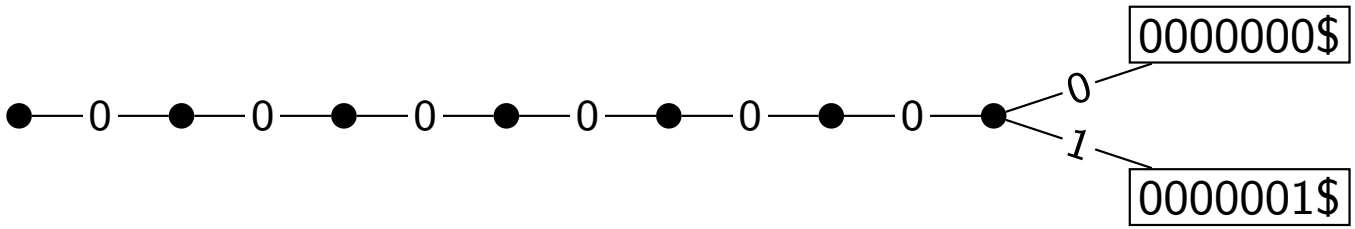


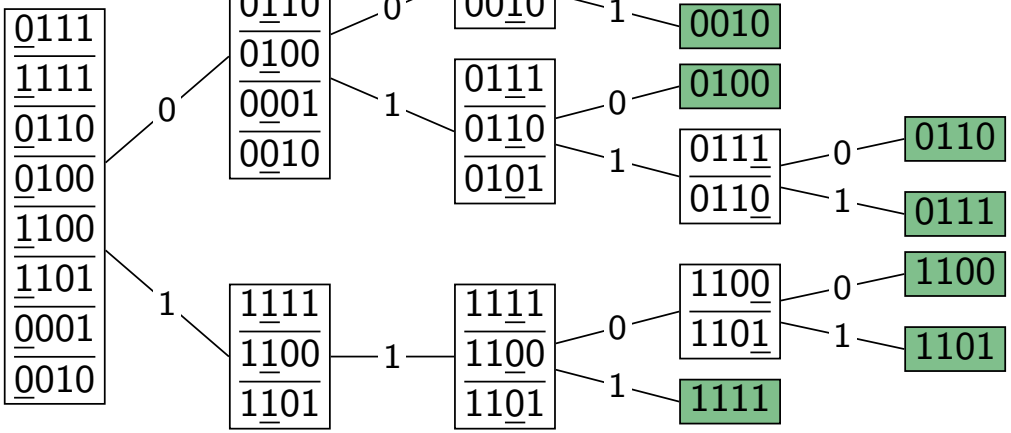
But the space can still be bad.

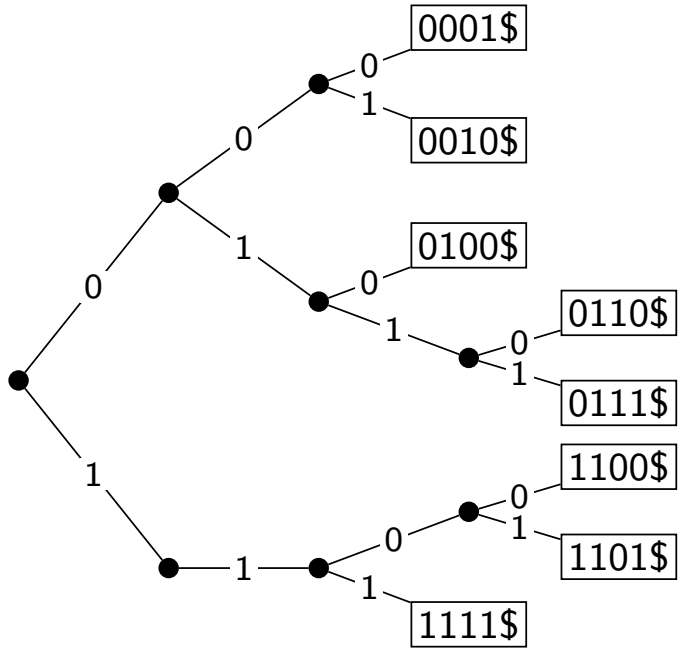


Pruned tries and MSD-radix sort

We have (implicitly) seen pruned tries before:

- For equal-length bitstrings:
  Pruned trie equals recursion tree of MSD radix-sort.

