Module 8: Range-Searching in Dictionaries for Points

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Based on lecture notes by many previous cs240 instructors

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References: Goodrich & Tamassia 21.1, 21.3
Outline

1. Range-Searching in Dictionaries for Points
   - Range Searches
   - Multi-Dimensional Data
   - Quadtrees
   - kd-Trees
   - Range Trees
   - Conclusion
Outline

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Range searches

- So far: $search(k)$ looks for one specific item.
- New operation $RangeSearch$: look for all items that fall within a given range.
  - Input: A range, i.e., an interval $I = (x, x')$
    - It may be open or closed at the ends.
  - Want: Report all KVPs in the dictionary whose key $k$ satisfies $k \in I$

**Example:**

<table>
<thead>
<tr>
<th>5</th>
<th>10</th>
<th>11</th>
<th>17</th>
<th>19</th>
<th>33</th>
<th>45</th>
<th>51</th>
<th>55</th>
<th>59</th>
</tr>
</thead>
</table>

$RangeSearch((18,45])$ should return \{19, 33, 45\}
Range searches

- So far: \textit{search}(k) looks for \textit{one} specific item.
- New operation \textbf{RangeSearch}: look for \textit{all} items that fall within a given range.
  - Input: A \textit{range}, i.e., an interval \( I = (x, x') \)
    - It may be open or closed at the ends.
  - Want: Report all KVPs in the dictionary whose key \( k \) satisfies \( k \in I \)

\[ \begin{array}{cccccccc}
5 & 10 & 11 & 17 & 19 & 33 & 45 & 51 & 55 & 59 \\
\end{array} \]

\textbf{Example:} \texttt{RangeSearch}((18,45]) should return \{19, 33, 45\}

- Let \( s \) be the \textbf{output-size}, i.e., the number of items in the range.
- We need \( \Omega(s) \) time simply to report the items.
- Note that sometimes \( s = 0 \) and sometimes \( s = n \); we therefore keep it as a separate parameter when analyzing the run-time.
Range searches in existing dictionary realizations

**Unsorted list/array/hash table**: Range search requires $\Omega(n)$ time: We have to check for each item explicitly whether it is in the range.

**Sorted array**: Range search in $A$ can be done in $O(\log n + s)$ time:

\[
\text{RangeSearch}( (18,45] ) \begin{array}{cccccccc}
5 & 10 & 11 & 17 & 19 & 33 & 45 & 51 & 55 & 59 \\
\uparrow i & \uparrow i' \\
\end{array}
\]

- Using binary search, find $i$ such that $x$ is at (or would be at) $A[i]$.
- Using binary search, find $i'$ such that $x'$ is at (or would be at) $A[i']$
- Report all items $A[i+1...i'-1]$.
- Report $A[i]$ and $A[i']$ if they are in range.

**BST**: Range searches can similarly be done in time $O(\text{height}+s)$ time. We will see this in detail later.
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Multi-Dimensional Data

Range searches are of special interest for **multi-dimensional data**.

**Example**: flights that leave between 9am and noon, and cost $300-$500

- Each item has \( d \) **aspects** (coordinates): \((x_0, x_1, \cdots, x_{d-1})\)
- Aspect values \((x_i)\) are numbers
- Each item corresponds to a point in \( d \)-dimensional space
- We concentrate on \( d = 2 \), i.e., points in Euclidean plane
(Orthogonal) *d-dimensional range search*: Given a query rectangle \( A \), find all points that lie within \( A \).

The time for range searches depends on how the points are stored.

- Could store a 1-dimensional dictionary (where the key is some combination of the aspects.)
  Problem: Range search on one aspect is not straightforward

- Could use one dictionary for each aspect
  Problem: inefficient, wastes space

- **Better idea**: Design new data structures specifically for points.
  - Quadtrees
  - \( kd \)-trees
  - range-trees

- **Assumption**: Points are in general position: No two \( x \)-coordinates or \( y \)-coordinates are the same.
  - Simplifies presentation; data structures can be generalized.
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Quadtrees

We have \( n \) points \( S = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\} \) in the plane.

