CS 240 – Data Structures and Data Management

Module 3: Sorting and Randomized Algorithms

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Based on lecture notes by many previous cs240 instructors

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Winter 2021
Outline

- Sorting and Randomized Algorithms
  - QuickSelect
  - Randomized Algorithms
  - QuickSort
  - Lower Bound for Comparison-Based Sorting
  - Non-Comparison-Based Sorting
Outline

- Sorting and Randomized Algorithms
  - QuickSelect
    - Randomized Algorithms
    - QuickSort
  - Lower Bound for Comparison-Based Sorting
  - Non-Comparison-Based Sorting
Selection Problem

- Given array $A$ of $n$ numbers, and $0 \leq k < n$, find the element that would be at position $k$ if $A$ was sorted
  - ‘select $k$’
  - $k$ elements are smaller or equal, $n - 1 - k$ elements are larger or equal

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- Special case: median finding ($k = \left\lfloor \frac{n}{2} \right\rfloor$)

- Heap-based selection can be done in $\Theta(n + k \log n)$
  - this is $\Theta(n \log n)$ for median finding
    - the same cost as our best sorting algorithms

- Question: can we do selection in linear time?
  - yes, with quick-select (average case analysis)
  - subroutines for quick-select also useful for sorting algorithms
Crucial Subroutines

- **quick-select** and related algorithm **quick-sort** rely on two subroutines
  - **choose-pivot**(A)
    - return an index \( p \) in \( A \)
    - use **pivot-value** \( v \leftarrow A[p] \) to rearrange the array
  - **partition** \((A, p)\) rearranges \( A \) so that
    - all items in \( A[0, ..., i - 1] \) are \( \leq v \)
    - pivot-value \( v \) is in \( A[i] \)
    - all items in \( A[i + 1, ..., n - 1] \) are \( \geq v \)
    - index \( i \) is called **pivot-index** \( i \)
    - **partition**(A, \( p \)) returns **pivot-index** \( i \)
      - \( i \) is a correct location of \( v \) in sorted \( A \)
      - if we were interested in select(\( i \)), then \( v \) would be the answer
Choosing Pivot

- Simplest idea for *choose-pivot*
  - always select rightmost element in array

```python
choose-pivot1(A)
return A.size() - 1
```

- Will consider more sophisticated ideas later
Partition Algorithm

\[ \text{partition}(A, p) \]

\( A: \) array of size \( n \), \( p: \) integer s.t. \( 0 \leq p < n \)

- create empty lists \( \text{small} \), \( \text{equal} \) and \( \text{large} \)

\[ v \leftarrow A[p] \]

for each element \( x \) in \( A \)

- if \( x < v \) then \( \text{small}.append(x) \)
- else if \( x > v \) then \( \text{large}.append(x) \)
- else \( \text{equal}.append(x) \)

\[ i \leftarrow \text{small}.size \]
\[ j \leftarrow \text{equal}.size \]

overwrite \( A[0 \ldots i - 1] \) by elements in \( \text{small} \)
overwrite \( A[i \ldots i + j - 1] \) by elements in \( \text{equal} \)
overwrite \( A[i + j \ldots n - 1] \) by elements in \( \text{large} \)

return \( i \)

- Easy linear-time implementation using extra (auxiliary) \( \Theta(n) \) space
- More challenging: partition \textit{in-place}, i.e. \( O(1) \) auxiliary space
Efficient In-Place partition (Hoare)

almost done, just swap with pivot $v$

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$v=70$

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$v=70$
Efficient In-Place partition (Hoare)

- **Idea Summary:** Keep swapping the outer-most wrongly-positioned pairs

  \[
  \begin{array}{c|ccc}
  & ? & & v \\
  i & & j & \\
  \end{array}
  \]

- One possible implementation

  \[
  \begin{align*}
  & \text{do } i \leftarrow i + 1 \text{ while } i < n \text{ and } A[i] \leq v \\
  & \text{do } j \leftarrow j - 1 \text{ while } j > 0 \text{ and } A[j] \geq v
  \end{align*}
  \]

- More efficient (for quickselect and quicksort) when many repeating elements

  \[
  \begin{align*}
  & \text{do } i \leftarrow i + 1 \text{ while } i < n \text{ and } A[i] < v \\
  & \text{do } j \leftarrow j - 1 \text{ while } j > 0 \text{ and } A[j] > v
  \end{align*}
  \]

- Can simplify the loop bounds

  \[
  \begin{align*}
  & \text{do } i \leftarrow i + 1 \text{ while } A[i] < v \\
  & \text{do } j \leftarrow j - 1 \text{ while } j \geq i \text{ and } A[j] > v
  \end{align*}
  \]
Efficient In-Place partition (Hoare)

\[\text{partition} \ (A, p)\]

\[A: \text{array of size } n\]

\[p: \text{integer s.t. } 0 \leq p < n\]

\[\text{swap}(A[n - 1], A[p])\]

\[i \leftarrow -1, \quad j \leftarrow n - 1, \quad v \leftarrow A[n - 1]\]

\[\text{loop}\]

\[\text{do } i \leftarrow i + 1 \ \text{while } A[i] < v\]

\[\text{do } j \leftarrow j - 1 \ \text{while } j \geq i \ \text{and} \ A[j] > v\]

\[\text{if } i \geq j \ \text{then break}\]

\[\text{else } \text{swap}(A[i], A[j])\]

\[\text{end loop}\]

\[\text{swap}(A[n - 1], A[i])\]

\[\text{return } i\]

- Running time is \(\Theta(n)\)
Efficient In-Place partition (Hoare)

partition $(A, p)$

$A$: array of size $n$

$p$: integer s.t. $0 \leq p < n$

$\text{swap}(A[n - 1], A[p])$

$i \leftarrow -1, \quad j \leftarrow n - 1, \quad v \leftarrow A[n - 1]$

loop

$\text{do } i \leftarrow i + 1 \text{ while } A[i] < v$

$\text{do } j \leftarrow j - 1 \text{ while } j \geq i \text{ and } A[j] > v$

if $i \geq j$ then break

else $\text{swap}(A[i], A[j])$

end loop

$\text{swap}(A[n - 1], A[i])$

return $i$

- Running time is $\Theta(n)$
Quick Select Algorithm

- Find item that would be in $A[k]$ if $A$ was sorted
- Similar to quick-sort, but recurse only on one side ("quick-sort with pruning")
- Example: \text{select}(k = 4)
  - [the correct answer is 40 in this case]

\begin{tabular}{cccccccccc}
30 & 60 & 10 & 0 & 50 & 80 & 90 & 20 & 40 & $v=70$
\end{tabular}

\begin{itemize}
  \item $i > k$, search recursively in the left side to select $k$
\end{itemize}
Quick Select Algorithm

- Example continued: `select(k = 4)`

```
30  60  10  0  50  40  ν=20
```

- \( i < k \), search recursively on the right, select \( k - (i + 1) \)
  - \( k = 1 \) in our example
Quick Select Algorithm

- Example continued: `select(k = 1)`

![Diagram showing the quick select algorithm example]

- $i > k$, search on the left to select $k$
Quick Select Algorithm

- Example continued: \( \text{select}(k = 1) \)

\[
\begin{align*}
30 & \quad 50 & v=40 \\
\end{align*}
\]

\[
\text{partition, } v=40
\]

\[
i=1
\]

\[
\begin{align*}
30 & \quad 40 & 50 \\
\end{align*}
\]