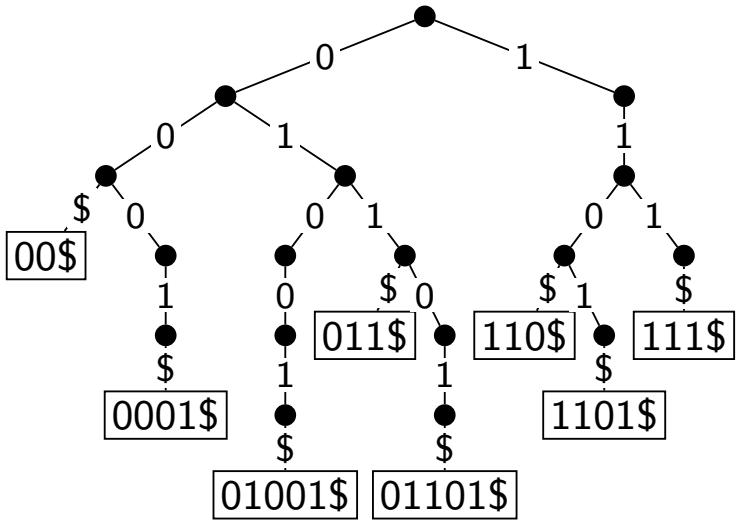


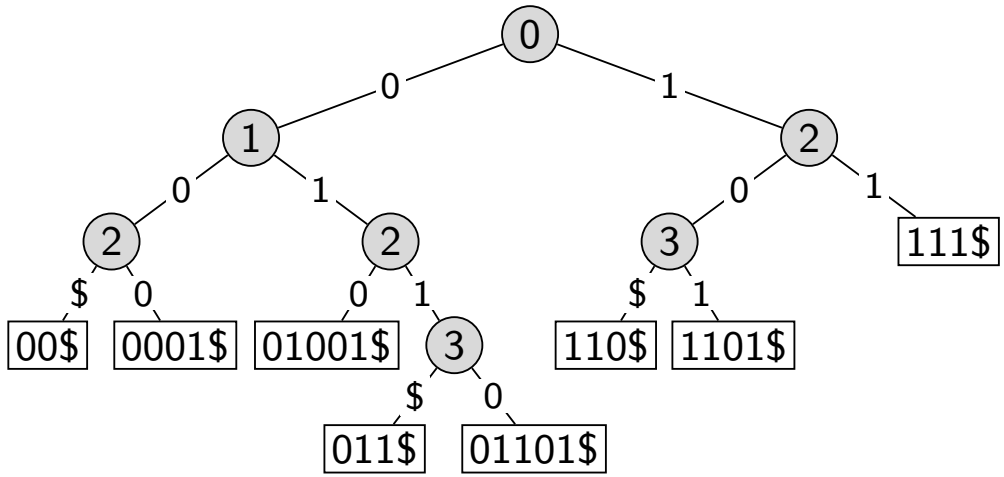


Compressed tries

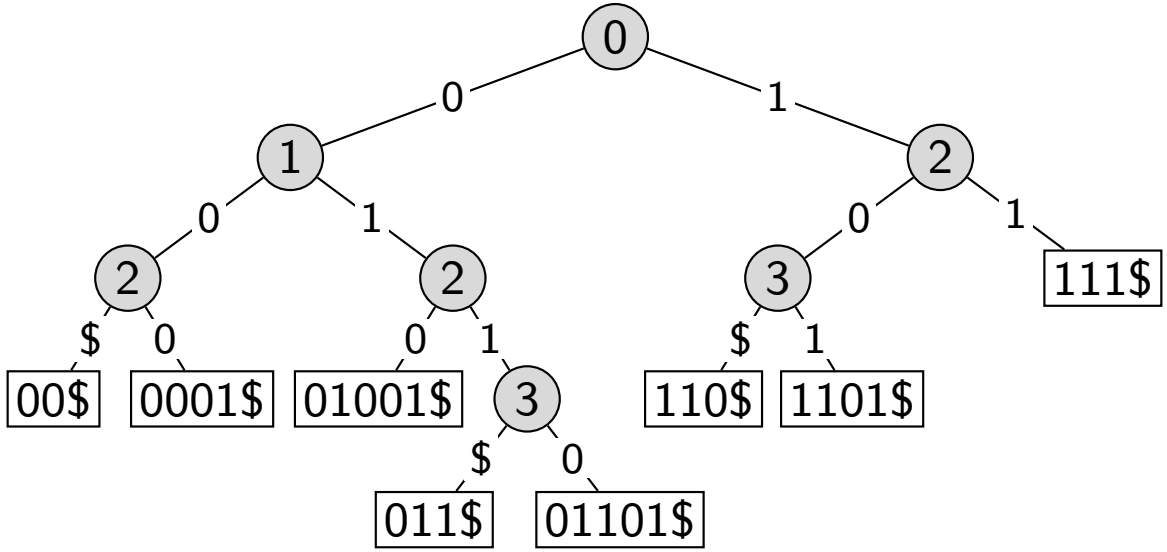
Another (important!) variation:

- Compress paths of nodes with only one child.
- Each node stores an *index*, corresponding to the level of the node in the uncompressed trie. (On level d, we searched for link with $w[d]$.)