We need a **bounding box** \( R \): a square containing all points.

- Can find \( R \) by computing minimum and maximum \( x \) and \( y \) values in \( S \)
- The width/height of \( R \) should be a power of 2

**Structure** (and also how to *build* the quadtree that stores \( S \)):

- Root \( r \) of the quadtree is associated with region \( R \)
- If \( R \) contains 0 or 1 points, then root \( r \) is a leaf that stores point.
- Else **split**: Partition \( R \) into four equal subsquares (**quadrants**) \( R_{NE}, R_{NW}, R_{SW}, R_{SE} \)
- Partition \( S \) into sets \( S_{NE}, S_{NW}, S_{SW}, S_{SE} \) of points in these regions.
  - **Convention**: Points on split lines belong to right/top side
- Recursively build tree \( T_i \) for points \( S_i \) in region \( R_i \) and make them children of the root.
Quadtrees example

\[\begin{array}{cccc}
p_3 & p_9 & \bullet & p_4 \\
\bullet & p_1 & \bullet & p_8 \\
p_0 & p_6 & \bullet & p_5 \\
p_2 & \bullet & p_7 & \end{array}\]

\(([0, 16] \times [0, 16])\)
Quadtrees example

Easier for humans: omit empty sub-trees, label edges
Quadtrees example

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Easier for humans: omit empty subtrees, label edges
Quadtree Dictionary Operations

- **search**: Analogous to binary search trees and tries
- **insert**:
  - Search for the point
  - Split the leaf while there are two points in one region
- **delete**:  
  - Search for the point
  - Remove the point
  - If its parent has only one point left: delete parent (and recursively all ancestors that have only one point left)
Quadtree Insert example

\[ \text{insert}(p_{10}) \]
Quadtree Insert example

\[
\begin{align*}
\text{\textit{insert}}(p_{10})
\end{align*}
\]
Quadtree Range Search

```cpp
QTree::RangeSearch(r ← root, A)

r: The root of a quadtree, A: Query-rectangle
1. \( R ← \) region associated with node \( r \)
2. \( \text{if} \ (R \subseteq A) \text{ then} \) // inside node
3. \( \text{report all points below } r; \text{ return} \)
4. \( \text{if} \ (R \cap A \text{ is empty}) \text{ then} \) // outside node
5. \( \text{return} \)
6. \( \text{if} \ (r \text{ is a leaf}) \text{ then} \)
7. \( p ← \) point stored at \( r \)
8. \( \text{if} \ p \text{ is in } A \text{ return } p \)
9. \( \text{else return} \)
10. \( \text{for each child } v \text{ of } r \text{ do} \)
11. \( QTree::RangeSearch(v, A) \)
```

Note: We assume here that each node of the quadtree stores the associated square. Alternatively, these could be re-computed during the search (space-time tradeoff).
Quadtree range search example

Red: Search stopped due to $R \cap A = \emptyset$.
Green: Search stopped due to $R \subseteq A$.
Blue: Must continue search in children / evaluate.

$[0, 16) \times [0, 16)$
Quadtree range search example

- Red: Search stopped due to $R \cap A = \emptyset$.
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- Blue: Must continue search in children / evaluate.
Quadtree Analysis

- Crucial for analysis: what is the height of a quadtree?
  - Can have very large height for bad distributions of points

- **spread factor** of points $S$:

  $$\beta(S) = \frac{\text{sidelength of } R}{\text{minimum distance between points in } S}$$

- Can show: height $h$ of quadtree is in $\Theta(\log \beta(S))$

- Complexity to build initial tree: $\Theta(nh)$ worst-case
- Complexity of range search: $\Theta(nh)$ worst-case even if the answer is $\emptyset$
- But in practice much faster.
Quadtrees in other dimensions

- Quad-tree of 1-dimensional points:

<table>
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<tr>
<th>“Points:”</th>
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Same as a trie (with splitting stopped once key is unique)

