CS 240 – Data Structures and Data Management

Module 5: Other Dictionary Implementations

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Based on lecture notes by many previous cs240 instructors

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Outline

- Dictionaries with Lists Revisited
  - Dictionary ADT
    - implementations so far
  - Skip Lists
  - Re-ordering items

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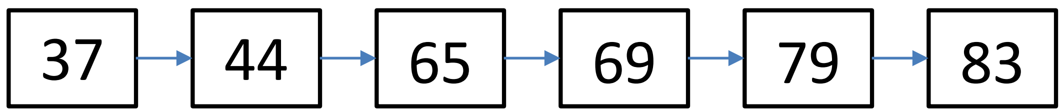
Dictionary ADT: Implementations thus far

- A *dictionary* is a collection of *key-value pairs* (KVPs)
    - *search*, *insert*, and *delete*
- Realizations
    - **Balanced search trees** (AVL trees)
        - $\Theta(\log n)$ search, insert, and delete
        - complex code and not necessarily the fastest running time in practice
    - **Binary search trees**
        - $\Theta(height)$ search, insert and delete
        - simpler than AVL tree
        - randomization helps efficiency
    - **Ordered array**
        - simple implementation
        - $\Theta(\log n)$ search
        - $\Theta(n)$ insert and delete
    - **Ordered linked list**



        - simple implementation
        - $\Theta(n)$ search, insert and delete
        - search is the bottleneck, insert and delete would be $\Theta(1)$ if do search first and account for its running time separately
        - efficient search (like binary search) in ordered linked list?

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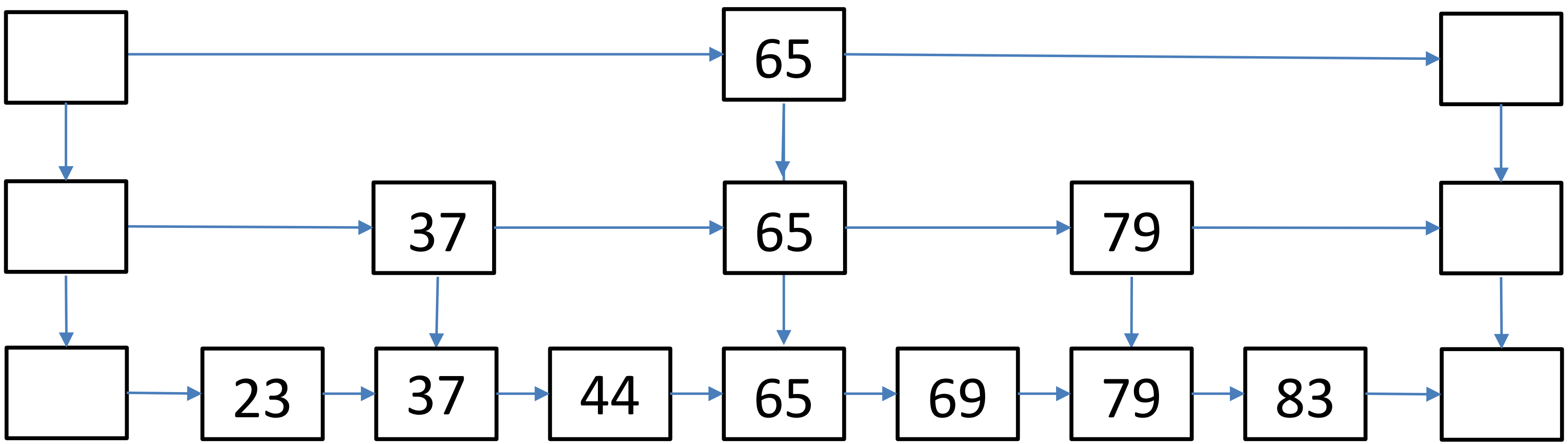
Skip Lists: Motivation

