CS 240 – Data Structures and Data Management

Module 9: String Matching

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Based on lecture notes by many previous cs240 instructors

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Winter 2021

version *2021-03-22 10:18*

Outline

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Pattern Matching Definition [1]

- Search for a string (pattern) in a large body of text
- $T[0..n-1]$ – The text (or haystack) being searched within
- $P[0..m-1]$ – The pattern (or needle) being searched for
- Strings over alphabet Σ
- Return smallest i such that

$$P[j] = T[i+j] \quad \text{for} \quad 0 \le j \le m-1$$

- This is the first occurrence of P in T
- If P does not occur in T, return `FAIL`
- Applications:
  - Information Retrieval (text editors, search engines)
  - Bioinformatics
  - Data Mining

Pattern Matching Definition [2]

Example:

- $T =$ "`Where is he?`"
- $P_1 =$ "`he`"
- $P_2 =$ "`who`"

Definitions:

- Substring $T[i..j]$ $0 \leq i \leq j < n$: a string of length $j - i + 1$ which consists of characters $T[i], \ldots T[j]$ in order
- A prefix of T:
  a substring $T[0..i]$ of T for some $0 \leq i < n$
- A suffix of T:
  a substring $T[i..n-1]$ of T for some $0 \leq i \leq n - 1$

General Idea of Algorithms

Pattern matching algorithms consist of guesses and checks:

- A **guess** or **shift** is a position i such that P might start at $T[i]$. Valid guesses (initially) are $0 \leq i \leq n - m$.
- A **check** of a guess is a single position j with $0 \leq j < m$ where we compare $T[i + j]$ to $P[j]$. We must perform m checks of a single correct guess, but may make (many) fewer checks of an incorrect guess.

We will diagram a single run of any pattern matching algorithm by a matrix of checks, where each row represents a single guess.

Brute-force Algorithm

Idea: Check every possible guess.

```
Bruteforce::patternMatching(T[0..n − 1], P[0..m − 1])
T: String of length n (text), P: String of length m (pattern)
    for i ← 0 to n − m do
        if strcmp(T[i..i+m−1], P) = 0
            return "found at guess i"
    return FAIL
```

Note: *strcmp* takes $\Theta(m)$ time.

```
strcmp(T[i..i+m−1], P[0..m − 1])
    for j ← 0 to m − 1 do
        if T[i + j] is before P[j] in Σ then return -1
        if T[i + j] is after P[j] in Σ then return 1
    return 0
```

Brute-Force Example

- Example: $T =$ `abbbababbab`, $P =$ `abba`

a	b	b	b	a	b	a	b	b	a	b
a	b	b	a							
	a									
		a								
			a							
				a	b	b				
					a					
						a	b	b	a	

- What is the worst possible input?
  $P = a^{m-1}b$, $T = a^n$
- Worst case performance $\Theta((n-m+1)m)$
- This is $\Theta(mn)$ e.g. if $m = n/2$.

How to improve?

More sophisticated algorithms

- Do extra preprocessing on the pattern P
  - **Karp-Rabin**
  - **Boyer-Moore**
  - Deterministic finite automata (**DFA**), **KMP**
  - We eliminate guesses based on completed matches and mismatches.
- Do extra preprocessing on the text T
  - **Suffix-trees**
  - We create a data structure to find matches easily.

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   - Introduction
   - Karp-Rabin Algorithm
   - String Matching with Finite Automata
   - Knuth-Morris-Pratt algorithm
   - Boyer-Moore Algorithm
   - Suffix Trees
   - Suffix Arrays
   - Conclusion

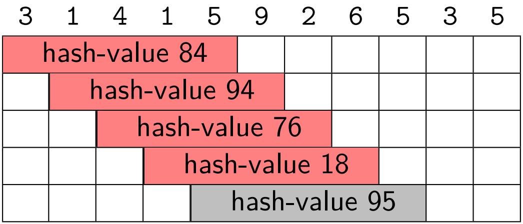
Karp-Rabin Fingerprint Algorithm – Idea

Idea: use hashing to eliminate guesses

- Compute hash function for each guess, compare with pattern hash
- If values are unequal, then the guess cannot be an occurrence
- Example: $P = 5\ 9\ 2\ 6\ 5$, $\qquad T = 3\ 1\ 4\ 1\ 5\ 9\ 2\ 6\ 5\ 3\ 5$
  - Use standard hash-function: flattening + modular (radix $R = 10$):

$$h(x_0 \ldots x_4) = \big(x_0x_1x_2x_3x_4\big)_{10} \bmod 97$$

  - $h(P) = 59265 \bmod 97 = 95.$



Karp-Rabin Fingerprint Algorithm – First Attempt

```
Karp-Rabin-Simple::patternMatching(T, P)
    h_P ← h(P[0..m−1)])
    for i ← 0 to n − m
        h_T ← h(T[i..i+m−1])
        if h_T = h_P
            if strcmp(T[i..i+m−1], P) = 0
                return "found at guess i"
    return FAIL
```

- Never misses a match: $h(T[i..i+m-1]) \neq h(P) \Rightarrow$ guess i is not P
- $h(T[i..i+m-1])$ depends on m characters, so naive computation takes $\Theta(m)$ time per guess
- Running time is $\Theta(mn)$ if P not in T (how can we improve this?)

Karp-Rabin Fingerprint Algorithm – Fast Update

The initial hashes are called **fingerprints**.

Crucial insight: We can update these fingerprints in constant time.