Quadtrees also easily generalize to higher dimensions (octrees, etc.) but are rarely used beyond dimension 3.
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<td>00000</td>
<td>01001</td>
<td>01100</td>
<td>01110</td>
<td>11000</td>
<td>11010</td>
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Quadtrees in other dimensions

- Quad-tree of 1-dimensional points:

```
“Points:”     0   9  12  14  24  26  28
(in base-2)  00000 01001 01100 01110 11000 11010 11100
```

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- Quadtrees also easily generalize to higher dimensions (octrees, etc.) but are rarely used beyond dimension 3.
Quadtree summary

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (bit-shift!) if the width/height of $R$ is a power of 2
- Space potentially wasteful, but good if points are well-distributed
- Variation: We could stop splitting earlier and allow up to $S$ points in a leaf (for some fixed bound $S$).
- Variation: Store pixelated images by splitting until each region has the same color.
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We have \( n \) points \( S = \{ (x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}) \} \).

Quadtrees split square into quadrants regardless of where points are.

(Point-based) kd-tree idea: Split the region such that (roughly) half the points are in each subtree.

Each node of the kd-tree keeps track of a splitting line in one dimension (2D: either vertical or horizontal).

**Convention:** Points on split lines belong to right/top side.

Continue splitting, switching between vertical and horizontal lines, until every point is in a separate region.

(There are alternatives, e.g., split by the dimension that has better aspect ratios for the resulting regions. No details.)
kd-tree example

For ease of drawing, we will usually not show the associated regions.
kd-tree example

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\[ \mathbb{R}^2 \]

\[ x < p_8.x? \]
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Constructing \( k \)-d-trees

Build \( k \)-d-tree with initial split by \( x \) on points \( S \):

- If \( |S| \leq 1 \) create a leaf and return.
- Else \( X := \text{quick-select}(S, \lfloor n/2 \rfloor) \) (select by \( x \)-coordinate)
- Partition \( S \) by \( x \)-coordinate into \( S_{x<X} \) and \( S_{x\geq X} \)
  - \( \lfloor n/2 \rfloor \) points on one side and \( \lceil n/2 \rceil \) points on the other.
    (Recall: Points in general position.)
- Create left subtree recursively (splitting by \( y \)) for points \( S_{x<X} \).
- Create right subtree recursively (splitting by \( y \)) for points \( S_{x\geq X} \).

Building with initial \( y \)-split symmetric.
Constructing kd-trees

**Run-time:**
- Find $X$ and partition $S$ in $\Theta(n)$ expected time using *randomized-quick-select*.
- Both subtrees have $\approx n/2$ points.

$$T^{\exp}(n) = 2T^{\exp}(n/2) + O(n) \quad \text{(sloppy recurrence)}$$

This resolves to $\Theta(n \log n)$ expected time.
- This can be reduced to $\Theta(n \log n)$ *worst-case* time by pre-sorting (no details).

**Height:** $h(1) = 0, \ h(n) \leq h(\lceil n/2 \rceil) + 1$.
- This resolves to $O(\log n)$ (specifically $\lceil \log n \rceil$).
**kd-tree Dictionary Operations**

- **search** (for single point): as in binary search tree using indicated coordinate
- **insert**: search, insert as new leaf.
- **delete**: search, remove leaf.

**Problem**: After insert or delete, the split might no longer be at exact median and the height is no longer guaranteed to be $\lceil \log_2 n \rceil$.