- \( i = k \), found our item, done!
- In our example, we got to subarray of size 3
- Often stop much sooner than that
  - running time?
QuickSelect Algorithm

\[ quick-select1(A, k) \]

- \( A \): array of size \( n \), \( k \): integer s.t. \( 0 \leq k < n \)
- \( p \leftarrow choose-pivot1(A) \)
- \( i \leftarrow partition(A, p) \)
- if \( i = k \) then
  - return \( A[i] \)
- else if \( i > k \) then
  - return \( quick-select1(A[0, 1, \ldots, i - 1], k) \)
- else if \( i < k \) then
  - return \( quick-select1(A[i + 1, \ldots, n - 1], k - (i + 1)) \)

- **Best case**
  - first chosen pivot could have pivot-index \( k \)
  - no recursive calls, total cost \( \Theta(n) \)

- **Worst case**: recurrence equation

\[
T(n) = \begin{cases} 
  cn + T(n - 1) & n > 1 \\
  c & n = 1 
\end{cases}
\]
QuickSelect Algorithm

- **Worst case**: recurrence equation \( T(n) = \begin{cases} cn + T(n - 1) & n > 1 \\ c & n = 1 \end{cases} \)

- Solution: repeatedly expand until we see a pattern forming

\[
T(n) = cn + T(n - 1)
\]

\[
T(n - 1) = c(n - 1) + T(n - 2)
\]

\[
T(n) = cn + c(n - 1) + T(n - 2)
\] after 1 expansion

\[
T(n - 2) = c(n - 2) + T(n - 3)
\]

\[
T(n) = cn + c(n - 1) + c(n - 2) + T(n - 3)
\] after 2 expansions

- After \( i \) expansions

\[
T(n) = cn + c(n - 1) + c(n - 2) + \cdots + c(n - i) + T(n - (i + 1))
\]

- Stop expanding when get to base case \( T(n - (i + 1)) = T(1) \)

- Happens when \( n - (i + 1) = 1 \), or, rewriting, \( i = n - 2 \)

- Thus \( T(n) = cn + c(n - 1) + c(n - 2) + \cdots + c \cdot 2 + T(1) \)

\[
= cn + c(n - 1) + c(n - 2) + \cdots + c \cdot 2 + c
\]

\[
= c(n + (n - 1) + \cdots + 2 + 1) \in \Theta(n^2)
\]
Average-Case Analysis of *quick-select1*

\[ T_{avr}(n) = \frac{1}{\text{# instances of size } n} \sum_{I: \text{size}(I) = n} T(I) \]

*infinitely many*

- Need to make some assumptions
- First assumption
  - all input numbers are distinct
  - this assumption is just for simpler analysis, can prove the same thing without this assumption
Average-Case Analysis of *quick-select* 1

- **QuickSelect** is *comparison-based*
  - only cares if \( A[i] < A[j] \) for \( i, j \)
  - does not care what the actual values of \( A[i], A[j] \) are

\[
\begin{array}{cccc}
I_1 & 30 & 60 & 0 & 10 \\
I_2 & 20 & 50 & 10 & 15 \\
\end{array}
\]

- **QuickSelect** makes exactly the same sequences of steps on \( I_1 \) and \( I_2 \)
  - therefore \( T(I_1) = T(I_2) \)

- Any comparison based algorithm has exactly the same running time for arrays that have the same relative order of elements, regardless of actual array values

- Second assumption: we are sorting integers 0, ..., \( n - 1 \)
  - now there are \( n! \) possible input instances \( I \)
  - more formal proof uses *sorting permutations*

- permutation \( \pi \) for which \( A[\pi(0)] \leq A[\pi(1)] \leq ... \leq A[\pi(n - 1)] \)
- for \( I_1 \) (and \( I_2 \)) sorting permutation is \( \pi = (2, 3, 0, 1) \)
- assume *each sorting permutation is equally likely*
- \( n! \) possible permutations
Average-Case Analysis of \textit{quick-select1}

\[ T_{avr}(n) = \frac{1}{\text{# instances of size } n} \sum_{I: \text{size}(I) = n} T(I) \]

- Example for \( n = 3 \), using all the assumptions

\[ T_{avr}(3) = \frac{1}{3!} (T([0,1,2]) + T([0,2,1]) + T([1,0,2]) + T([1,2,0]) + T([2,0,1]) + T([2,1,0])) \]
Average-Case Analysis of \textit{quick-select1}

- Recall that pivot is last array element
- Pivot index is equal to pivot value due to assuming we sort 0, ..., \( n - 1 \)

\[
A = \begin{bmatrix}
0 & 1 & 2 & 3 \\
2 & 3 & 0 & v=1
\end{bmatrix}
\quad \text{for } v=1, \text{ pivot index } i = 1
\]

- Partition sum over different pivot indexes

\[
T^{\text{avr}}(n) = \frac{1}{n!} \sum_{I: \text{Size}(I) = n} T(I) = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{I: \text{size}(I) = n, \text{pivot is } i} T(I)
\]

- Example for \( n = 3 \)

\[
T^{\text{avr}}(3) = \frac{1}{3!} (T(\{0,1,2\}) + T(\{0,2,1\}) + T(\{1,0,2\}) + T(\{1,2,0\}) + T(\{2,0,1\}) + T(\{2,1,0\}))
\]

\[
T^{\text{avr}}(3) = \frac{1}{3!} (T(\{1,2,0\}) + T(\{2,1,0\})) + (T(\{0,2,1\}) + T(\{2,0,1\})) + (T(\{0,1,2\}) + T(\{1,0,2\}))
\]
Average-Case Analysis of quick-select

- Partition sum over different pivots
  \[ T^{avr}(n) = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\text{size}(I)=n, \ pivot \ is \ i} T(I) \]

- There are \((n - 1)!\) input instances \(I\) with pivot index \(i\)

\[
\begin{array}{|c|c|c|c|}
\hline
A & \text{choice of } n-1 & \text{choice of } n-2 \text{ items:} & \text{‘choice’ of} \\
    & \text{items: anything but } i & \text{anything but } i \text{ and } A[0] & 1 \text{ items} \\
\text{no choice} & \text{anything but } i & \ldots & \text{no choice} \\
\hline
\end{array}
\]

- One can show (will only hint at the proof with example for \(n = 4, i = 1\))
  \[ \sum_{\text{size}(I)=n, \ pivot \ is \ i} T(I) \leq (n - 1)! \cdot cn + (n - 1)! \cdot \max{T^{avr}(i), T^{avr}(n - i - 1)} \]

- Therefore
  \[ T^{avr}(n) \leq cn + \frac{1}{n} \sum_{i=0}^{n-1} \max{T^{avr}(i), T^{avr}(n - i - 1)} \]
Average-Case Analysis of quick-select

- Let \( n = 4, i = 1 \)

\[
T(I) = \sum_{I: \text{size}(I) = 4, \ pivot \ is \ 1} T(I) = T(\{0,2,3, 1\}) + T(\{0,3,2, 1\}) + T(\{2,0,3, 1\}) + T(\{2,3,0, 1\}) + T(\{3,0,2, 1\}) + T(\{3,2,0, 1\})
\]