- Use previous hash to compute next hash
- $O(1)$ time per hash, except first one

Example:

- Pre-compute: $10000 \bmod 97 = 9$
- Previous hash: $41592 \bmod 97 = 76$
- Next hash: $15926 \bmod 97 = ??$

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Example:

- Pre-compute: $10000 \bmod 97 = 9$
- Previous hash: $41592 \bmod 97 = 76$
- Next hash: $15926 \bmod 97 = ??$

Observe: $15926 = (41592 - 4 \cdot 10\,000) \cdot 10 + 6$

$$15926 \bmod 97 = \Big((\underbrace{41592 \bmod 97}_{76 \text{ (previous hash)}} - 4 \cdot \underbrace{10000 \bmod 97}_{9 \text{ (pre-computed)}} \Big) \cdot 10 + 6) \bmod 97$$

$$= \Big((76 - 4 \cdot 9) \cdot 10 + 6 \Big) \bmod 97 = 18$$

Karp-Rabin Fingerprint Algorithm – Conclusion

Karp-Rabin-RollingHash::patternMatching(T, P)

```
M ← suitable prime number
h_P ← h(P[0..m−1)])
h_T ← h(T[0..m−1)])
s ← 10^(m−1) mod M
for i ← 0 to n − m
    if h_T = h_P
        if strcmp(T[i..i+m−1], P) = 0
            return "found at guess i"
    if i < n − m // compute hash-value for next guess
        h_T ← ((h_T − T[i] · s) · 10 + T[i+m]) mod M
return "FAIL"
```

- Choose "table size" M at random to be huge prime
- Expected running time is $O(m + n)$
- $\Theta(mn)$ worst-case, but this is (unbelievably) unlikely

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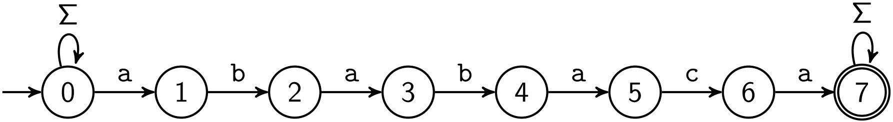
String Matching with Finite Automata

Example: Automaton for the pattern $P =$ `ababaca`

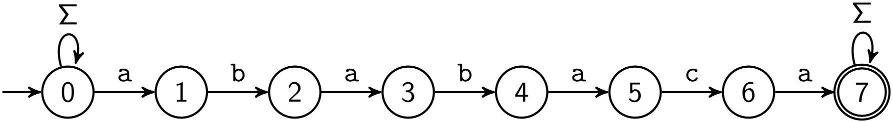


You should be familiar with:

- finite automaton, DFA, NFA, converting NFA to DFA
- transition function δ, states Q, accepting states F

String Matching with Finite Automata

Example: Automaton for the pattern $P = \texttt{ababaca}$



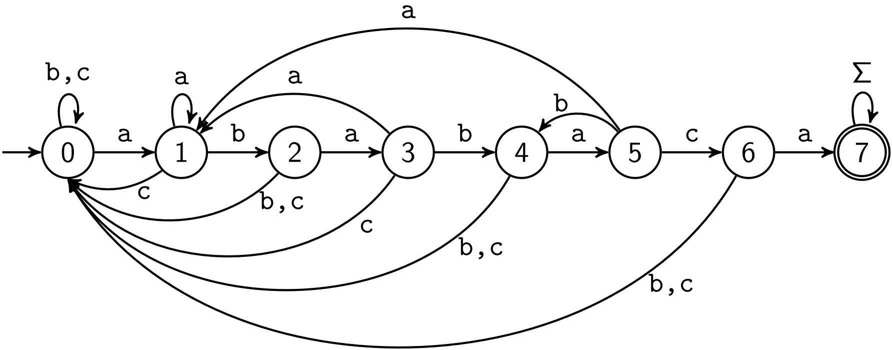
You should be familiar with:

- finite automaton, DFA, NFA, converting NFA to DFA
- transition function δ, states Q, accepting states F

- The above finite automation is an **NFA**
- State q expresses "we have seen $P[0..q-1]$"
  - NFA accepts T if and only if T contains $\texttt{ababaca}$
  - But evaluating NFAs is very slow.

String matching with DFA

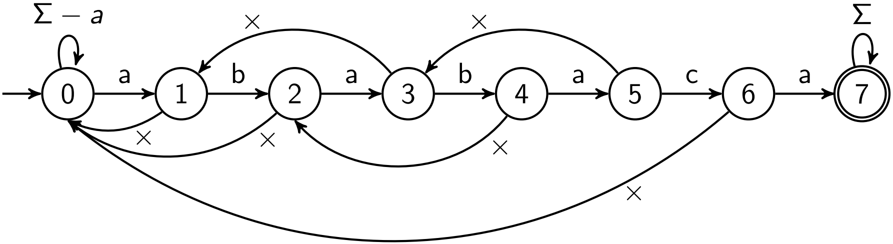
Can show: There exists an equivalent small DFA.



- Easy to test whether P is in T.
- But how do we find the arcs?
- We will not give the details of this since there is an even better automaton.

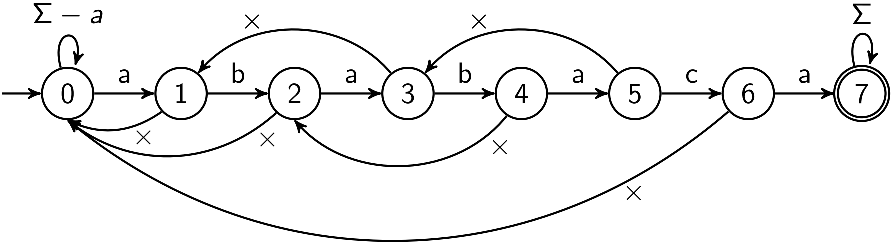
Outline

Knuth-Morris-Pratt Motivation



- Use a new type of transition $\times$ (*"failure"*):
  - Use this transition only if no other fits.
  - Does not consume a character.
  - With these rules, computations of the automaton are deterministic. (But it is formally not a valid DFA.)