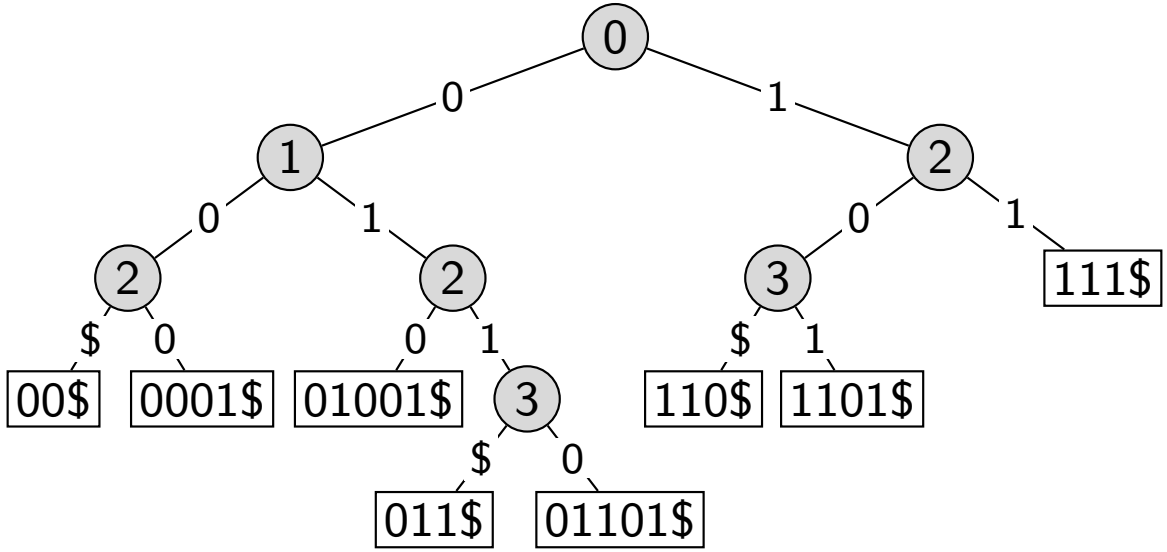


Compressed tries



- **Invariant:** At any node z where $d = z.\mathit{index}$,
  all words in the subtree rooted at z have the same initial d bits.

Compressed tries



- **Invariant:** At any node z where $d = z.\mathit{index}$,
  all words in the subtree rooted at z have the same initial d bits.
- **Observe:** Any compressed trie with n words has $O(n)$ nodes.
  - #nodes = # leaves + # internal node.
  - Every internal node has two or more children.
  - **Can show:** Therefore more leaves than internal nodes.
  - So #nodes $\leq 2n - 1$.

  In particular we use $O(n)$ auxiliary space.

Compressed tries: Search

- As for tries, follow links that corresponds to current bits in w
- Main difference: stored indices say which bits to compare.
- Also: must compare w to word found at the leaf (why?)

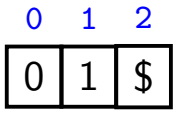
CompressedTrie::get-path-to(w)

1. $P \leftarrow$ empty stack; $z \leftarrow$ *root*; P.*push*(z)
2. **while** z is not a leaf and ($d \leftarrow z$.*index* $\leq |w|$) **do**
3. **if** (z has a child-link labelled with $w[d]$) **then**
4. $z \leftarrow$ child at this link; P.*push*(z)
5. **else break**
6. **return** P

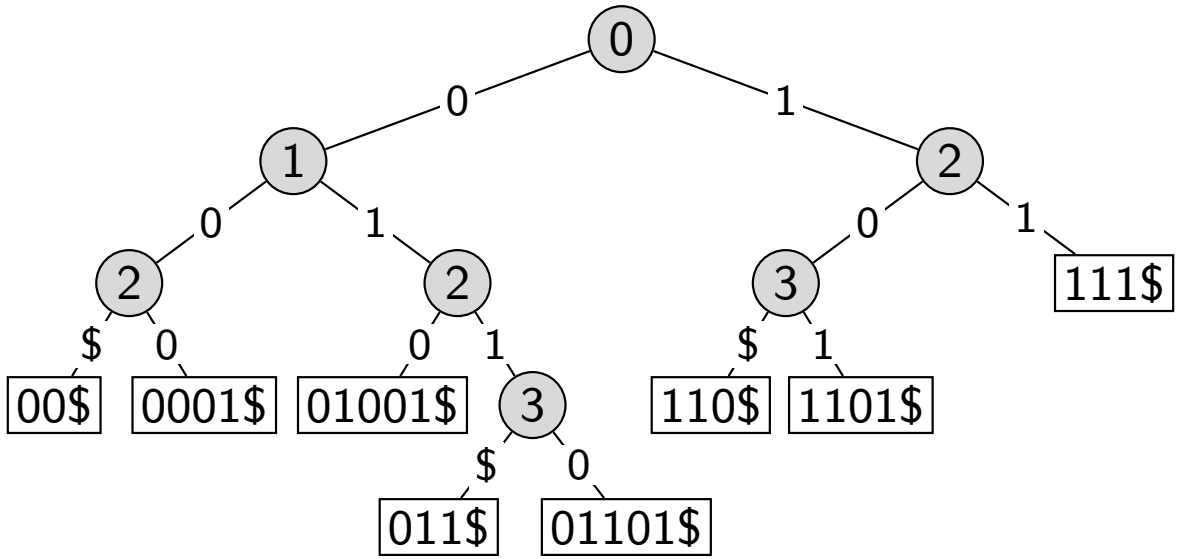
CompressedTrie::search(w)

1. $P \leftarrow$ *get-path-to*(w), $z \leftarrow P$.*top*
2. **if** (z is not a leaf or word stored at z is not w) **then**
3. **return** "not found"
4. **return** key-value pair at z

Compressed tries search: Example

Example 1: CompressedTrie::search(

)