- Total work is proportional to comparisons, will count comparisons

<table>
<thead>
<tr>
<th>comparisons to partition</th>
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<td>instances</td>
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<td>{0,3,2, 1}</td>
<td>{2,0,3, 1}</td>
<td>{2,3,0, 1}</td>
<td>{3,0,2, 1}</td>
<td>{3,2,0, 1}</td>
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</tr>
<tr>
<td>partitions (assume stable order)</td>
<td>{0}</td>
<td>{2,3}</td>
<td>{0}</td>
<td>{3,2}</td>
<td>{0}</td>
<td>{2,3}</td>
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Total: 3(3)!
Average-Case Analysis of \textit{quick-select1}

- Let $n = 4, i = 1$
  \[ T(I) = \sum_{I: \text{size}(I) = 4, \text{pivot is 1}} T(I) = T(\{0,2,3,1\}) + T(\{0,3,2,1\}) + T(\{2,0,3,1\}) + T(\{2,3,0,1\}) + T(\{3,0,2,1\}) + T(\{3,2,0,1\}) \]

- Total work is proportional to comparisons, will count comparisons

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Case 1: $k > i$

\[ T(\{2,3\}) + T(\{3,2\}) + T(\{2,3\}) + T(\{2,3\}) + T(\{3,2\}) + T(\{3,2\}) = T(\{0,1\}) + T(\{1,0\}) + T(\{0,1\}) + T(\{0,1\}) + T(\{1,0\}) + T(\{1,0\}). \]

Since only relative order matters.
Average-Case Analysis of quick-select1

- Let \( n = 4, i = 1 \)

\[
\sum_{I : \text{size}(I) = 4, \text{pivot is 1}} T(I) = T(\{0,2,3, 1\}) + T(\{0,3,2, 1\}) + T(\{2,0,3, 1\}) + T(\{2,3,0, 1\}) + T(\{3,0,2, 1\}) + T(\{3,2,0, 1\})
\]

- Total work is proportional to comparisons, will count comparisons

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Case 1: \( k > i \)

\[
T(\{2,3\}) + T(\{3,2\}) + T(\{2,3\}) + T(\{2,3\}) + T(\{3,2\}) + T(\{3,2\})
\]

\[
= T(\{0,1\}) + T(\{1,0\}) + T(\{0,1\}) + T(\{1,0\}) + T(\{0,1\}) + T(\{1,0\})
\]

\[
\underbrace{2! T^{avr}(2)} + \underbrace{2! T^{avr}(2)} + \underbrace{2! T^{avr}(2)}
\]

Total recursive comparisons

\[
\frac{3!}{2!} 2! T^{avr}(2) = 3! T^{avr}(2)
\]
Average-Case Analysis of *quick-select*1

- Let $n = 4, i = 1$

  \[ T(I) = \sum_{i: \text{size}(I) = 4, \text{pivot is } 1} T(I) \]

  \[ = T(\{0,2,3, 1\}) + T(\{0,3,2, 1\}) + T(\{2,0,3, 1\}) + T(\{2,3,0, 1\}) + T(\{3,0,2, 1\}) + T(\{3,2,0, 1\}) \]

- Total work is proportional to comparisons, will count comparisons

  \[
  \begin{array}{ccccccc}
  \text{comparisons to partition:} & 3 & 3 & 3 & 3 & 3 & 3 & \text{Total: } 3(3)! \\
  \text{instances} & \{0,2,3, 1\} & \{0,3,2, 1\} & \{2,0,3, 1\} & \{2,3,0, 1\} & \{3,0,2, 1\} & \{3,2,0, 1\} \\
  \text{partitions} & \{0\} & \{2,3\} & \{0\} & \{3,2\} & \{0\} & \{2,3\} & \{0\} & \{3,2\} & \{0\} & \{3,2\} \\
  \end{array}
  \]

- Case 2: $k < i$

  \[
  T(\{0\}) + T(\{0\}) + T(\{0\}) + T(\{0\}) + T(\{0\}) + T(\{0\}) \\
  = \frac{3!}{1!} T^{avr}(1) = 3! T^{avr}(1) \\
  \]

  \[
  \frac{3!}{1!} 1! T^{avr}(1) = 3! T^{avr}(1) \\
  \]

  \[
  [\text{Case 1, total recursive comparisons:} 3! T^{avr}(2)] \\
  \]

- Combining both cases, total recursive comparisons:

  \[
  \leq 3(3)! + 3! \max\{T^{avr}(1), T^{avr}(2)\} \\
  \]

- Adding comparisons to partition:

  \[
  \leq 3(3)! + 3! \max\{T^{avr}(1), T^{avr}(2)\} \\
  \]
Average-Case Analysis of quick-select1

- Let \( n = 4, i = 1 \)

\[
T(I) = \sum_{I: \text{size}(I) = 4, \text{pivot is 1}} T(I) = T\{0,2,3, 1\} + T\{0,3,2, 1\} + T\{2,0,3, 1\} + T\{2,3,0, 1\} + T\{3,0,2, 1\} + T\{3,2,0, 1\}
\]

- Total work is proportional to comparisons, will count comparisons

<table>
<thead>
<tr>
<th>comparisons to partition</th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>3</th>
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<th>3</th>
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<tbody>
<tr>
<td>instances</td>
<td>{0,2,3, 1}</td>
<td>{0,3,2, 1}</td>
<td>{2,0,3, 1}</td>
<td>{2,3,0, 1}</td>
<td>{3,0,2, 1}</td>
<td>{3,2,0, 1}</td>
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<tr>
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<td>{2,3}</td>
<td>{0}</td>
<td>{3,2}</td>
<td>{0}</td>
<td>{2,3}</td>
<td>{0}</td>
</tr>
</tbody>
</table>

- Case 2: \( k < i \)

\[
T\{0\} + T\{0\} + T\{0\} + T\{0\} + T\{0\} + T\{0\}
\]

\[
\text{Total recursive comparisons} = 3! T^{avr}(1) = 3! T^{avr}(1)
\]

Adding comparisons to partition:

\[
\sum_{I: \text{size}(I) = n, \text{pivot is } i} T(I) \leq (n - 1)! cn + (n - 1)! \max\{T^{avr}(i), T^{avr}(n - i - 1)\}
\]

\[
\leq 3(3)! + 3! \max\{T^{avr}(1), T^{avr}(2)\}
\]
Average-Case Analysis of *quick-select* 1

\[ T(n) \leq c \cdot n + \frac{1}{n} \sum_{i=0}^{n-1} \max\{T(i), T(n-i-1)\} \]

**Theorem:** \( T(n) \in O(n) \)

**Proof:**

- will prove \( T(n) \leq 4cn \) by induction on \( n \)
- base case, \( n = 1 \): \( T(1) = c \leq 4c \cdot 1 \)
- induction hypothesis: assume \( T(m) \leq 4cm \) for all \( m < n \)
- need to show \( T(n) \leq 4cn \)

\[
T(n) \leq c \cdot n + \frac{1}{n} \sum_{i=0}^{n-1} \max\{T(i), T(n-i-1)\}
\]

\[
\leq c \cdot n + \frac{1}{n} \sum_{i=0}^{n-1} \max\{4ci, 4c(n-i-1)\}
\]

\[
\leq c \cdot n + \frac{4c}{n} \sum_{i=0}^{n-1} \max\{i, n-i-1\}
\]
Average-Case Analysis of *quick-select*1