Knuth-Morris-Pratt Motivation



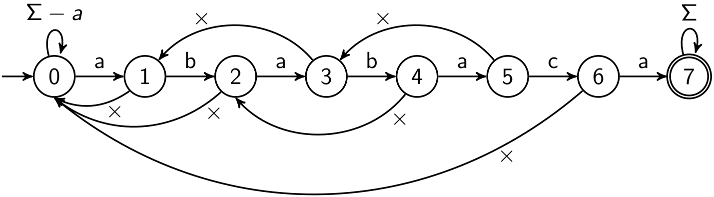
- Use a new type of transition $\times$ (*"failure"*):
  - Use this transition only if no other fits.
  - Does not consume a character.
  - With these rules, computations of the automaton are deterministic. (But it is formally not a valid DFA.)
- Can store failure-function in an array $F[0..m-1]$
  - The failure arc from state j leads to $F[j-1]$
- Given the failure-array, we can easily test whether P is in T: Automaton accepts T if and only if T contains ababaca

Knuth-Morris-Pratt Algorithm

```
KMP::patternMatching(T, P)
    F ← failureArray(P)
    i ← 0 // current character of T to parse
    j ← 0 // current state: we have seen P[0..j−1]
    while i < n do
        if P[j] = T[i]
            if j = m − 1
                return "found at guess i − m + 1"
            else
                i ← i + 1
                j ← j + 1
        else // i.e. P[j] ≠ T[i]
            if j > 0
                j ← F[j − 1]
            else
                i ← i + 1
    return FAIL
```

String matching with KMP – Example

Example: $T =$ `abababbcabababaca`...

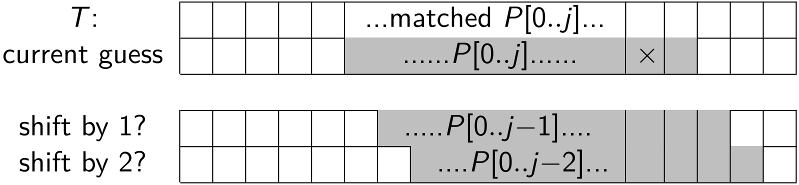


T :	a	b	a	b	a	b	b	c	a	b	a	b	a	c	a
	a	b	a	b	a	×									
			(a)	(b)	(a)	b	×								
					(a)	(b)	×								
							×								
								×							
									a	b	a	b	a	c	a
q :	1	2	3	4	5	3, 4	2, 0	0	1	2	3	4	5	6	7

(after reading this character)

String matching with KMP – Failure-function

Assume we reach state $j+1$ and now have mismatch.



- Can eliminate "shift by 1" if $P[1..j] \neq P[0..j-1]$.
- Can eliminate "shift by 2" if $P[1..j]$ does not end with $P[0..j-2]$.
- Generally eliminate guess if that prefix of P is not a suffix of $P[1..j]$.
- So want longest prefix $P[0..\ell-1]$ that is a suffix of $P[1..j]$.
- The ℓ characters of this prefix are matched, so go to state ℓ.

$$
\begin{aligned}
F[j] &= \text{head of failure-arc from state } j+1 \\
&= \text{length of the longest prefix of } P \text{ that is a suffix of } P[1..j].
\end{aligned}
$$

KMP Failure Array – Example

$F[j]$ is the length of the longest prefix of P that is a suffix of $P[1..j]$.

Consider $P =$ ababaca

j	$P[1..j]$	Prefixes of P	longest	$F[j]$
0	Λ	Λ, a, ab, aba, abab, ababa, ...	Λ	0
1	b	Λ, a, ab, aba, abab, ababa, ...	Λ	0
2	ba	Λ, a, ab, aba, abab, ababa, ...	a	1
3	bab	Λ, a, ab, aba, abab, ababa, ...	ab	2
4	baba	Λ, a, ab, aba, abab, ababa, ...	aba	3
5	babac	Λ, a, ab, aba, abab, ababa, ...	Λ	0
6	babaca	Λ, a, ab, aba, abab, ababa, ...	a	1

This can clearly be computed in $O(m^3)$ time, but we can do better!

Computing the Failure Array

KMP::failureArray(P)
P: String of length m (pattern)

$F[0] \leftarrow 0$
$j \leftarrow 1$ // index within parsed text
$\ell \leftarrow 0$ // reached state
while $j < m$ **do**
  if $P[j] = P[\ell]$
    $\ell \leftarrow \ell + 1$
    $F[j] \leftarrow \ell$
    $j \leftarrow j + 1$
  else if $\ell > 0$
    $\ell \leftarrow F[\ell - 1]$
  else
    $F[j] \leftarrow 0$
    $j \leftarrow j + 1$

Correctness-idea: $F[j]$ is defined via pattern matching of P in $P[1..j]$. So KMP uses itself! Already-built parts of $F[\cdot]$ are used to expand it.

KMP – Runtime

failureArray

- Consider how $2j - \ell$ changes in each iteration of the while loop
  - j and ℓ both increase by 1 $\Rightarrow 2j - \ell$ increases –OR–
  - ℓ decreases $(F[\ell - 1] < \ell) \Rightarrow 2j - \ell$ increases –OR–
  - j increases $\Rightarrow 2j - \ell$ increases
- Initially $2j - \ell \geq 0$, at the end $2j - \ell \leq 2m$
- So no more than $2m$ iterations of the while loop.
- Running time: $\Theta(m)$

KMP – Runtime

failureArray

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  - j and ℓ both increase by 1 $\Rightarrow 2j - \ell$ increases –OR–
  - ℓ decreases ($F[\ell - 1] < \ell$) $\Rightarrow 2j - \ell$ increases –OR–
  - j increases $\Rightarrow 2j - \ell$ increases
- Initially $2j - \ell \geq 0$, at the end $2j - \ell \leq 2m$
- So no more than $2m$ iterations of the while loop.
- Running time: $\Theta(m)$

KMP main function

- failureArray can be computed in $\Theta(m)$ time
- Same analysis gives at most $2n$ iterations of the while loop since $2i - j \leq 2n$.
- Running time KMP altogether: $\Theta(n + m)$

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Boyer-Moore Algorithm

Fastest pattern matching on English text.

Important components:

- Reverse-order searching: Compare P with a guess moving backwards

When a mismatch occurs, choose the better of the following two options:

- Bad character jumps: Eliminate guesses based on mismatched characters of T.
- Good suffix jumps: Eliminate guesses based on matched suffix of P.