- Ordered array has efficient binary search

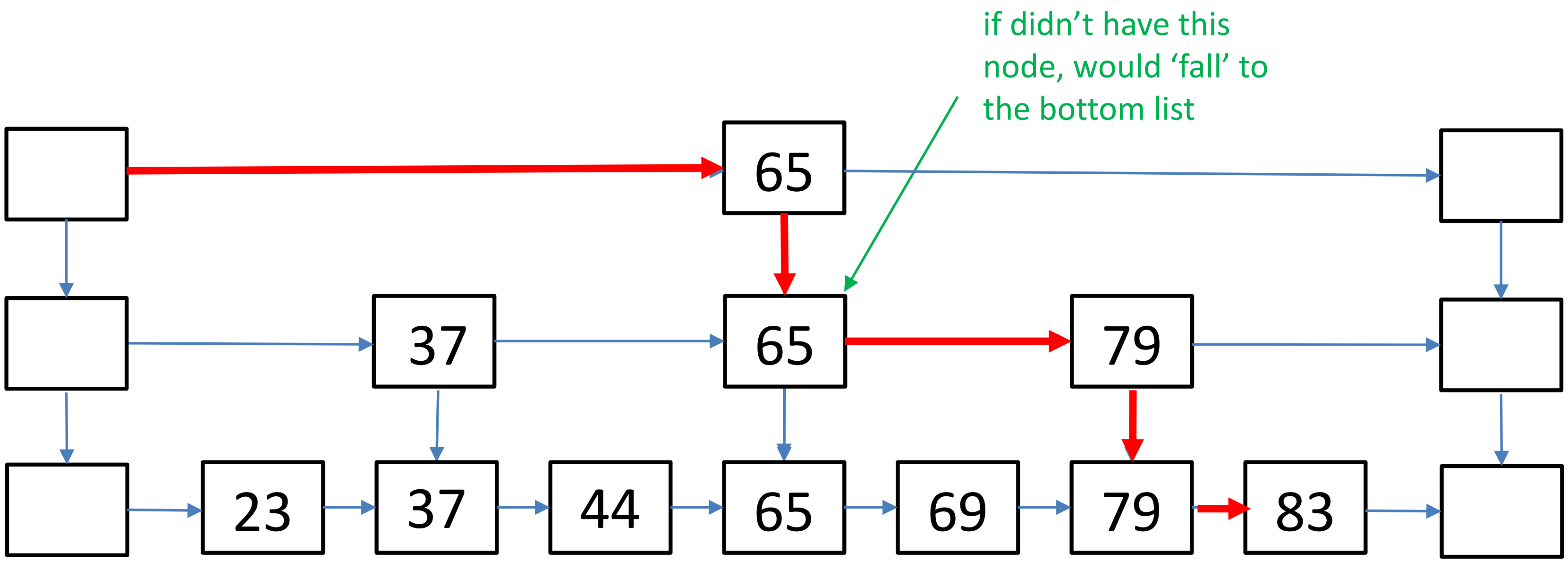
0	1	2	3	4	5	6
23	37	44	65	69	79	83

- Can we imitate binary search in an ordered linked list?