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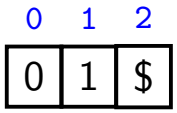
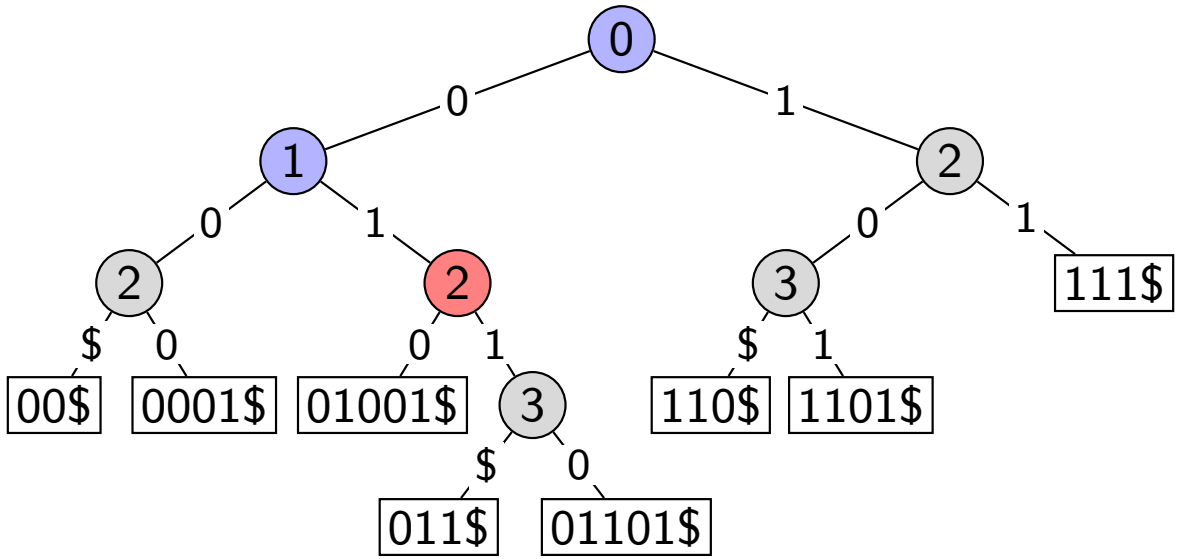
Example 1: CompressedTrie::search(

) unsuccessful (no $-child)



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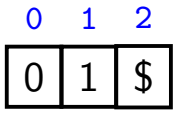
Example 1: CompressedTrie::search(

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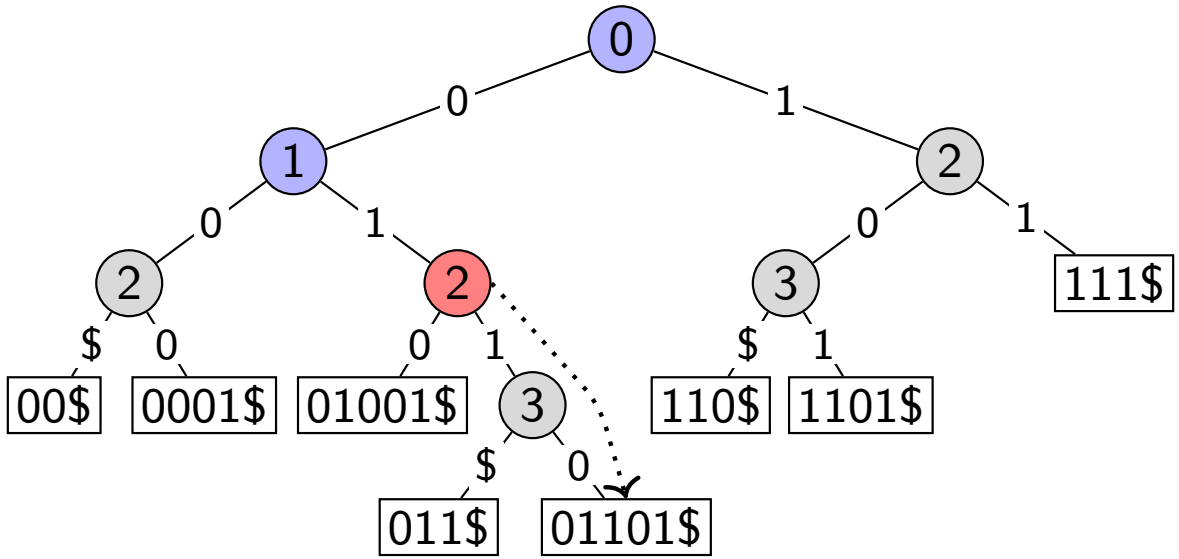
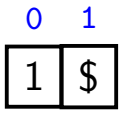


prefix-search(w): Compare w to z.*leaf* at last visited node z.