We can maintain $O(\log n)$ height by occasionally re-building entire subtrees. (No details.) But *rangeSearch* will be slower.

kd-trees do not handle insertion/deletion well.
**kd-tree Range Search**

- Range search is *exactly* as for quad-trees, except that there are only two children.

```cpp
kdTree::RangeSearch(r ← root, A)

1. R ← region associated with node r
2. if (R ⊆ A) then report all points below r; return
3. if (R ∩ A is empty) then return
4. if (r is a leaf) then
5.   p ← point stored at r
6.   if p is in A return p
7.   else return
8. for each child v of r do
9.   kdTree::RangeSearch(v, A)
```

- We assume again that each node stores its associated region.
- To save space, we could instead pass the region as a parameter and compute the region for each child using the splitting line.
kd-tree: Range Search Example

- \( p_0 \)
- \( p_1 \)
- \( p_2 \)
- \( p_3 \)
- \( p_4 \)
- \( p_5 \)
- \( p_6 \)
- \( p_7 \)
- \( p_8 \)
- \( p_9 \)

**Red**: Search stopped due to \( \mathcal{R} \cap \mathcal{A} = \emptyset \).

**Green**: Search stopped due to \( \mathcal{R} \subseteq \mathcal{A} \).
kd-tree: Range Search Example

Red: Search stopped due to $R \cap A = \emptyset$. Green: Search stopped due to $R \subseteq A$. 
The complexity is $O(s + Q(n))$ where

- $s$ is the output-size
- $Q(n)$ is the number of “boundary” nodes (blue):
  - `kdTree::RangeSearch` was called.
  - Neither $R \subseteq A$ nor $R \cap A = \emptyset$

**Can show:** $Q(n)$ satisfies the following recurrence relation (no details):

$$Q(n) \leq 2Q(n/4) + O(1)$$

This solves to $Q(n) \in O(\sqrt{n})$

Therefore, the complexity of range search in kd-trees is $O(s + \sqrt{n})$
kd-tree: Higher Dimensions

- kd-trees for $d$-dimensional space:
  - At the root the point set is partitioned based on the first coordinate
  - At the subtrees of the root the partition is based on the second coordinate
  - At depth $d-1$ the partition is based on the last coordinate
  - At depth $d$ we start all over again, partitioning on first coordinate

- **Storage**: $O(n)$
- **Height**: $O(\log n)$
- **Construction time**: $O(n \log n)$
- **Range search time**: $O(s + n^{1-1/d})$

This assumes that $d$ is a constant.
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Towards Range Trees

- Both Quadtrees and kd-trees are intuitive and simple.
- But: both may be very slow for range searches.
- Quadtrees are also potentially wasteful in space.

New idea: **Range trees**

- Somewhat wasteful in space, but much faster range search.
- **Tree of trees** (a *multi-level* data structure)
2-dimensional Range Trees

**Primary structure:**
Balanced binary search tree $T$ that stores $P$ and uses $x$-coordinates as keys.

Each node $v$ of $T$ stores an **associate structure** $T(v)$:
- Let $P(v)$ be all points in subtree of $v$ in $T$ (including point at $v$)
- $T(v)$ stores $P(v)$ in a balanced binary search tree, using the $y$-coordinates as key
- Note: $v$ is not necessarily the root of $T(v)$
Range tree example
Range tree example
Range tree example

Not all associate trees are shown.
Range Tree Space Analysis

- Primary tree uses $O(n)$ space.
- Associate tree $T(v)$ uses $O(|P(v)|)$ space (where $P(v)$ are the points at descendants of $v$ in $T$)
- **Key insight:** $w \in P(v)$ means that $v$ is an ancestor of $w$ in $T$
  - Every node $w$ has $O(\log n)$ ancestors in $T$ (Recall that we assume $T$ to be balanced.)
  - Every node $w$ belongs to $O(\log n)$ sets $P(v)$
  - So $\sum_v |P(v)| \leq \sum_w \#\{\text{ancestors of } w\} \in O(n \log n)$

Therefore: A range-tree with $n$ points uses $O(n \log n)$ space.
Range Trees Operations

- **search**: search by $x$-coordinate in $T$
- **insert**: First, insert point by $x$-coordinate into $T$. Then, walk back up to the root and insert the point by $y$-coordinate in *all* associate trees $T(v)$ of nodes $v$ on path to the root.
- **delete**: analogous to insertion