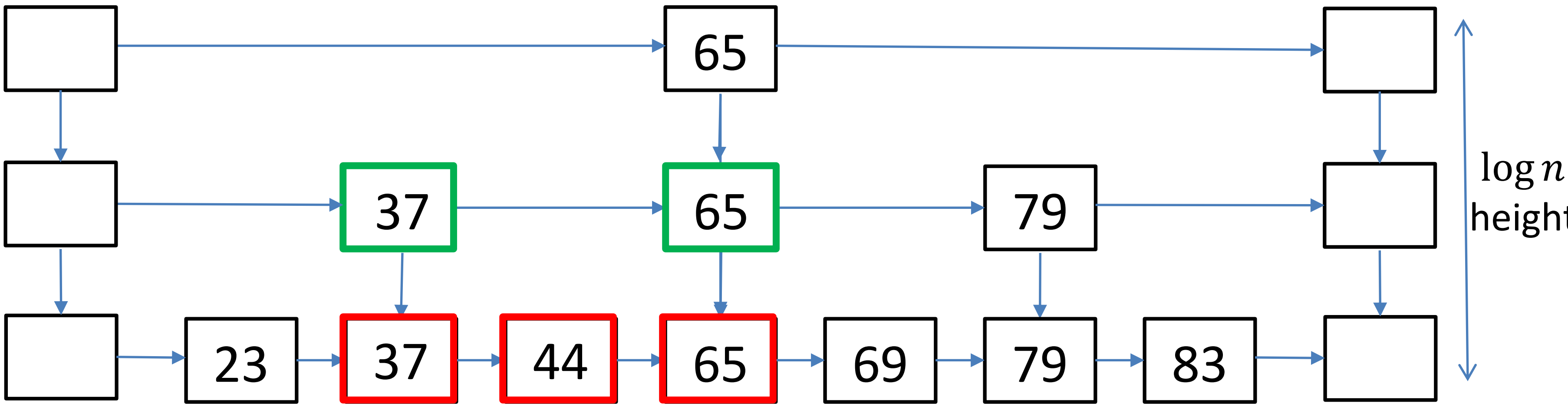
Skip Lists: Motivation

- Search(83)



Skip Lists: Motivation

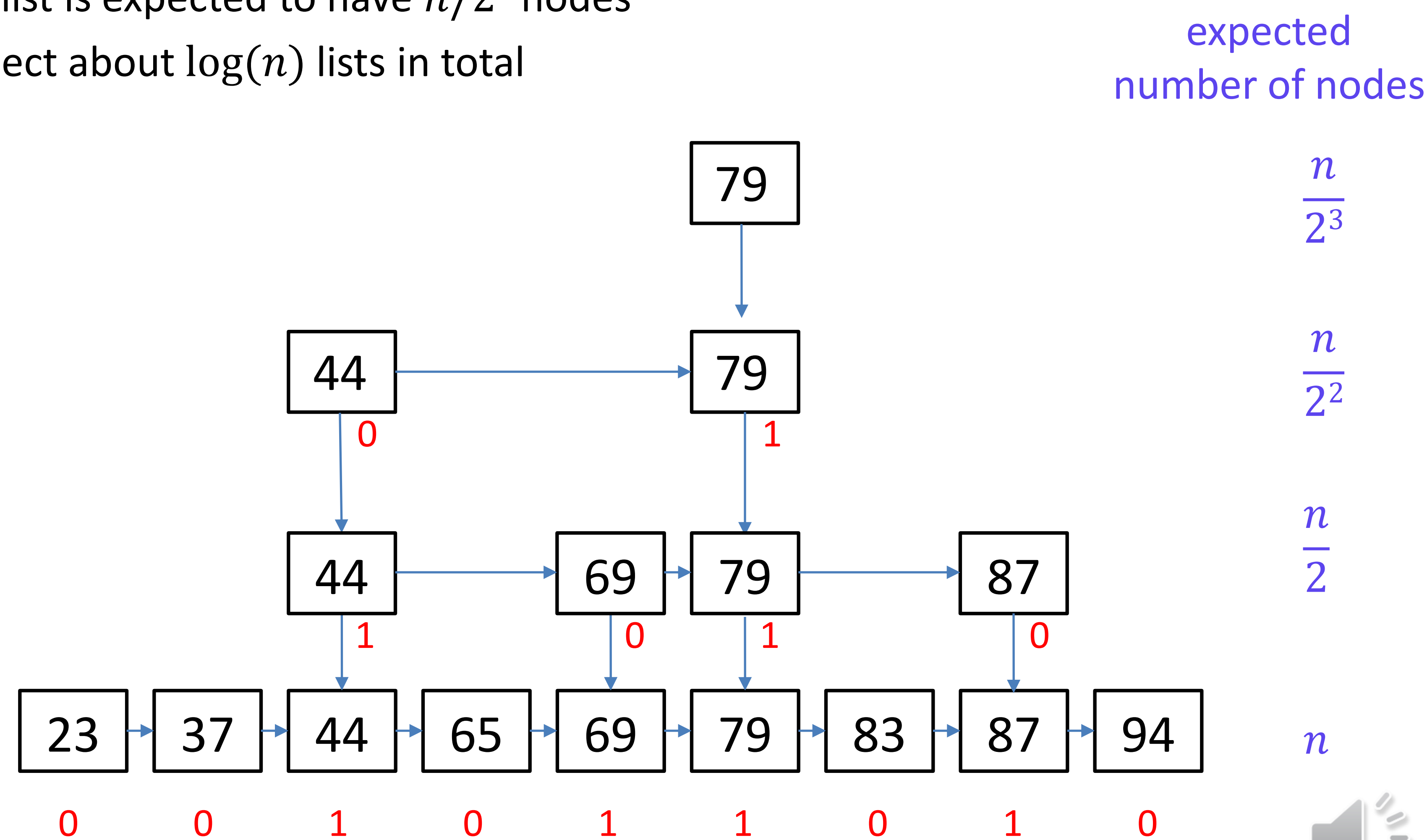
- Imitating binary search with a *hierarchy* of linked lists
  - build from bottom to top, each higher up list has 1/2 of previous list items
  - $\log n$ height (total number of linked lists needed)