Compressed tries search: Example

Example 2: CompressedTrie::search(

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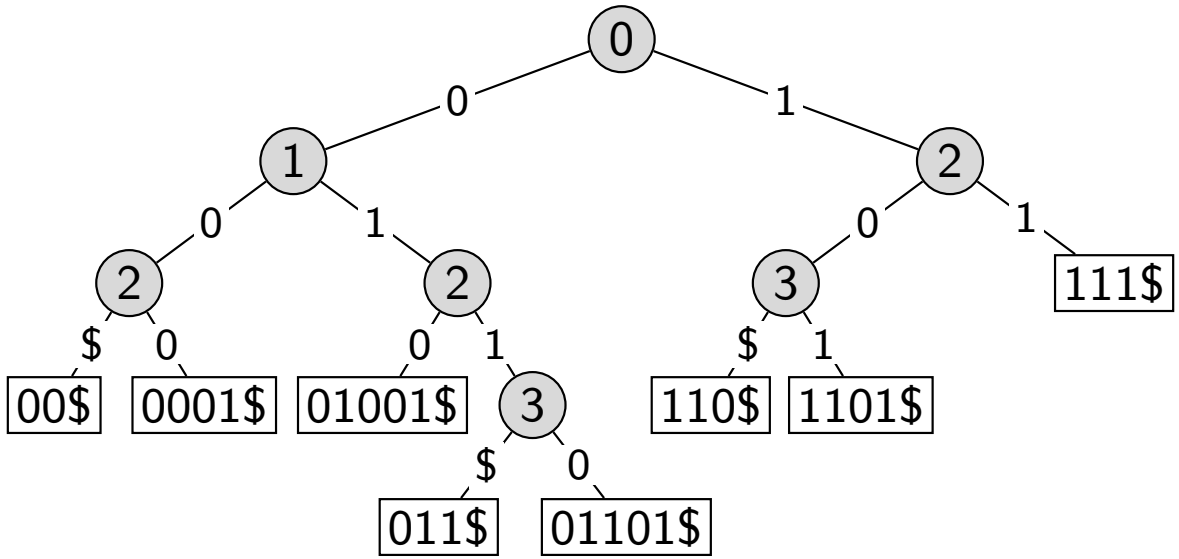
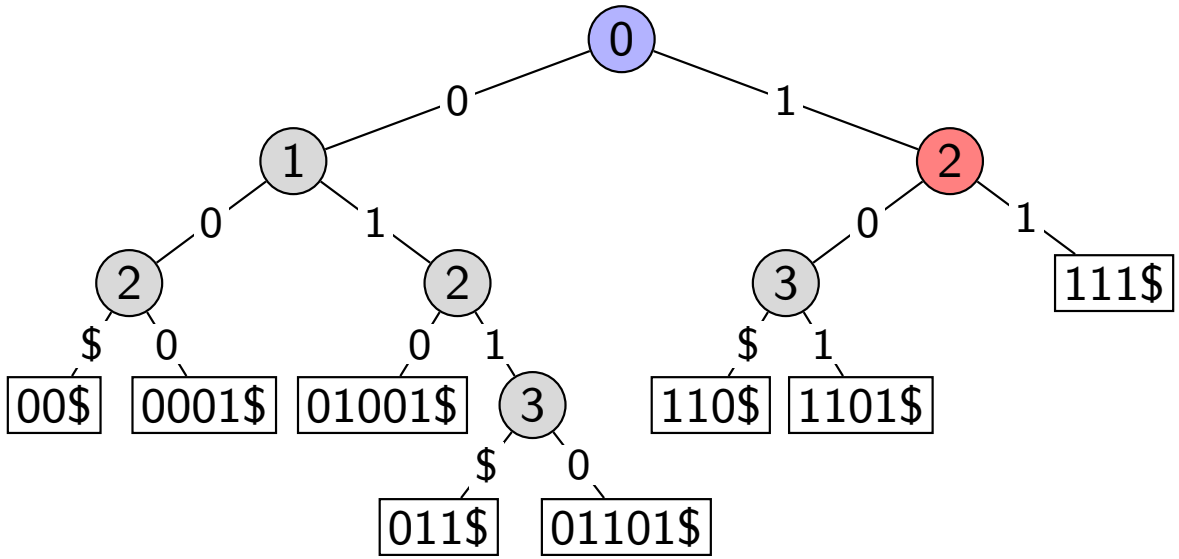
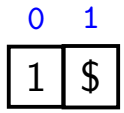


Compressed tries search: Example

Example 2: CompressedTrie::search($\begin{array}{cc} 0 & 1 \\ \boxed{1} & \boxed{\$} \end{array}$) unsuccessful (d too big)



Compressed tries search: Example

Example 2: CompressedTrie::search(

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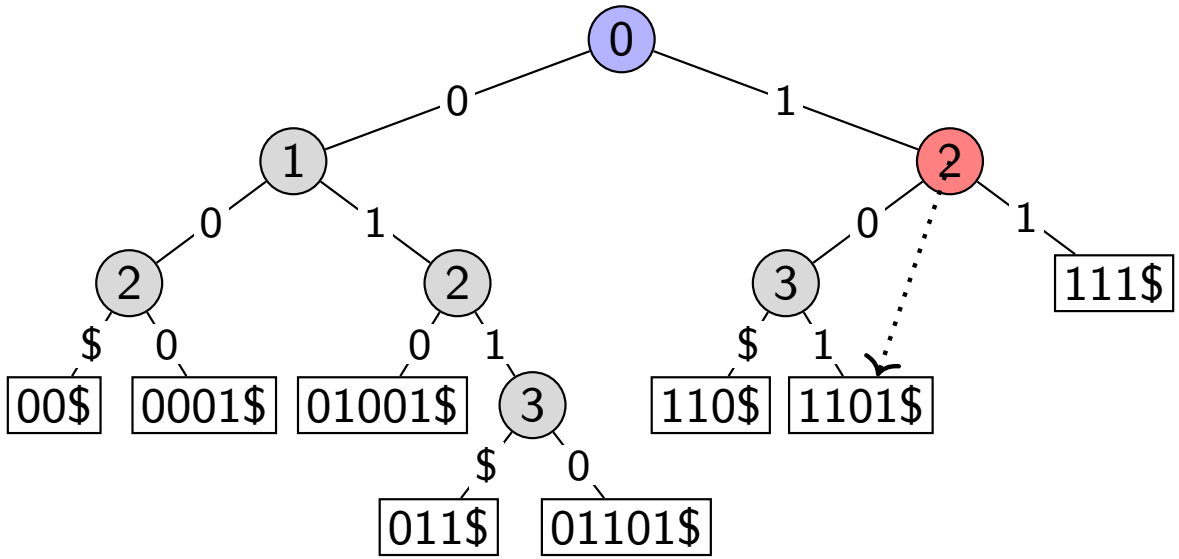
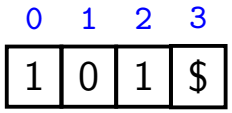


prefix-search(w): Compare w to z.*leaf* at last visited node z.