**Problem**: We want the binary search trees to be balanced.
  - This makes *insert/delete* very slow if we use AVL-trees.
  - (A rotation at $v$ changes $P(v)$ and hence requires a re-build of $T(v)$.)
  - **Solution**: Completely rebuild highly unbalanced subtrees (no details)
Range Trees Operations

- **search**: search by $x$-coordinate in $T$
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- This makes **insert/delete** very slow if we use AVL-trees. (A rotation at $v$ changes $P(v)$ and hence requires a re-build of $T(v)$.)
- **Solution**: Completely rebuild highly unbalanced subtrees (no details)

- **range-search**: search by $x$-range in $T$. Among found points, search by $y$-range in some associated trees.
- Must understand first: How to do (1-dimensional) range search in binary search tree?
BST Range Search

**BST::RangeSearch**($r \leftarrow root, x_1, x_2$)

*r*: root of a binary search tree, $x_1, x_2$: search keys

Returns keys in subtree at $r$ that are in range $[x_1, x_2]$

1. **if** $r = \text{NIL}$ **then** return
2. **if** $x_1 \leq r.key \leq x_2$ **then**
   3. $L \leftarrow BST::RangeSearch(r.left, x_1, x_2)$
   4. $R \leftarrow BST::RangeSearch(r.right, x_1, x_2)$
   5. **return** $L \cup r\{\text{key}\} \cup R$
3. **if** $r.key < x_1$ **then**
   7. **return** $BST::RangeSearch(r.right, x_1, x_2)$
5. **if** $r.key > x_2$ **then**
   9. **return** $BST::RangeSearch(r.left, x_1, x_2)$

Keys are reported in in-order, i.e., in sorted order.
BST Range Search example

$BST::\text{RangeSearch}(T, 28, 43)$

Note: Search from 39 was unnecessary: all its descendants are in range.
BST Range Search example

\textit{BST::RangeSearch}(T, 28, 43)

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BST Range Search example

$BST::RangeSearch(T, 28, 43)$

Note: Search from 39 was unnecessary: all its descendants are in range.
BST Range Search re-phrased

- Search for left boundary \( x_1 \): this gives path \( P_1 \)
- Search for right boundary \( x_2 \): this gives path \( P_2 \)
- This partitions \( T \) into three groups: outside, on, or between the paths.
**boundary nodes**: nodes in $P_1$ or $P_2$
  - For each boundary node, test whether it is in the range.

**outside nodes**: nodes that are left of $P_1$ or right of $P_2$
  - These are *not* in the range, we stop the search at the topmost.

**inside nodes**: nodes that are right of $P_1$ and left of $P_2$
  - We stop the search at the topmost inside node.
  - All descendants of such a node are *in* the range.
  For a 1d range search, report them.
BST Range Search analysis

Assume that the binary search tree is balanced:

- Search for path $P_1$: $O(\log n)$
- Search for path $P_2$: $O(\log n)$
- $O(\log n)$ boundary nodes
- We spend $O(1)$ time on each.
BST Range Search analysis

Assume that the binary search tree is balanced:

- Search for path $P_1$: $O(\log n)$
- Search for path $P_2$: $O(\log n)$
- $O(\log n)$ boundary nodes
- We spend $O(1)$ time on each.

- We spend $O(1)$ time per topmost outside node.
  - They are children of boundary nodes, so this takes $O(\log n)$ time.
- We spend $O(1)$ time per topmost inside node $v$.
  - They are children of boundary nodes, so this takes $O(\log n)$ time.
- For 1d range search, also report the descendants of $v$.
  - We have $\sum_{v \text{ topmost inside}} \#\{\text{descendants of } v\} \leq s$ since subtrees of topmost inside nodes are disjoint. So this takes time $O(s)$ overall.