Forward-searching vs. reverse-searching

P: aldo

T: whereiswaldo

Forward-searching:

w	h	e	r	e	i	s	w	a	l	d	o

Reverse-searching:

w	h	e	r	e	i	s	w	a	l	d	o

Forward-searching vs. reverse-searching

P: aldo

T: whereiswaldo

Forward-searching:

w	h	e	r	e	i	s	w	a	l	d	o
a											

- `w` does not occur in P.
  $\Rightarrow$ shift pattern past `w`.

Reverse-searching:

w	h	e	**r**	e	i	s	w	a	l	d	o
			o								

- `r` does not occur in P.
  $\Rightarrow$ shift pattern past `r`.

Forward-searching vs. reverse-searching

P: aldo
T: whereiswaldo

Forward-searching:

w	**h**	e	r	e	i	s	w	a	l	d	o
a											
	a										

- `w` does not occur in P.
  $\Rightarrow$ shift pattern past `w`.
- `h` does not occur in P.
  $\Rightarrow$ shift pattern past `h`.

Reverse-searching:

w	h	e	r	e	i	s	**w**	a	l	d	o
			o								
							o				

- `r` does not occur in P.
  $\Rightarrow$ shift pattern past `r`.
- `w` does not occur in P.
  $\Rightarrow$ shift pattern past `w`.

Forward-searching vs. reverse-searching

P: aldo
T: whereiswaldo

Forward-searching:

w	h	e	r	e	i	s	w	a	l	d	o
a											
	a										
		a									

- w does not occur in P.
  $\Rightarrow$ shift pattern past w.
- h does not occur in P.
  $\Rightarrow$ shift pattern past h.

With forward-searching, no guesses are ruled out.

Reverse-searching:

w	h	e	r	e	i	s	w	a	l	d	o
			o								
							o				
								a	l	d	o

- r does not occur in P.
  $\Rightarrow$ shift pattern past r.
- w does not occur in P.
  $\Rightarrow$ shift pattern past w.

This *bad character heuristic* works well with reverse-searching.

Bad character heuristic details

P : p a p e r

T : f e e d a l l p o o r p a r r o t s

Bad character heuristic details

P : p a p e r

T:	f	e	e	d	**a**	l	l	p	o	o	r	p	a	r	r	o	t	s
					r													

- Mismatched character in the text is `a`

Bad character heuristic details

P : p a p e r

T : f e e d **a** l l p o o r p a r r o t s

				r													
				[a]													

- Mismatched character in the text is `a`
- Shift the guess until `a` in *P* aligns with `a` in *T*
  - All skipped guessed are impossible since they do not match `a`

Bad character heuristic details

P : p a p e r

T : f e e d a l l **p** o o r p a r r o t s

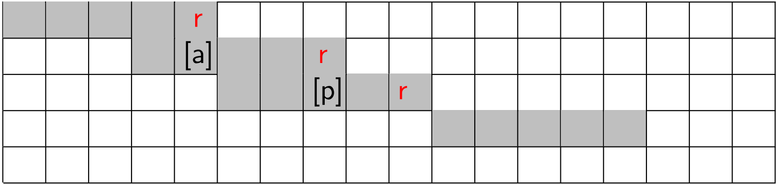
				r													
				[a]			r										
							[p]										

- Mismatched character in the text is a
- Shift the guess until a in P aligns with a in T
  - All skipped guessed are impossible since they do not match a
- Shift the guess until *last* p in P aligns with p in T
  - Use "last" since we cannot rule out this guess.

Bad character heuristic details

P : p a p e r

T : f e e d a l l p o **o** r p a r r o t s



- Mismatched character in the text is `a`
- Shift the guess until `a` in P aligns with `a` in T
  - All skipped guessed are impossible since they do not match `a`
- Shift the guess until *last* `p` in P aligns with `p` in T
  - Use "last" since we cannot rule out this guess.
- As before, shift completely past `o` since `o` is not in P.

Bad character heuristic details

P : p a p e r

T :	f	e	e	d	a	l	l	p	o	o	r	p	a	**r**	r	o	t	s
					r													
					[a]			r										
								[p]		r								
														e	r			

- Mismatched character in the text is a
- Shift the guess until a in P aligns with a in T
  - All skipped guessed are impossible since they do not match a
- Shift the guess until *last* p in P aligns with p in T
  - Use "last" since we cannot rule out this guess.
- As before, shift completely past o since o is not in P.
- Finding r does not help $\Rightarrow$ shift by one unit.
  - Here the other strategy will do better.

Last-Occurrence Array

- Build the last-occurrence array L mapping Σ to integers
- $L[c]$ is the largest index i such that $P[i] = c$
- We will see soon: If c is not in P, then we should set $L[c] = -1$

Pattern: `paper`

char	p	a	e	r	all others
$L[\cdot]$	2	1	3	4	-1

- We can build this in time $O(m + |\Sigma|)$ with simple for-loop

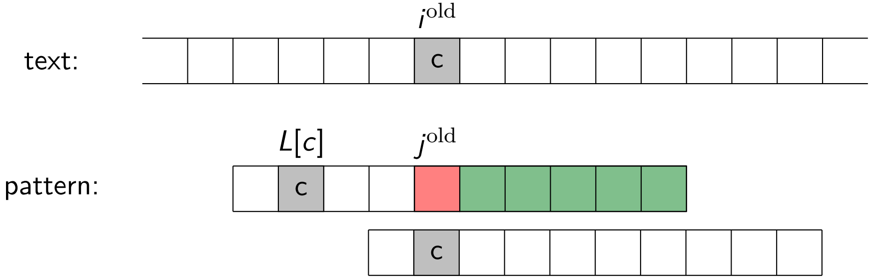
BoyerMoore::lastOccurrenceArray$(P[0..m-1])$
1. initialize array L indexed by Σ with all -1
2. **for** $j \leftarrow 0$ **to** $m-1$ **do** $L[P[j]] \leftarrow j$
3. **return** L

- But how should we do the update?

Bad character heuristic formula

We will always compare $T[i]$ and $P[j]$. How to update at a mismatch?

"Good" case: $L[c] < j$, so c is left of $P[j]$.