**Proof: (cont.)**  \[ T(n) \leq c \cdot n + \frac{4c}{n} \sum_{i=0}^{n-1} \max\{i, n-i-1\} \leq c \cdot n + \frac{4c}{n} \cdot \frac{3}{4} n^2 = 4cn \]

\[
\sum_{i=0}^{n-1} \max\{i, n-i-1\} = \sum_{i=0}^{\frac{n-1}{2}} \max\{i, n-i-1\} + \sum_{i=\frac{n}{2}}^{n-1} \max\{i, n-i-1\}
\]

\[
= \max\{0, n-1\} + \max\{1, n-2\} + \max\{2, n-3\} + \cdots + \max\left\{\frac{n}{2} - 1, \frac{n}{2}\right\}
\]

\[
+ \max\left\{\frac{n}{2}, \frac{n}{2} - 1\right\} + \max\left\{\frac{n}{2} + 1, \frac{n}{2} - 2\right\} + \cdots + \max\{n-1, 0\}
\]

\[
= (n-1) + (n-2) + \cdots + \frac{n}{2} + \frac{n}{2} + \left(\frac{n}{2} + 1\right) + \cdots (n-1) = \left(\frac{3n}{2} - 1\right) \frac{n}{2}
\[
\leq \frac{3}{4} n^2
\]
Average-Case Analysis of quick-select1

- Proved average case time $T(n)$ is $O(n)$
- Average case is also $\Omega(n)$ since have to perform $\text{partition}(A, p)$
- Therefore average case is $T(n)$ is $\Theta(n)$
Outline

- Sorting and Randomized Algorithms
  - QuickSelect
  - Randomized Algorithms
    - QuickSort
    - Lower Bound for Comparison-Based Sorting
    - Non-Comparison-Based Sorting
Randomized Algorithms

- A *randomized algorithm* is one which relies on some random numbers in addition to the input.
- The cost will depend on both the input and the random numbers used.
- **Goal**
  - shift the dependency of run-time from what we cannot control (the input), to what we can control (random numbers).
  - no more bad instances, just unlucky numbers
    - if running time is long on some instance, it’s because we generated unlucky random numbers, not because of the instance itself.
- **Side note**
  - computers cannot generate truly random numbers
  - we assume there is a pseudo-random number generator (PRNG), a deterministic program that uses an initial value or seed to generate a sequence of seemingly random numbers
  - quality of randomized algorithm depends on the quality of the PRNG
Expected Running Time

- How do we measure the running time of a randomized algorithm?
  - It depends on the input $I$ and on $R$, the sequence of random numbers an algorithm chooses during execution.

- Define $T(I, R)$ to be the running time of a randomized algorithm for instance $I$ and $R$.

- The expected running time $T_{\text{exp}}(I)$ for instance $I$ is the expected value for $T(I, R)$.
  \[
  T_{\text{exp}}(I) = \mathbb{E}[T(I, R)] = \sum_{\text{all possible sequences } R} T(I, R) \cdot \Pr[R]
  \]

- **Worst-case expected running time**
  \[
  T_{\text{exp}}(n) = \max_{\{I: \text{size}(I) = n\}} T_{\text{exp}}(I)
  \]

- **Average-case expected running time**
  \[
  T_{\text{exp}}(n) = \frac{1}{|\{I: \text{size}(I) = n\}|} \sum_{I: \text{size}(I) = n} T_{\text{exp}}(I)
  \]

- Usually design $A$ so that all instances of size $n$ have the same expected run time.

- Thus the average and worst case expected run times are the same, and we just compute the worst case expected time.
Expected Running Time

- How do we measure the running time of a randomized algorithm?
  - It depends on the input $I$ and on $R$, the sequence of random numbers an algorithm chooses during execution.

- Define $T(I, R)$ to be running time of randomized algorithm for instance $I$ and $R$.

- The expected running time $T^{\text{exp}}(I)$ for instance $I$ is expected value for $T(I, R)$.

  $$T^{\text{exp}}(I) = \mathbb{E}[T(I, R)] = \sum_{\text{all possible sequences } R} T(I, R) \cdot \Pr[R]$$

- Worst-case expected running time
  $$T^{\text{exp}}(n) = \max_{\{I: \text{size}(I) = n\}} T^{\text{exp}}(I)$$

- Average-case expected running time
  $$T^{\text{exp}}(n) = \frac{1}{|\{I: \text{size}(I) = n\}|} \sum_{I: \text{Size}(I) = n} T^{\text{exp}}(I)$$

- Usually design $A$ so that all instances of size $n$ have the same expected run time.
- Thus average and worst case expected run times are usually the same.
  - Just compute the worst case expected time.

- Sometimes we also want to know the running time if we got really unlucky with the random numbers $R$ we generate during the execution, or, formally:

  $$\max_R \max_{\{I: \text{size}(I) = n\}} T(I, R)$$
Randomized QuickSelect: Shuffle

- **Goal**: create a randomized version of *QuickSelect* for which all input has the same expected run-time
- **First idea**: first randomly permute input using *shuffle* and then run selection algorithm

```plaintext
shuffle(A)
A : array of size n
for i ← 0 to n − 1 do
    swap(A[i], A[random(i + 1)])
```

- `random(n)` returns an integer uniformly sampled from `{0, 1, 2, ..., n − 1}`
- can show that expected running time is $\Theta(n)$, the same as average running time
Randomized QuickSelect: Shuffle

- **Goal**: create a randomized version of *QuickSelect* for which all input has the same expected run-time

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  ```
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  A : array of size n
  for i ← 0 to n − 1 do
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  ```

- *random*(n) returns an integer uniformly sampled from \{0, 1, 2, ..., n − 1\}

- can show that expected running time is \(\Theta(n)\), the same as average running time

- if we get very unlucky with random numbers, we could get a sorted or almost sorted array after shuffle, resulting in \(O(n^2)\) performance for selection algorithm
  - probability of this happening is almost zero

- whereas the user is quite likely to give instance which is sorted or almost sorted to the selection algorithm
  - probability is far from zero, humans often produce almost sorted data
Randomized QuickSelect: Random Pivot

- **Second idea**: select a random pivot from \( \{0, 1, 2, \ldots, n - 1\} \)

  
  ```
  choose-pivot2(A)
  return random(A.size())
  ```

- Simpler and more efficient than shuffling the array
- Usually fastest in practice
- Expected running time is again \( \Theta(n) \)
Efficiency of Randomized QuickSelect

Assume all elements of $A$ are distinct

Select pivot with equal probability at each recursive call, and independently from other recursive calls

- $P(\text{pivot has index } i) = \frac{1}{n}$ for any instance of size $n$

$T^{\text{exp}}(I)$ depends only on the size of $I$, not the contents of $I$

Let $T^{\text{exp}}(n)$ be expected time on an instance of size $n$

Running time to partition array is $cn$, and with probability $\frac{1}{n}$ pivot-index is $i$

<table>
<thead>
<tr>
<th>$T^{\text{exp}}(i)$</th>
<th>$v$</th>
<th>$T^{\text{exp}}(n - i - 1)$</th>
</tr>
</thead>
</table>

size $i$

size $n - i - 1$

running time if pivot index is $i \leq c \cdot n + \max\{T^{\text{exp}}(i), T^{\text{exp}}(n - i - 1)\}$
Efficiency of Randomized QuickSelect

running time if pivot-index is \( i \leq c \cdot n + \max\{T^{exp}(i), T^{exp}(n - i - 1)\} \)