- When searching, go through the highest level possible
  - thus visit at most two items at each level
- Easy to implement if data structure is *static*
  - know all items beforehand, no need to insert or delete, but in static case an ordered array will work, and is more efficient (no links)
- To enable insert and delete, use *randomization*

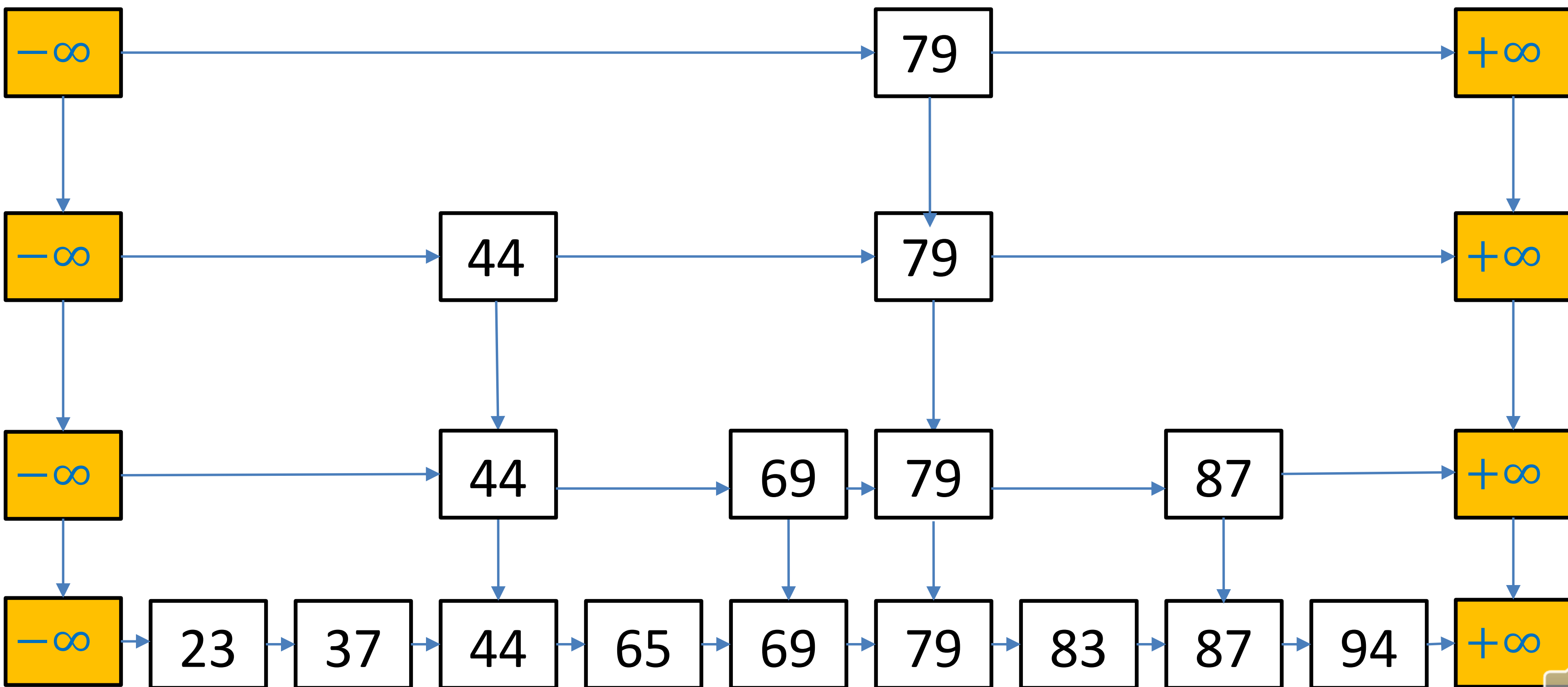
Skip Lists: Motivation

- For next level, choose each item from previous level with probability ½ (coin toss)
- ith list is expected to have $n/2^i$ nodes
- Expect about $\log(n)$ lists in total