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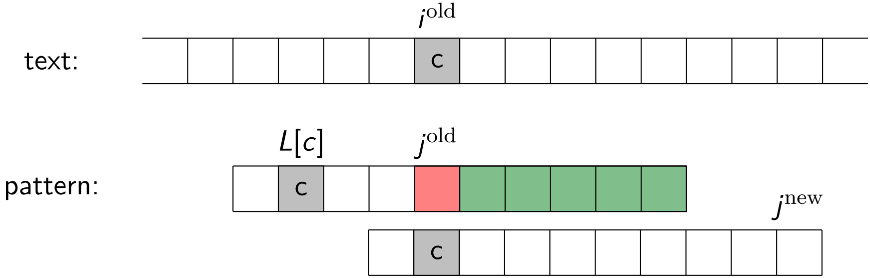

- $j^{\text{new}} = m-1$ (we re-start the search from the right end)

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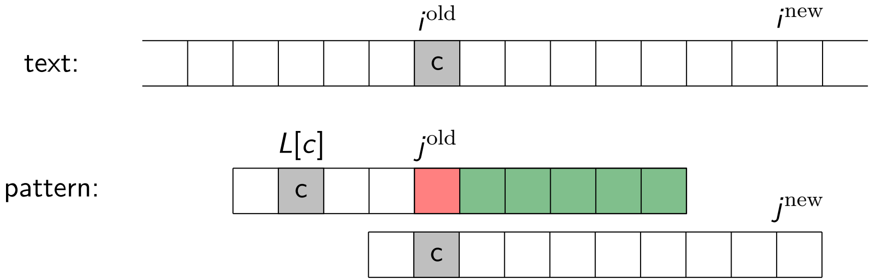

- $j^{\text{new}} = m-1$ (we re-start the search from the right end)
- $i^{\text{new}} =$ corresponding index in T. What is it?

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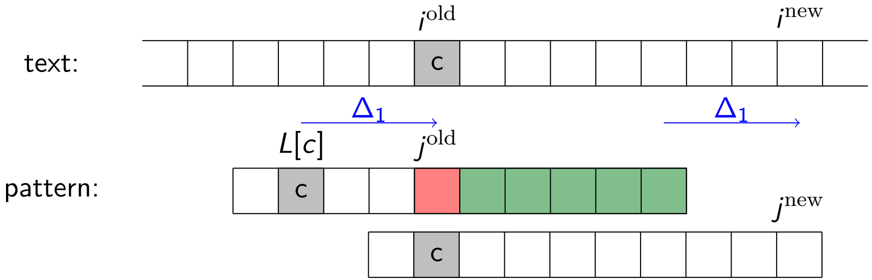

- $j^{\text{new}} = m-1$ (we re-start the search from the right end)
- i^{new} = corresponding index in T. What is it?
  - Δ_1 = amount that we should shift $= j^{\text{old}} - L[c]$

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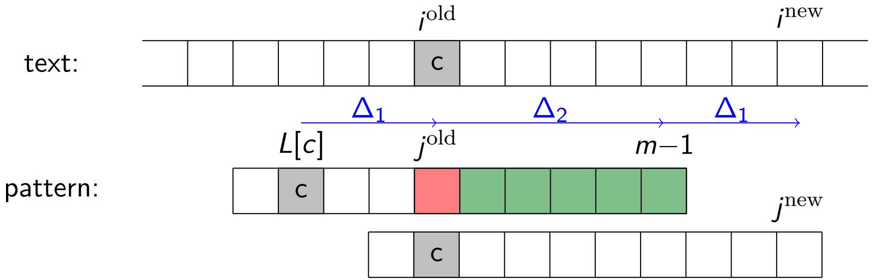
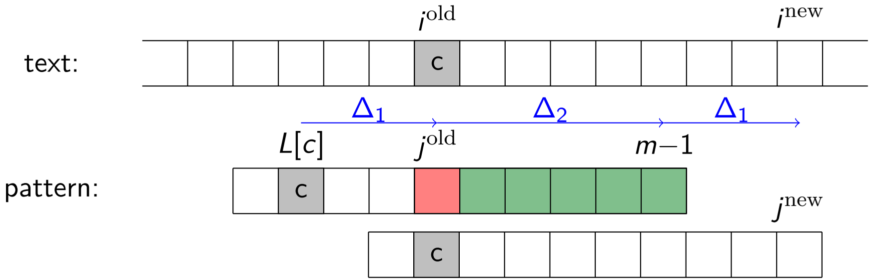


- $j^{\text{new}} = m-1$ (we re-start the search from the right end)
- i^{new} = corresponding index in T. What is it?
  - Δ_1 = amount that we should shift = $j^{\text{old}} - L[c]$
  - Δ_2 = how much we had compared = $(m-1) - j^{\text{old}}$

Bad character heuristic formula

We will always compare $T[i]$ and $P[j]$. How to update at a mismatch?

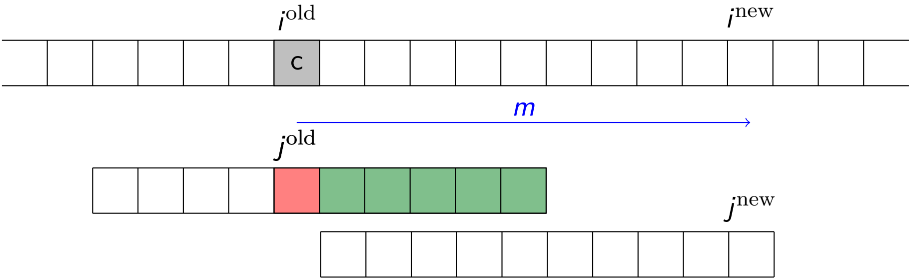
"Good" case: $L[c] < j$, so c is left of $P[j]$.