Run-time for 1d range search: $O(\log n + s)$. This is no faster overall, but topmost inside nodes will be important for 2d range search.
Range Trees: Range Search

Range search for $A = [x_1, x_2] \times [y_1, y_2]$ is a two stage process:

- Perform a range search (on the $x$-coordinates) for the interval $[x_1, x_2]$ in primary tree $T$ ($BST::RangeSearch(T, x_1, x_2)$)
- Get boundary, topmost outside and topmost inside nodes as before.
- For every boundary node, test to see if the corresponding point is within the region $A$.
- For every topmost inside node $v$:
  - Let $P(v)$ be the points in the subtree of $v$ in $T$.
  - We know that all $x$-coordinates of points in $P(v)$ are within range.
  - Recall: $P(v)$ is stored in $T(v)$.
  - To find points in $P(v)$ where the $y$-coordinates are within range as well, perform a range search in $T(v)$: $BST::RangeSearch(T(v), y_1, y_2)$
Range tree range search example
Range tree range search example

![Diagram of range tree]

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Range tree range search example

primary tree $T$
Range tree range search example

Diagram of a range tree with nodes labeled from 1 to 16, illustrating how points are indexed and queried.
Range tree range search example

![Range Tree Diagram]

- $T(6) \rightarrow (1, 5) \rightarrow (2, 7) \rightarrow (3, 1) \rightarrow (4, 4)$
- $T(6) \rightarrow (5, 13) \rightarrow (6, 15)$
- $T(6) \rightarrow (7, 11) \rightarrow (8, 10) \rightarrow (9, 6)$
- $T(6) \rightarrow (10, 12)$
- $T(6) \rightarrow (11, 8)$
- $T(6) \rightarrow (12, 14)$
- $T(6) \rightarrow (13, 2)$
- $T(6) \rightarrow (14, 9)$
- $T(6) \rightarrow (15, 16)$
Range tree range search example

![Diagram of range tree range search example]
Range Trees: Range Search Run-time

- $O(\log n)$ time to find boundary and topmost inside nodes in primary tree.
- There are $O(\log n)$ such nodes.
- $O(\log n + s_v)$ time for each topmost inside node $v$, where $s_v$ is the number of points in $T(v)$ that are reported
- Two topmost inside nodes have no common point in their trees
  $\Rightarrow$ every point is reported in at most one associate structure
  $\Rightarrow \sum_v \text{topmost inside } s_v \leq s$

Time for range search in range-tree is proportional to

$$\sum_{v \text{ topmost inside}} (\log n + s_v) \in O(\log^2 n + s)$$

(There are ways to make this even faster. No details.)
Range Trees: Higher Dimensions

- Range trees can be generalized to $d$-dimensional space.

**Space** \( O(n (\log n)^{d-1}) \)

**Construction time** \( O(n (\log n)^d) \)

**Range search time** \( O(s + (\log n)^d) \)

(Note: \( d \) is considered to be a constant.)
Range Trees: Higher Dimensions

- Range trees can be generalized to $d$-dimensional space.

<table>
<thead>
<tr>
<th></th>
<th>Space</th>
<th>Construction time</th>
<th>Range search time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$O(n (\log n)^{d-1})$</td>
<td>$O(n (\log n)^d)$</td>
<td>$O(s + (\log n)^d)$</td>
</tr>
<tr>
<td>kd-trees</td>
<td>$O(n)$</td>
<td>$O(n \log n)$</td>
<td>$O(s + n^{1-1/d})$</td>
</tr>
</tbody>
</table>

(Note: $d$ is considered to be a constant.)

- Space/time trade-off compared to kd-trees.
Outline

1. Range-Searching in Dictionaries for Points
   - Range Searches
   - Multi-Dimensional Data
   - Quadtrees
   - kd-Trees
   - Range Trees
   - Conclusion
Range search data structures summary

- **Quadtrees**
  - simple (also for dynamic set of points)
  - work well only if points evenly distributed
  - wastes space for higher dimensions

- **kd-trees**
  - linear space
  - range search time $O(\sqrt{n} + s)$
  - inserts/deletes destroy balance and range search time (no simple fix)

- **range-trees**
  - range search time $O(\log^2 n + s)$
  - wastes some space
  - inserts/deletes destroy balance (can fix this with occasional rebuilt)

**Convention:** Points on split lines belong to right/top side.