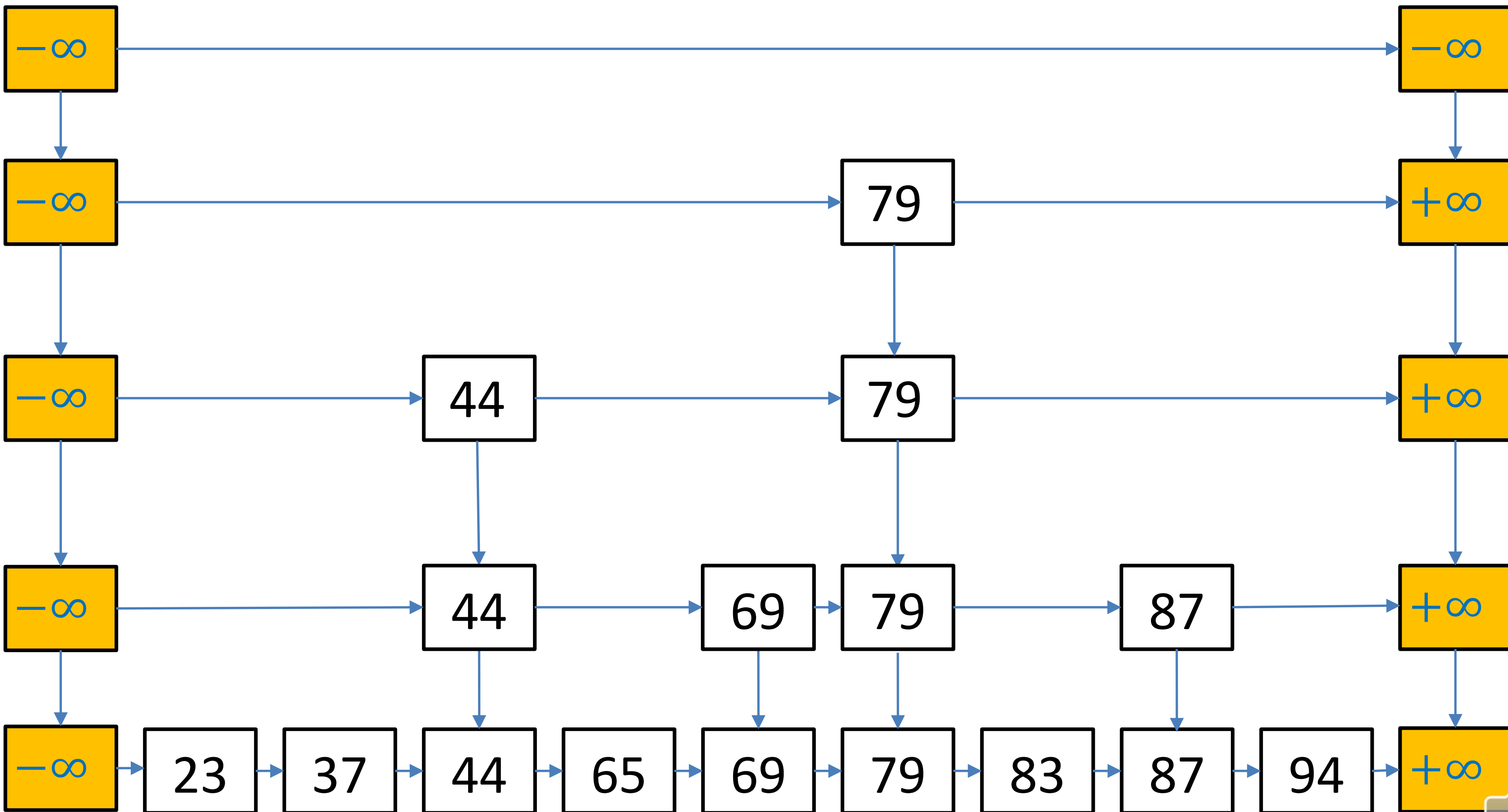
Skip Lists: Motivation

- Insert 'boundary' nodes with special sentinel symbols $-\infty$ and $+\infty$
  - to simplify code for searching