- $j^{\text{new}} = m-1$ (we re-start the search from the right end)
- $i^{\text{new}} =$ corresponding index in T. What is it?
  - $\Delta_1 =$ amount that we should shift $= j^{\text{old}} - L[c]$
  - $\Delta_2 =$ how much we had compared $= (m-1) - j^{\text{old}}$
  - $i^{\text{new}} = i^{\text{old}} + \Delta_2 + \Delta_1 = i^{\text{old}} + (m-1) - L[c]$

Bad character heuristic formula

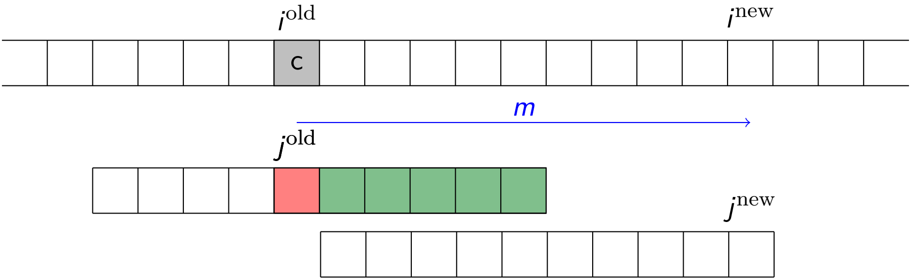
Bad case 1: c does not occur in P.



- We want to shift past $T[i^{\mathrm{old}}]$, so need $i^{\mathrm{new}} = i^{\mathrm{old}} + m$
- What value of $L[c]$ would achieve this automatically?

Bad character heuristic formula

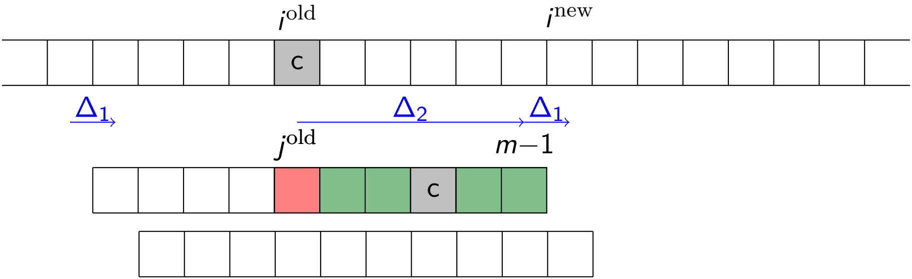
Bad case 1: c does not occur in P.



- We want to shift past $T[i^{\text{old}}]$, so need $i^{\text{new}} = i^{\text{old}} + m$
- What value of $L[c]$ would achieve this automatically?
  - formula was $i^{\text{new}} = i^{\text{old}} + (m{-}1) - L[c]$
  - $\Rightarrow$ set $L[c] := -1$

Bad character heuristic formula

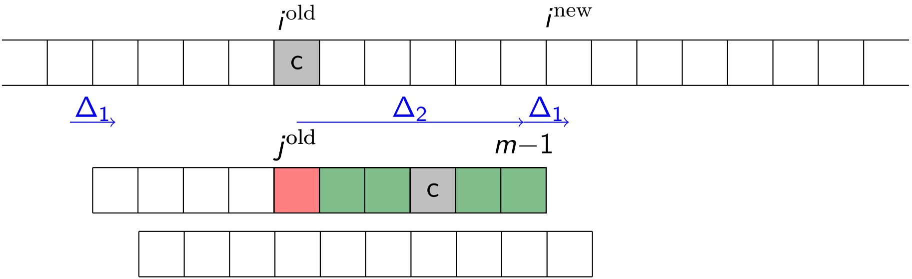
Bad case 2: $L[c] > j$, so c is right of $P[j]$.



- Bad character heuristic not helpful in this case.
- We want to shift by $\Delta_1 := 1$ units

Bad character heuristic formula

Bad case 2: $L[c] > j$, so c is right of $P[j]$.

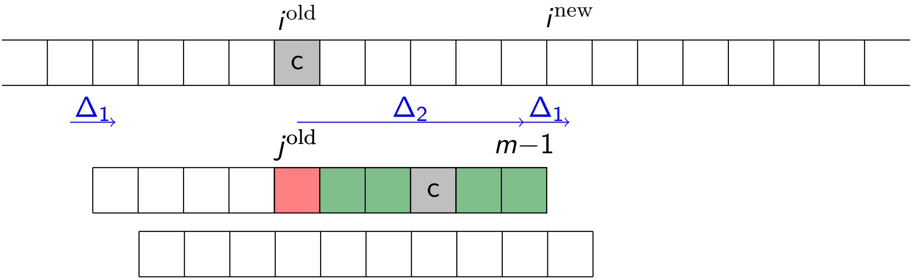


- Bad character heuristic not helpful in this case.
- We want to shift by $\Delta_1 := 1$ units

$$i^{\text{new}} = i^{\text{old}} + \Delta_2 + \Delta_1 = i^{\text{old}} + 1 + (m-1) - j^{\text{old}}$$

Bad character heuristic formula

Bad case 2: $L[c] > j$, so c is right of $P[j]$.



- Bad character heuristic not helpful in this case.
- We want to shift by $\Delta_1 := 1$ units

$$i^{\text{new}} = i^{\text{old}} + \Delta_2 + \Delta_1 = i^{\text{old}} + 1 + (m-1) - j^{\text{old}}$$

Unified formula for all cases: $\boxed{i^{\text{new}} = i^{\text{old}} + (m-1) - \min\left\{L[c], j^{\text{old}}-1\right\}}$

Boyer-Moore Algorithm

Boyer-Moore::patternMatching(T,P)

1. $L \leftarrow$ *lastOccurrenceArray*(P)
2. $S \leftarrow$ good suffix array computed from P
3. $i \leftarrow m-1, \quad j \leftarrow m-1$
4. **while** $i < n$ **and** $j \geq 0$ **do**
5.   **if** $T[i] = P[j]$
6. $i \leftarrow i-1$
7. $j \leftarrow j-1$
8.   **else**
9. $i \leftarrow i + m-1 - \min\{L[T[i]], j-1, S[j]\}$
10. $j \leftarrow m-1$
11. **if** $j = -1$ **return** "found at $T[i{+}1..i{+}m]$"
12. **else return** `FAIL`

S will be explained below.