Compressed tries search: Example

Example 3: CompressedTrie::search(

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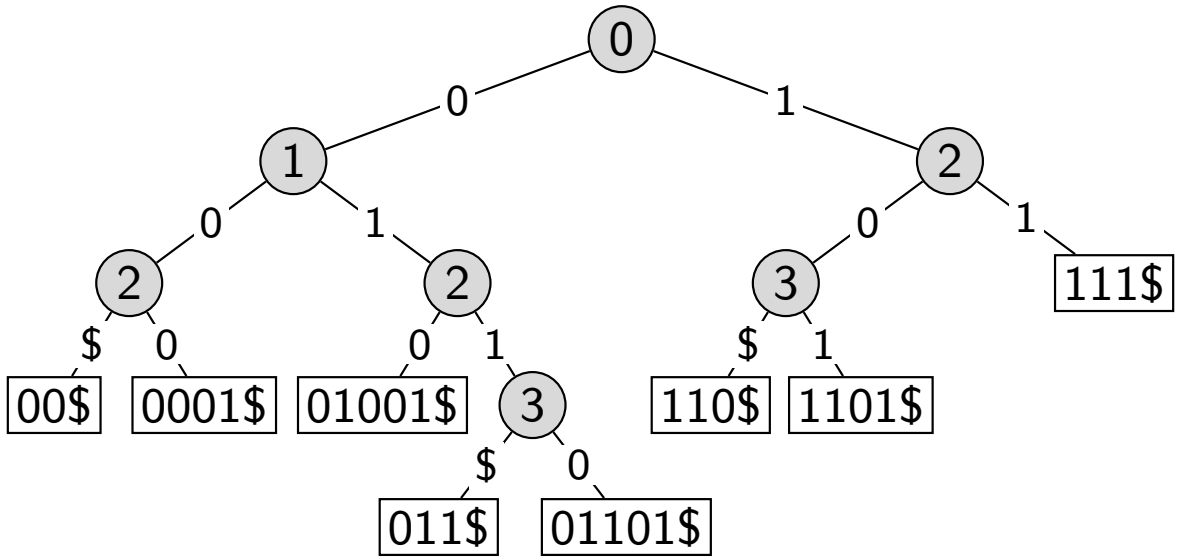
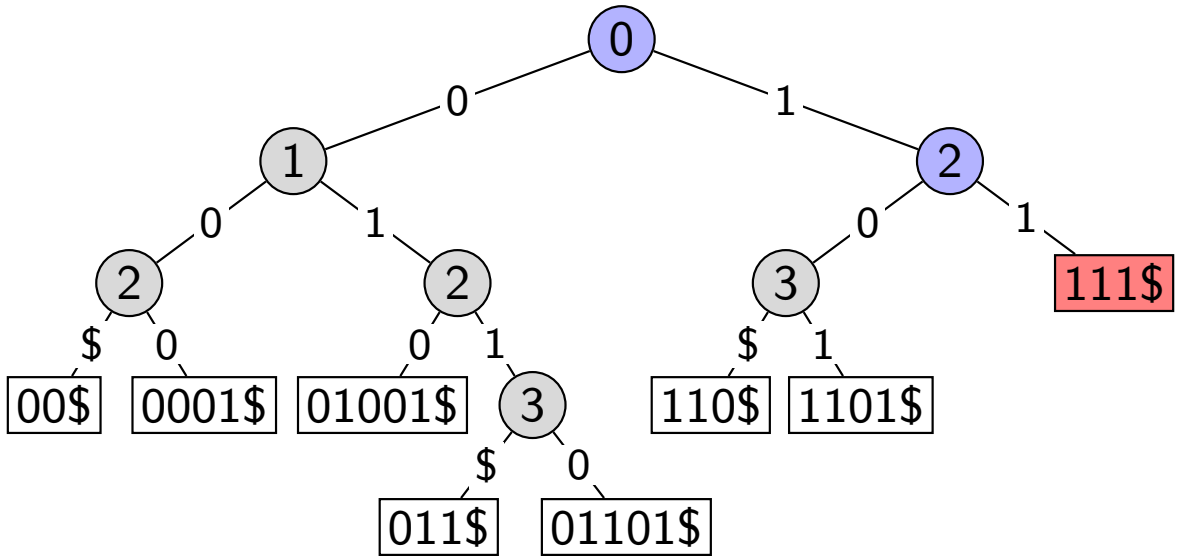


Compressed tries search: Example

Example 3: CompressedTrie::search(| 1 | 0 | 1 | $ |) unsuccessful (wrong word at leaf)