- Taking expectation over pivot index \( i \)

\[
T^{exp}(n) = \sum_{i=0}^{n-1} (\text{running time if pivot index is } i) \cdot P(\text{index of pivot is } i)
\]

\[
\leq \sum_{i=0}^{n-1} (cn + \max\{T^{exp}(i), T^{exp}(n - i - 1)\}) \cdot \frac{1}{n}
\]

\[
\leq cn + \sum_{i=0}^{n-1} \frac{1}{n} \max\{T^{exp}(i), T^{exp}(n - i - 1)\}
\]

- Same recurrence as for non-randomized average case
- Resolves to \( \Theta(n) \) expected time on instance of size \( n \)
- Side note
  - there is selection algorithm “Median of Medians” (cs341) that has worst-case running time \( O(n) \)
    - uses double recursion
    - slower in practice
QuickSelect: Badly Designed Randomization

\begin{align*}
\text{\texttt{choose-random-pivot-badly}(A)} \\
\text{if } A.size \geq 3 \text{ return random}(3) \\
\text{else return 0}
\end{align*}

\begin{align*}
T^{exp}(n) &= \max_{\{I:\text{size}(I)=n\}} T^{exp}(I) \\
T^{exp}(I_n) &= \begin{cases} 
  cn + \frac{1}{3} T^{exp}(I_{n-1}) + \frac{1}{3} T^{exp}(I_{n-2}) + \frac{1}{3} T^{exp}(I_{n-3}) & \text{if } n \geq 3 \\
  \text{c} & \text{if } n < 3
\end{cases} \\
T^{exp}(I_n) &\geq cn + T(I_{n-3}) \text{ if } n \geq 3 \\
\text{Resolves to } \Theta(n^2) \\
\text{Worst case expected time is } \Theta(n^2)
\end{align*}
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QuickSort

- Hoare developed *partition* and *quick-select* in 1960
- He also used them to *sort* based on partitioning

\[
quick-sort_1(A)
\]

Input: array \(A\) of size \(n\)

\[
\begin{align*}
\text{if } n \leq 1 & \text{ then return} \\
p & \leftarrow \text{choose-pivot}_1(A) \\
i & \leftarrow \text{partition}(A, p) \\
quick-sort_1(A[0, 1, \ldots, i - 1]) & \\
quick-sort_1(A[i + 1, \ldots, n - 1])
\end{align*}
\]

- Let \(T(n)\) to be the runtime on size \(n\) array
- If we know pivot-index \(i\), then \(T(n) = cn + T(i) + T(n - i - 1)\)
- Worst case \(T(n) = T(n - 1) + cn\)
  - recurrence solved in the same way as *quick-select*, \(\Theta(n^2)\)
- Best case \(T(n) = T([n/2]) + T([n/2]) + cn\)
  - solved in the same way as *merge-sort*, \(\Theta(n \log n)\)
Average-case analysis of quick-sort1

- Make the same assumptions as for quick-select1
- Deriving recurrence equation is similar to quick-select1, but recurse on both sides

Using the same approach as for quick-select1, average running time is

\[ T(n) = \frac{1}{n} \sum_{i=0}^{n-1} (cn + T(i) + T(n - i - 1)), \quad n \geq 2 \]

Running time is proportional to the number of comparisons

Recurrence for counting comparisons

\[ T(n) = \frac{1}{n} \sum_{i=0}^{n-1} (n + T(i) + T(n - i - 1)), \quad n \geq 2 \]
Average-case analysis of quick-sort

First let us get a simpler recursive expression for $T(n)$

$$T(n) = \frac{1}{n} \sum_{i=0}^{n-1} (n + T(i) + T(n - i - 1))$$

$$= n + \frac{1}{n} \sum_{i=0}^{n-1} T(i) + \frac{1}{n} \sum_{i=0}^{n-1} T(n - i - 1)$$

$T(0) + T(1) + \cdots + T(n - 1)$

$T(n - 1) + T(n - 2) + \cdots + T(0)$

$$= n + \frac{2}{n} \sum_{i=0}^{n-1} T(i)$$

Thus $T(n) = n + \frac{2}{n} \sum_{i=0}^{n-1} T(i)$
Average-case analysis of quick-sort

\[ T(n) = n + \frac{2}{n} \sum_{i=0}^{n-1} T(i) \text{ is } \Theta(n \log n) \]

Proof

Multiply by \( n \):

\[ nT(n) = n^2 + 2 \sum_{i=0}^{n-1} T(i) \]

Plug in \( n - 1 \):

\[ (n - 1)T(n - 1) = (n - 1)^2 + 2 \sum_{i=0}^{n-2} T(i) \]

Subtract:

\[ nT(n) - (n - 1)T(n - 1) = 2n - 1 + 2T(n - 1) \]

Rearrange:

\[ nT(n) = (n + 1)T(n - 1) + 2n - 1 \]

Divide by \((n + 1)n\):

\[ \frac{T(n)}{n + 1} = \frac{T(n - 1)}{n} + \frac{2n - 1}{n(n + 1)} \]

Let \( A(n) = \frac{T(n)}{n+1} \):

\[ A(n) = A(n - 1) + \frac{2n - 1}{n(n+1)} = A(n - 2) + \frac{2(n - 1) - 1}{(n-1)n} + \frac{2n - 1}{n(n + 1)} \]

\[ = \ldots = \sum_{i=1}^{n} \frac{2i - 1}{i(i + 1)} = \sum_{i=1}^{n} \frac{2}{i + 1} - \sum_{i=1}^{n} \frac{1}{i(i + 1)} \]

Therefore:

\[ A(n) = c \log n \]

Finally:

\[ T(n) = (n + 1)A(n) = c(n + 1) \log n \in \Theta(n \log n) \]
Improvement ideas for QuickSort

- Randomize by using choose-pivot2, giving \( \Theta(n \log n) \) expected time for quick-sort2
- The auxiliary space is \( \Omega(\text{recursion depth}) \)
  - \( \Theta(n) \) in the worst-case
  - can be reduce to \( \Theta(\log n) \) worst-case by
    - recurse in smaller sub-array first
    - replacing the other recursion by a while-loop (tail call elimination)
- Stop recursion when, say \( n \leq 10 \)
  - array is not completely sorted, but almost sorted
  - at the end, run insertionSort, it sorts in just \( O(n) \) time since all items are within 10 units of the required position
- Arrays with many duplicates sorted faster by changing partition to produce three subsets
  - \(< v \quad = v \quad > v > v\rangle < v \quad = v \quad > v \rangle
- Programming tricks
  - instead of passing full arrays, pass only the range of indices
  - avoid recursion altogether by keeping an explicit stack
QuickSort with Tricks

quick-sort3(A, n)

initialize a stack S of index-pairs with {(0, n − 1)}

while S is not empty

(l, r) ← S.pop() // get the next subproblem

while r − l + 1 > 10 // work on it if it’s larger than 10

p ← choose-pivot2(A, l, r)

i ← partition (A, l, r, p)

if i − l > r − i do // is left side larger than right?