Can show: '$j-1$' is not needed in line 9 since $\min\{L[T[i]], S[j]\} \leq j-1$

Good Suffix Heuristic

$S[j]$ expresses

"since $P[j+1..m-1]$ was matched, how much should we shift?"

P:	o	n	o	b	o	b	o												
T:	o	n	o	o	o	b	o	o	o	i	b	b	o	u	n	d	a	r	y
				b	o	b	o												
	Do smallest shift so that obo fits in the new guess.																		
					(o)	(b)	(o)												

- Doing examples is easy, but the formula is complicated (no details)
- $S[\cdot]$ computable (similar to KMP failure function) in $\Theta(m)$ time.

Summary:

- Boyer-Moore performs very well (even without good suffix heuristic).
- On typical *English text* Boyer-Moore looks at only $\approx 25\%$ of T
- Worst-case run-time for is $O(mn)$, but in practice much faster.
  [There are ways to ensure $O(n)$ run-time. No details.]

Outline

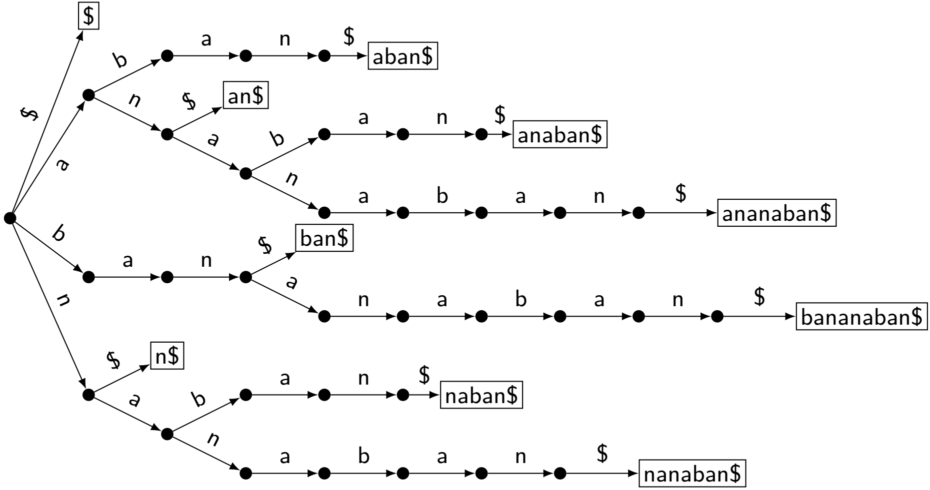
Tries of Suffixes and Suffix Trees

- What if we want to search for many patterns P within the same fixed text T?
- Idea: Preprocess the text T rather than the pattern P
- Observation: P is a substring of T if and only if P is a prefix of some suffix of T.
- So want to store all suffixes of T in a trie.
- To save space:
  - Use a compressed trie.
  - Store suffixes implicitly via indices into T.
- This is called a **suffix tree**.

Trie of suffixes: Example

T = `bananaban` has suffixes

{`bananaban`, `ananaban`, `nanaban`, `anaban`, `naban`, `aban`, `ban`, `an`, `n`, Λ}

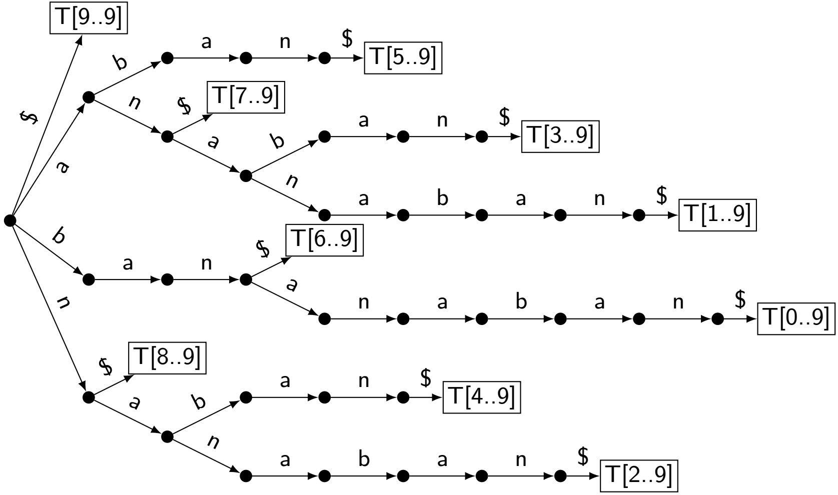


Tries of suffixes

Store suffixes via indices:

$T =$

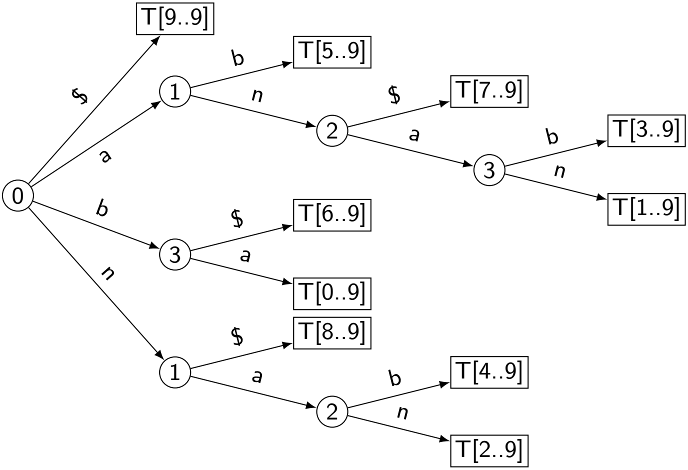
0	1	2	3	4	5	6	7	8	9
b	a	n	a	n	a	b	a	n	$



Suffix tree

Suffix tree: Compressed trie of suffixes

	0	1	2	3	4	5	6	7	8	9
$T =$	b	a	n	a	n	a	b	a	n	\$



More on Suffix Trees

Building:

- Text T has n characters and $n+1$ suffixes
- We can build the suffix tree by inserting each suffix of T into a compressed trie. This takes time $\Theta(n^2|\Sigma|)$.
- There *is* a way to build a suffix tree of T in $\Theta(n|\Sigma|)$ time. This is quite complicated and beyond the scope of the course.