Compressed tries search: Example

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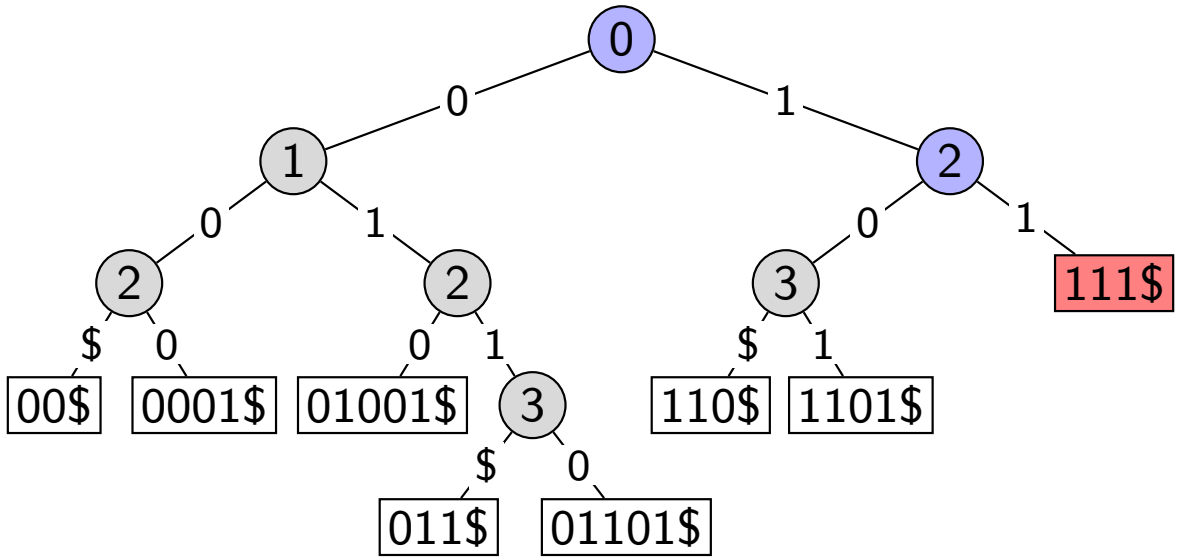


prefix-search(w): Compare w to word at reached leaf.

Compressed tries: Summary

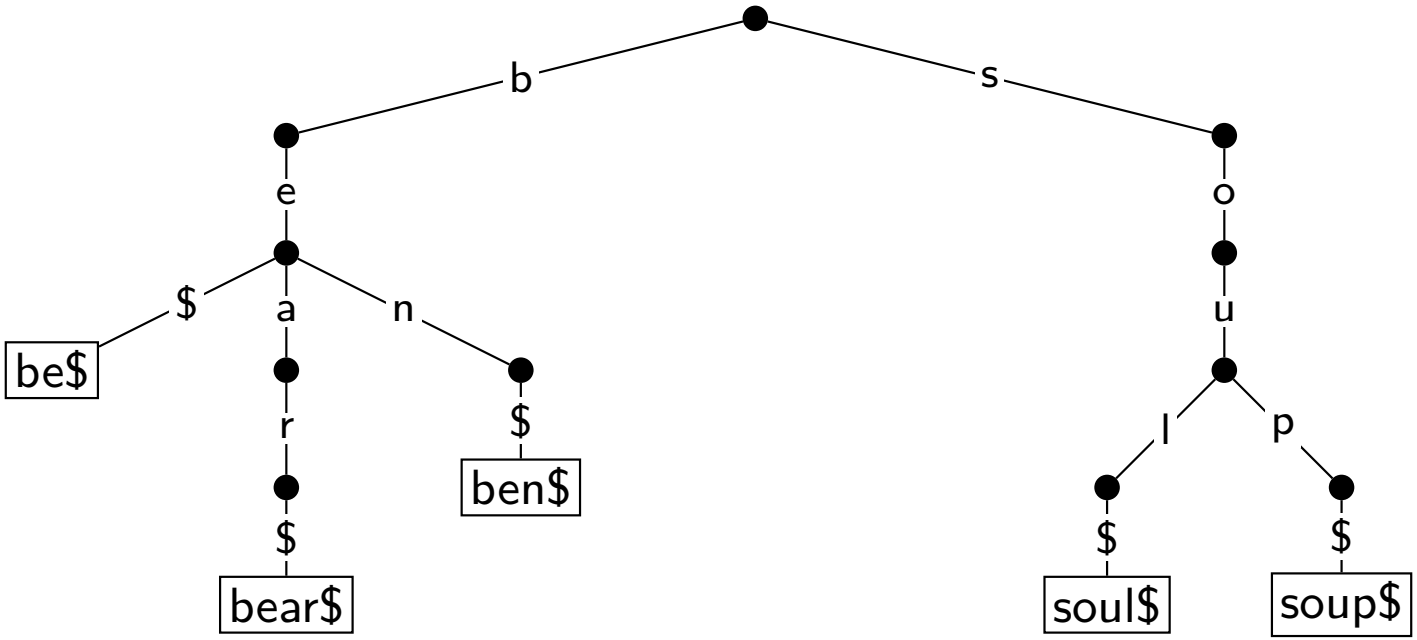
- *search*(w) and *prefix-search*(w) are fairly easy.
- *insert*(w) and *delete*(w) are conceptually simple by *uncompressing*:
  - Search for path P to word w (say we reach node z)
  - Uncompress this path (using characters of $z.leaf$)
  - Insert/Delete w as in an uncompressed trie.
  - Compress path from root to where change happened

  (Pseudocode gets more complicated and is omitted.)
- All operations take $O(|w|)$ time for a word w.
- Compressed tries use $O(n)$ space (better than other trie-variants)

Overall, code is more complicated, but space-savings are worth it if words are unevenly distributed.