S.push((l, i − 1)) // store larger problem in S for later

l ← i + 1 // next work on the right side

else

S.push((i + 1, r)) // store larger problem in S for later

r ← i − 1 // next work on the left side

InsertionSort(A)

- This is often the most efficient sorting algorithm in practice
Outline

- Sorting and Randomized Algorithms
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  - Randomized Algorithms
  - QuickSort
- Lower Bound for Comparison-Based Sorting
- Non-Comparison-Based Sorting
Lower bounds for sorting

- We have seen many sorting algorithms

<table>
<thead>
<tr>
<th>Sort</th>
<th>Running Time</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selection Sort</td>
<td>$\Theta(n^2)$</td>
<td>worst-case</td>
</tr>
<tr>
<td>Insertion Sort</td>
<td>$\Theta(n^2)$</td>
<td>worst-case</td>
</tr>
<tr>
<td>Merge Sort</td>
<td>$\Theta(n \log n)$</td>
<td>worst-case</td>
</tr>
<tr>
<td>Heap Sort</td>
<td>$\Theta(n \log n)$</td>
<td>worst-case</td>
</tr>
<tr>
<td>quick-sort1</td>
<td>$\Theta(n \log n)$</td>
<td>average-case</td>
</tr>
<tr>
<td>quick-sort2</td>
<td>$\Theta(n \log n)$</td>
<td>expected</td>
</tr>
</tbody>
</table>

**Question**: Can one do better than $\Theta(n \log n)$ running time?

**Answer**: *It depends on what we allow*

- No: comparison-based sorting lower bound is $\Omega(n \log n)$
  - no restriction on input, just must be able to compare
- Yes: non-comparison-based sorting can achieve $O(n)$
  - restrictions on input
The Comparison Model

- All sorting algorithms seen so far are in the comparison model.
- In the **comparison model** data can only be accessed in two ways:
  - comparing two elements
    - \( A[i] \leq A[j] \)
  - moving elements around (e.g. copying, swapping)
- This makes very few assumptions on the things we are sorting:
  - just count the number of above operations
- Under comparison model, will show that any sorting algorithm requires \( \Omega(n \log n) \) comparisons.
- This lower bound is not for an algorithm, it is for the sorting problem.
- How can we talk about problem without algorithm?
  - count number of comparisons any sorting algorithm has to perform.
Decision Tree

- Decision tree succinctly describes all the decisions that are taken during the execution of an algorithm and the resulting outcome.
- For each sorting algorithm we can construct a corresponding decision tree.
- Given decision tree, we can deduce the algorithm.
- Decision tree can be constructed for any algorithm, not just sorting.
Decision Tree Example

- Decision tree for a concrete comparison based sorting algorithm, with 3 non-repeating elements \([x_0, x_1, x_2]\)

Set of all possible inputs

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Comparison</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 1, 2</td>
<td>(x_0 &lt; x_1 &lt; x_2)</td>
<td>([x_0, x_1, x_2])</td>
</tr>
<tr>
<td>0, 2, 1</td>
<td>(x_0 &lt; x_2 &lt; x_1)</td>
<td>([x_0, x_2, x_1])</td>
</tr>
<tr>
<td>1, 0, 2</td>
<td>(x_1 &lt; x_0 &lt; x_2)</td>
<td>([x_1, x_0, x_2])</td>
</tr>
<tr>
<td>1, 2, 0</td>
<td>(x_2 &lt; x_0 &lt; x_1)</td>
<td>([x_2, x_0, x_1])</td>
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<td>2, 1, 0</td>
<td>(x_2 &lt; x_1 &lt; x_0)</td>
<td>([x_2, x_1, x_0])</td>
</tr>
</tbody>
</table>

- Have to determine which of the 6 inputs we are given before can give output unique output for each distinct input
Decision Tree

- Decision tree for a concrete comparison based sorting algorithm, with 3 non-repeating elements

- Root corresponds to the set of all possible inputs
- Interior nodes are comparisons: each comparison splits the set of possible inputs into two
- Know correct sorting order only when the set of possible inputs shrinks to size one
  - nodes where possible input shrunk to size one are leaves, when reach them, can output sorting result
- Sorting algorithm will traverse a path starting at root and ending at a leaf
  - length of the path is the number of comparisons to be made
- Tree height is the number of comparisons required for sorting in the worst case
 Decision Tree

- Decision tree for a concrete comparison based sorting algorithm, with 3 non-repeating elements

- Algorithm could do more comparisons than necessary
- Thus can have more leafs than possible inputs
- But the number of leaves must be \textit{at least} the number of possible inputs
Decision Tree

- **Decision tree** for any comparison-based sorting algorithm, $n$ non-repeating elements

Tree must have at least $n!$ leaves
- Binary tree with height $h$ has at most $2^h$ leaves
- Height $h$ must be at least such that $2^h \geq n!$
- Tree height is the number of comparisons required in the worst case
Lower bound for sorting in the comparison model

**Theorem:** Any correct comparison-based sorting algorithm requires at least \( \Omega(n \log n) \) comparisons.

**Proof:**

- There exists a set of \( n! \) possible inputs such that each leads to a different output.
- Decision tree must have at least \( n! \) leaves.
- Binary tree with height \( h \) has at most \( 2^h \) leaves.
- Height \( h \) must be at least such that \( 2^h \geq n! \).
- Taking logs of both sides:

\[
h \geq \log(n!) = \log(n(n-1) \ldots 1) = \log n + \cdots + \log \left( \frac{n}{2} + 1 \right) + \log \frac{n}{2} + \cdots + \log 1
\]

\[
\geq \log \frac{n}{2} + \cdots + \log \frac{n}{2} = \frac{n}{2} \log \frac{n}{2} = \frac{n}{2} \log n - \frac{n}{2} \in \Omega(n \log n)
\]
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Non-Comparison-Based Sorting

- Sort without comparing items to each other
- Non-comparison based sorting is less general than comparison based
- In particular, we need to make assumptions about items we sort
  - unlike in comparison based sorting, which sorts any data, as long as it can be compared
- Will assume we are sorting non-negative integers
  - can adapt to negative integers
  - also to some other data types, such as strings
  - but cannot sort arbitrary data
Non-Comparison-Based Sorting

- Simplest example
  - suppose all keys in $A$ are integers in range $[0, ..., L - 1]$
- For non-comparison sorting, running time depends on both
  - array size $n$
  - $L$
Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, ..., L - 1]$
- Use an auxiliary *bucket array* $B[0, ..., L - 1]$ to sort
  - i.e. array of initially empty linked lists, initialization is $\Theta(L)$
- Example with $L = 15$

<table>
<thead>
<tr>
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</tbody>
</table>
Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, \ldots, L - 1]$
- Use an axillary *bucket array* $B[0, \ldots, L - 1]$ to sort
  - i.e. array of linked lists, initialization is $\Theta(L)$
- Example with $L = 15$

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<p>| | | | | | | | | | | | | |</p>
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</tbody>
</table>
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$k = 0$

```
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<td>12</td>
<td>14</td>
</tr>
</tbody>
</table>
```

$B$
Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, \ldots, L - 1]$
- Use an axillary *bucket array* $B[0, \ldots, L - 1]$ to sort
  - i.e. array of linked lists, initialization is $\Theta(L)$
- Example with $L = 15$