Pattern Matching:

- Essentially *search* for P in compressed trie. Some changes are needed, since P may only be prefix of stored word.
- Run-time: $O(|\Sigma|m)$.

Summary: Theoretically good, but construction is slow or complicated, and lots of space-overhead $\rightsquigarrow$ rarely used.

Outline

Suffix Arrays

- Relatively recent development (popularized in the 1990s)
- Sacrifice some performence for simplicity:
  - Slightly slower (by a log-factor) than suffix trees.
  - Much easier to build.
  - Much simpler pattern matching.
  - Very little space; only one array.

Idea:

- Store suffixes implicitly (by storing start-indices)
- Store *sorting permutation* of the suffixes of T.

Suffix Array Example

	0	1	2	3	4	5	6	7	8	9
Text T:	b	a	n	a	n	a	b	a	n	$

i	suffix $T[i..n-1]$
0	bananaban$
1	ananaban$
2	nanaban$
3	anaban$
4	naban$
5	aban$
6	ban$
7	an$
8	n$
9	$

$\longrightarrow$

sort lexicographically

j	$A^s[j]$	
0	9	$
1	5	aban$
2	7	an$
3	3	anaban$
4	1	ananaban$
5	6	ban$
6	0	bananaban$
7	8	n$
8	4	naban$
9	2	nanaban$

	0	1	2	3	4	5	6	7	8	9
Suffix array:	9	5	7	3	1	6	0	8	4	2

Suffix Array Construction

- Easy to construct using *MSD-Radix-Sort*.
  - Fast in practice; suffixes are unlikely to share many leading characters.
  - But worst-case run-time is $\Theta(n^2)$
    - n rounds of recursions (have n chars)
    - Each round takes $\Theta(n)$ time (bucket-sort)

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  - But worst-case run-time is $\Theta(n^2)$
    - n rounds of recursions (have n chars)
    - Each round takes $\Theta(n)$ time (bucket-sort)
- **Idea:** We do not need n rounds!

  (
  - Consider sub-array after one round.
  - These have same leading char. Ties are broken by rest of words.
  - But rest of words are also suffixes $\rightsquigarrow$ sorted elsewhere
  - We can double length of sorted part every round.
)

  - $O(\log n)$ rounds enough $\Rightarrow$ $O(n \log n)$ **run-time**
- Construction-algorithm: MSD-radix-sort plus some bookkeeping
  - needs only one extra array
  - easy to implement
- You do not need to know details.

Pattern matching in suffix arrays

- Suffix array stores suffixes (implicitly) in sorted order.
- **Idea:** apply binary search!

$P =$ban:

	j	$A^s[j]$	$T[\,A^s[j]..n-1]$
$\ell \rightarrow$	0	9	\$
	1	5	aban\$
	2	7	an\$
	3	3	anaban\$
$\nu \rightarrow$	4	1	ananaban\$
	5	6	ban\$
	6	0	bananaban\$
	7	8	n\$
	8	4	naban\$
$r \rightarrow$	9	2	nanaban\$

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	4	1	ananaban\$
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	1	5	aban$
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	3	3	anaban$
	4	1	ananaban$
$\nu = \ell \rightarrow$	5	6	ban$ found
$r \rightarrow$	6	0	bananaban$
	7	8	n$
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Pattern matching in suffix arrays

- Suffix array stores suffixes (implicitly) in sorted order.
- **Idea:** apply binary search!

P =ban:

	j	$A^s[j]$	$T[A^s[j]..n-1]$
	0	9	\$
	1	5	aban\$
	2	7	an\$
	3	3	anaban\$
	4	1	ananaban\$
$\nu=\ell \rightarrow$	5	6	ban\$ found
$r \rightarrow$	6	0	bananaban\$
	7	8	n\$
	8	4	naban\$
	9	2	nanaban\$

- $O(\log n)$ comparisons.
- Each comparison is *strcmp*$(P, T[A^s[\nu]..A^s[\nu+m-1]])$
- $O(m)$ time per comparison $\Rightarrow$ **run-time** $O(m \log n)$

Pattern matching in suffix arrays

SuffixArray-search$(A^s[0...n-1], P[0..m-1])$

A^s: suffix array of T, P: pattern

$\ell \leftarrow 0$, $r \leftarrow n-1$

while $(\ell < r)$

  $\nu \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$

  $i \leftarrow A^s[\nu]$ // Suffix is $T[i..n-1]$

  $s \leftarrow$ *strcmp*$(T[i..i+m-1], P)$

    // Assuming *strcmp* handles "out of bounds" suitably

  if $(s < 0)$ **do** $\ell \leftarrow \nu + 1$

  else if $(s > 0)$ **do** $r \leftarrow \nu - 1$

  else return "found at guess $T[i..i+m-1]$"

if *strcmp*$(T, P, A^s[\ell], A^s[\ell]+m-1) = 0$

  return "found at guess $T[\ell..\ell+m-1]$"

return FAIL

Outline

String Matching Conclusion

	Brute-Force	Karp-Rabin	DFA	Knuth-Morris-Pratt	Boyer-Moore	Suffix Tree	Suffix Array
Preproc.	—	$O(m)$	$O(m\lvert\Sigma\rvert)$	$O(m)$	$O(m+\lvert\Sigma\rvert)$	$O(n^2\lvert\Sigma\rvert)$ [$O(n)\lvert\Sigma\rvert$]	$O(n\log n)$ [$O(n)$]
Search time	$O(nm)$	$O(n+m)$ expected	$O(n)$	$O(n)$	$O(n)$ or better	$O(m)$	$O(m\log n)$
Extra space	—	$O(1)$	$O(m\lvert\Sigma\rvert)$	$O(m)$	$O(m+\lvert\Sigma\rvert)$	$O(n)$	$O(n)$

- Our algorithms stopped once they have found one occurrence.
- Most of them can be adapted to find *all* occurrences within the same worst-case run-time.