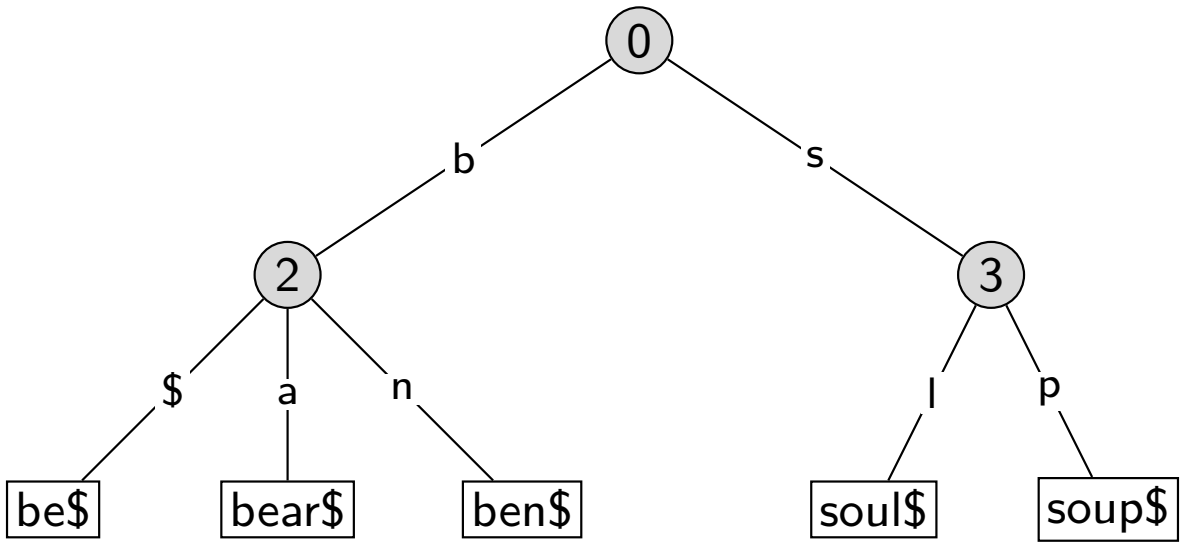
Multiway tries for larger alphabets

- To represent *strings* over any *fixed alphabet* Σ
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



Compressed multiway tries

- **Variation**: Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear$, ben$, be$, soul$, soup$}



Multiway tries: Summary

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- Run-time $O(|w| \cdot (\text{time to find the appropriate child}))$

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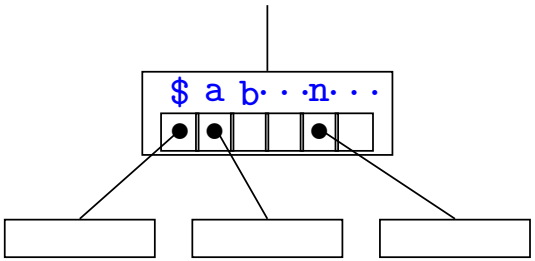
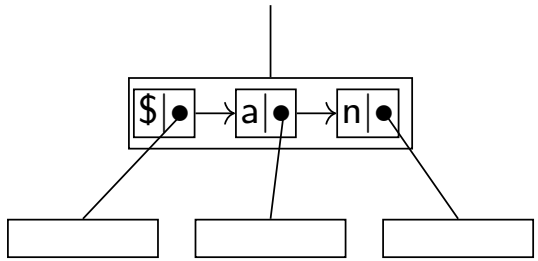
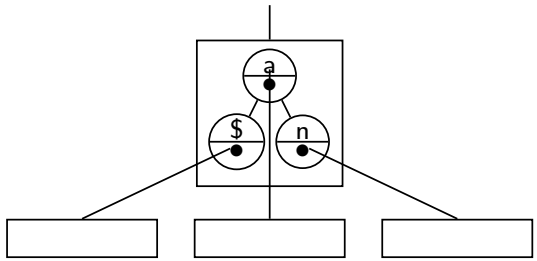
- Each node now has up to $|\Sigma| + 1$ children. How should they be stored?



Array?



List?



Dictionary?

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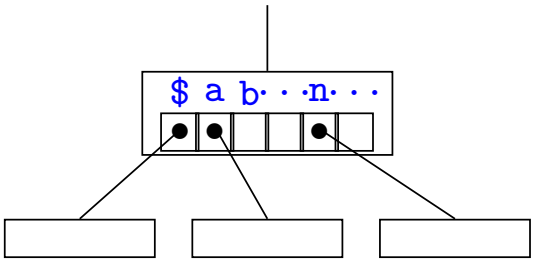

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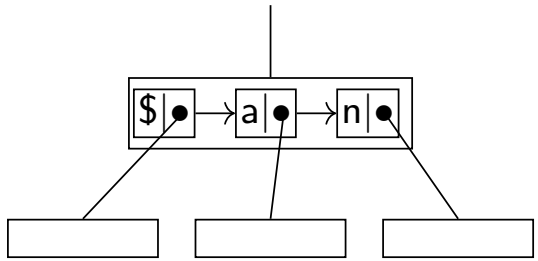

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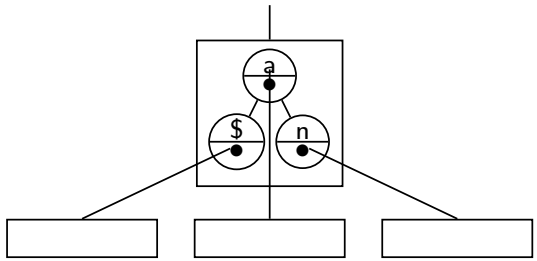

Dictionary?

- Time/space tradeoff: arrays are fast, lists are space-efficient.
- Dictionary best in theory, not worth it in practice unless $|\Sigma|$ is huge.
- In practice, use *direct addressing* or *hashing* ($\rightarrow$ module 07).