![Diagram of bucket sort with keys 12, 14, 7, 6, 7, 0, 10 and bucket array $B$ showing keys 12 and 14 in the 2nd bucket]
Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, \ldots, L - 1]$
- Use an axillary *bucket array* $B[0, \ldots, L - 1]$ to sort
  - i.e. array of linked lists, initialization is $\Theta(L)$
- Example with $L = 15$

![Diagram](image)

$A$

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</table>

$k = 2$

$B$

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<th>3</th>
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<tbody>
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</tbody>
</table>

Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, ..., L - 1]$
- Use an auxiliary *bucket array* $B[0, ..., L - 1]$ to sort
  - i.e. array of linked lists, initialization is $\Theta(L)$
- Example with $L = 15$

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>6</td>
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<tr>
<td>14</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
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<tr>
<td>0</td>
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<tr>
<td>10</td>
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</tr>
</tbody>
</table>

$k = 3$
Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, ..., L - 1]$.
- Use an auxiliary *bucket array* $B[0, ..., L - 1]$ to sort.
  - i.e. array of linked lists, initialization is $\Theta(L)$.
- Example with $L = 15$.

$$
\begin{array}{c}
A \\
12 \\
14 \\
7 \\
6 \\
0 \\
10 \\
\end{array}
\quad
\begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } & \underline{\ } \\
\end{array}
$$

$k = 4$

\[
\begin{array}{c}
6 \\
7 \\
12 \\
14 \\
\end{array}
\]
Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, \ldots, L - 1]$
- Use an auxiliary bucket array $B[0, \ldots, L - 1]$ to sort
  - i.e. array of linked lists, initialization is $\Theta(L)$
- Example with $L = 15$

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0, 7</td>
</tr>
<tr>
<td>14</td>
<td>6, 7</td>
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<tr>
<td>7</td>
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</table>
Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, ..., L - 1]$
- Use an auxiliary *bucket array* $B[0, ..., L - 1]$ to sort
  - i.e. array of linked lists, initialization is $\Theta(L)$
- Example with $L = 15$

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
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<tbody>
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</tbody>
</table>

$k = 6$
Bucket Sort

- Suppose all keys in $A$ are integers in range $[0, ..., L - 1]$
- Use an axillary *bucket array* $B[0, ..., L - 1]$ to sort
  - i.e. array of linked lists, initialization is $\Theta(L)$
- Example with $L = 15$
- Now iterate through $B$ and copy non-empty buckets to $A$

- Time complexity is $\Theta(L + n)$
  - $n$ is size of $A$
Digit Based Non-Comparison-Based Sorting

- Running time of bucket sort is $\Theta(L + n)$
  - $n$ is size of $A$
  - $L$ is range $[0, L)$ of integers in $A$
- What if $L$ is much larger than $n$?
  - i.e. $A$ has size 100, range of integers in $A$ is $[0, \ldots, 99999]$
- Assume at most $m$ digits in any key
  - pad with leading 0s
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</thead>
<tbody>
<tr>
<td>123</td>
<td>230</td>
<td>021</td>
<td>320</td>
<td>210</td>
<td>232</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>101</td>
</tr>
</tbody>
</table>

- Can sort ‘digit by digit’, can go
  - forward, from digit $1 \rightarrow m$ (more obvious)
  - backward, from digit $m \rightarrow 1$ (less obvious)
- bucket sort is perfect for sorting ‘by digit’
- Example: $A$ has size 100, range of integers in $A$ is $[0, \ldots, 99999]$
  - integers have at most 5 digits, need only 5 iterations of bucket sort
Bucket Sort on Last Digit

- Equivalent to normal bucket sort if we redefine comparison
  - \( a \leq b \) if the last digit of \( a \) is smaller than (or equal) to the last digit of \( b \)

\[ \begin{array}{c|c|c}
A & B & A \\
\hline
123 & B[0] & 230 \\
230 & B[1] & 320 \\
320 & B[3] & \end{array} \]

- Bucket sort is stable: equal items stay in original order
  - crucial for developing LSD radix sort later
Base $R$ number representation

- Number of distinct digits gives the number of buckets $R$
- Useful to control number of buckets
  - larger $R$ means less digits (less iterations), but more work per iteration (larger bucket array)
  - may want exactly 2, or 4, or even 128 buckets
- Can do so with base $R$ representation
  - digits go from 0 to $R - 1$
  - $R$ buckets
  - numbers are in the range $\{0, 1, \ldots, R^m - 1\}$
- From now on, assume keys are numbers in base $R$ ($R$: radix)
  - $R = 2, 10, 128, 256$ are common
- Example ($R = 4$)

| 123 | 230 | 21 | 320 | 210 | 232 | 101 |
Bucket-sort \((A, d)\)

\(A\) : array of size \(n\), contains numbers with digits in \(\{0, \ldots, R - 1\}\)

\(d\): index of digit by which we wish to sort

initialize array \(B[0, \ldots, R - 1]\) of empty lists (buckets)

\[\text{for } i \leftarrow 0 \text{ to } n - 1 \text{ do} \]

\[\text{next } \leftarrow A[i] \]

\[\text{append } next \text{ at end of } B[d\text{th digit of } next] \]

\(i \leftarrow 0\)

\[\text{for } j \leftarrow 0 \text{ to } R - 1 \text{ do} \]

\[\text{while } B[j] \text{ is non-empty do} \]

\[\text{move first element of } B[j] \text{ to } A[i + 1] \]

- Sorting is stable: equal items stay in original order
- Run-time \(\Theta(n + R)\)
- Auxiliary space \(\Theta(n + R)\)
  - \(\Theta(R)\) for array \(B\), and linked lists are \(\Theta(n)\)
Single Digit Bucket Sort

- **Bucket-sort**
  - $A$: array of size $n$, contains numbers with digits in $\{0, \ldots, R-1\}$
  - $d$: index of digit by which we wish to sort

- Initialize array $B[0], \ldots, B[R-1]$ of empty lists (buckets)

- For $i \leftarrow 0$ to $n-1$ do
  - $\text{next} \leftarrow A[i]$
  - Append next at end of $B[d]$th digit of next

- $i \leftarrow 0$

- For $j \leftarrow 0$ to $R-1$ do
  - While $B[j]$ is non-empty do
    - Move first element of $B[j]$ to $A[i]+1$

- Sorting is stable
- Run-time $\Theta(n + R)$
- Auxiliary space $\Theta(n + R)$
  - $\Theta(R)$ for array $B$, and linked lists are $\Theta(n)$
- Can replace lists by two auxiliary arrays of size $R$ and $n$, resulting in count-sort
  - no details
MSD-Radix-Sort

- Sorts multi-digit numbers from the most significant to the least significant
- Start by sorting the whole array by the first digit

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
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</tbody>
</table>
MSD-Radix-Sort

- Sorts multi-digit numbers from the most significant to the least significant
- Start by sorting the whole array by the first digit

<table>
<thead>
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<th>123</th>
</tr>
</thead>
<tbody>
<tr>
<td>232</td>
</tr>
<tr>
<td>021</td>
</tr>
<tr>
<td>320</td>
</tr>
<tr>
<td>210</td>
</tr>
<tr>
<td>230</td>
</tr>
<tr>
<td>101</td>
</tr>
</tbody>
</table>
**MSD-Radix-Sort**

- Sorts multi-digit numbers from the most significant to the least significant.
- Start by sorting the whole array by the first digit.

**Diagram:**

- Group 1: 021, 123, 101, 232
- Group 2: 101, 232
- Group 3: 210, 230
- Group 4: 320

- Cannot sort the whole array by the second digit, will mess up the order.
- Have to break down in groups by the first digit:
  - Each group can be safely sorted by the second digit.
  - Call sort recursively on each group, with appropriate array bounds.
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

- group 1
  - 021
  - 123
  - 101
  - 232
  - 210
  - 230
  - 320

- group 1
  - 021
  - 021

- recursion depth 0
- recursion depth 1
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

```
021
123
101
232
210
230
320
```

Recursion depth 0

Recursion depth 1
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

- group 1
  - 021
- group 2
  - 123
  - 101
  - 232
- group 3
  - 210
  - 230
- group 4
  - 320

- recursion depth 0
- recursion depth 1
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

```
group 1
  021
  123
  101
  232
  210
  230
  320
```

```
group 2
  101
  123
```

```
group 3
  101
  123
```

```
group 4
```

Recursion depth 0
Recursion depth 1
Recursion depth 2
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

![Diagram showing the MSD-Radix-Sort process]

- group 1
  - 021
  - 123
  - 101
  - 232

- group 2
  - 210
  - 230

- group 3
  - 101
  - 123

- group 4
  - 123

recursion depth 0
recursion depth 1
recursion depth 2
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

```
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

```

```plaintext
Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

```
**MSD-Radix-Sort**

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

![Diagram showing MSD-Radix-Sort process](image-url)
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

```
<table>
<thead>
<tr>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
<th>Group 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>021</td>
<td>123</td>
<td>101</td>
<td>232</td>
</tr>
<tr>
<td>123</td>
<td>101</td>
<td>210</td>
<td>230</td>
</tr>
<tr>
<td>232</td>
<td>210</td>
<td>230</td>
<td>320</td>
</tr>
</tbody>
</table>
```

Recursion depth:
- Depth 0
- Depth 1
- Depth 2
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

<table>
<thead>
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<th>group 2</th>
<th>group 3</th>
<th>group 4</th>
</tr>
</thead>
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recursion depth 0
recursion depth 1
recursion depth 2
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

![Diagram of MSD-Radix-Sort]
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group

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Recursion depth:
- Depth 0: 021
- Depth 1: 101, 123
- Depth 2: 210, 230, 232, 320
MSD-Radix-Sort

- Recursively sorts multi-digit numbers
  - sort by leading digit, group by next digit, then call sort recursively on each group
MSD-Radix-Sort Space Analysis

- Bucket-sort
  - auxiliary space $\Theta(n + R)$
- Recursion depth is $m - 1$
  - auxiliary space $\Theta(m)$
- Total auxiliary space $\Theta(n + R + m)$
MSD-Radix-Sort Time Analysis

- Time spent for each recursion depth
  - Depth 0
    - one bucket sort on $n$ items
    - $\Theta(n + R)$
  - All other depths
    - lets $k$ be the number of bucket sorts at each depth
      - $k \leq n$
        - cannot have more bucket sorts than the array size
    - each bucket sort is on $n_i$ items
    - $\sum_{i=0}^{k} n_i = n$
    - each bucket sort is $n_i + R$
    - $\sum_{i=0}^{k}(n_i + R) = n + \sum_{i=0}^{k} R \leq n + nR$
    - total time at any depth is $O(nR)$

- Number of depths is at most $m - 1$
- Total time $O(mnR)$
MSD-Radix-Sort Time Analysis

- Total time $O(mnR)$
- This is $O(n)$ if sort items in limited range
  - suppose $R = 2$, and we sort are $n$ integers in the range $[0, 2^{10})$
  - then $m = 10$, $R = 2$, and sorting is $O(n)$
    - note that $n$, the number of items to sort, can be arbitrarily large
MSD-Radix-Sort Time Analysis

- Total time $O(mnR)$
- This is $O(n)$ if sort items in limited range
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  - then $m = 10$, $R = 2$, and sorting is $O(n)$
    - note that $n$, the number of items to sort, can be arbitrarily large
- This does not contradict $\Omega(n\log n)$ bound on the sorting problem, since the bound applies to comparison-based sorting
### MSD-Radix-Sort Pseudocode

- Sorts array of \( m \)-digit radix-\( R \) numbers recursively
- Sort by leading digit, then each group by next digit, etc.

#### MSD-Radix-sort

\[ \text{MSD-Radix-sort}(A, \ l \leftarrow 0, r \leftarrow n - 1, d \leftarrow \text{leading digit index}) \]

\( l, r \) : indexes between which to sort, \( 0 \leq l, r \leq n - 1 \)

- \( \text{if } l < r \)
  - \( \text{bucket-sort}(A[\ l \ldots r], \ d) \)
  - \( \text{if } \) there are digits left
    - \( l' \leftarrow l \)
  - \( \text{while } (l' < r) \text{ do} \)
    - let \( r' \geq l' \) be the maximal \( s.t \) \( A[\ l' \ldots r'] \) have the same \( d \)th digit
    - \( \text{MSD-Radix-sort}(A, l', r', d + 1) \)
    - \( l' \leftarrow l \)

- Run-time \( O(mnR) \)
- Auxiliary space is \( \Theta(m + n + R) \) for bucket sort and recursion stack
- Drawback of \( \text{MSD-Radix-sort} \) is many recursions
LSD-Radix-Sort

- **Idea**: apply single digit bucket sort from least significant digit to the most significant digit
- Observe that digit bucket sort is stable
  - equal elements stay in the original order
  - therefore, we can apply single digit bucket sort to the **whole array**, and the output will be sorted after iterations over all digits
LSD-Radix-Sort

- Prepare to sort by last digit
- Last digit sorted
- Prepare to sort by middle digit
- Last two digits sorted
- Prepare to sort by first digit
- Last three digits sorted

- $m$ bucket sorts, on $n$ items each, one bucket sort is $\Theta(n + R)$
- Total time cost $\Theta(m(n + R))$
LSD-Radix-Sort

\[ \text{LSD-radix-sort}(A) \]

\(A\): array of size \(n\), contains \(m\)-digit radix-\(R\) numbers

\(\text{for } d \leftarrow \text{least significant down to most significant digit } \text{do} \)

\[ \text{bucket-sort}(A, d) \]

- Loop invariant: after iteration \(i\), \(A\) is sorted w.r.t. the last \(i\) digits of each entry
- Time cost \(\Theta(m(n + R))\)
- Auxiliary space \(\Theta(n + R)\)
Summary

- Sorting is an important and very well-studied problem.
- Can be done in $\Theta(n\log n)$ time.
  - Faster is not possible for general input.
- HeapSort is the only $\Theta(n\log n)$ time algorithm we have seen with $O(1)$ auxiliary space.
- MergeSort is also $\Theta(n\log n)$ time.
- Selection and insertion sorts are $\Theta(n^2)$.
- QuickSort is worst-case $\Theta(n^2)$, but often the fastest in practice.
- BucketSort and RadixSort can achieve $o(n\log n)$ if the input is special.
- Best-case, worst-case, average-case can all differ.
- Randomized algorithms can eliminate “bad cases”, resulting in the same expected time for all cases.