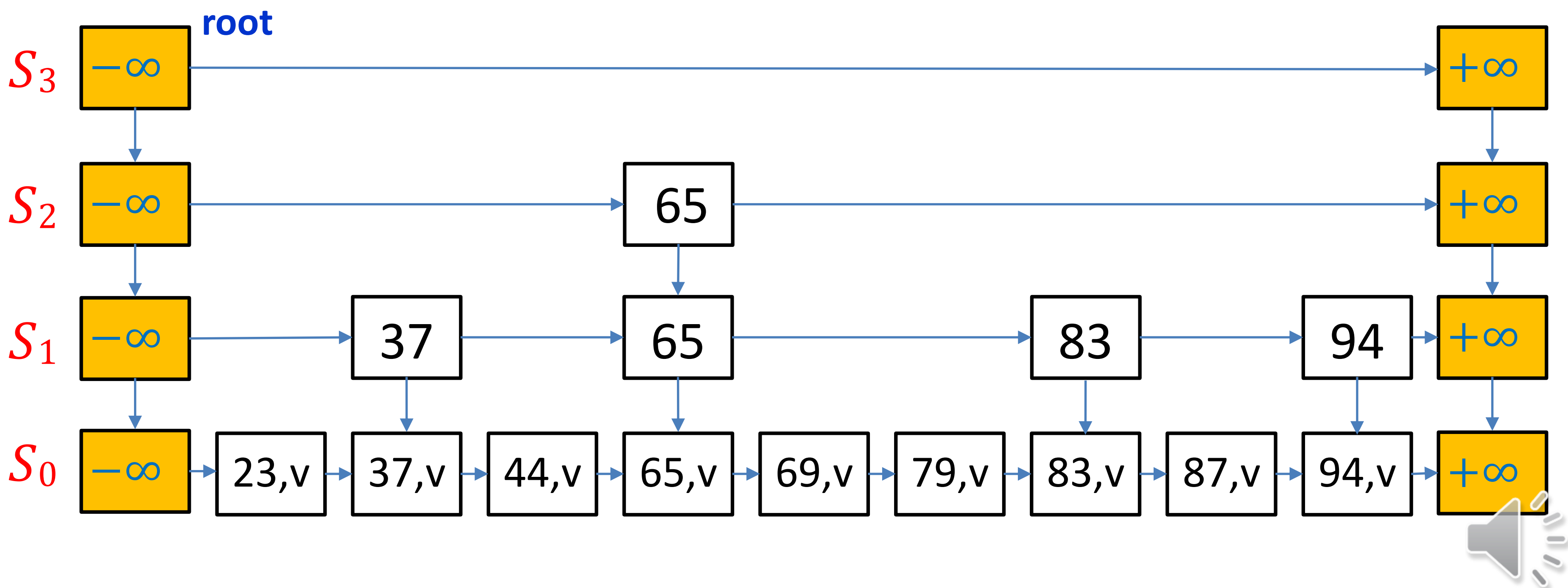
Skip Lists: Motivation

- Insert sentinel only level, with only $-\infty$ and $+\infty$
  - to simplify code for searching



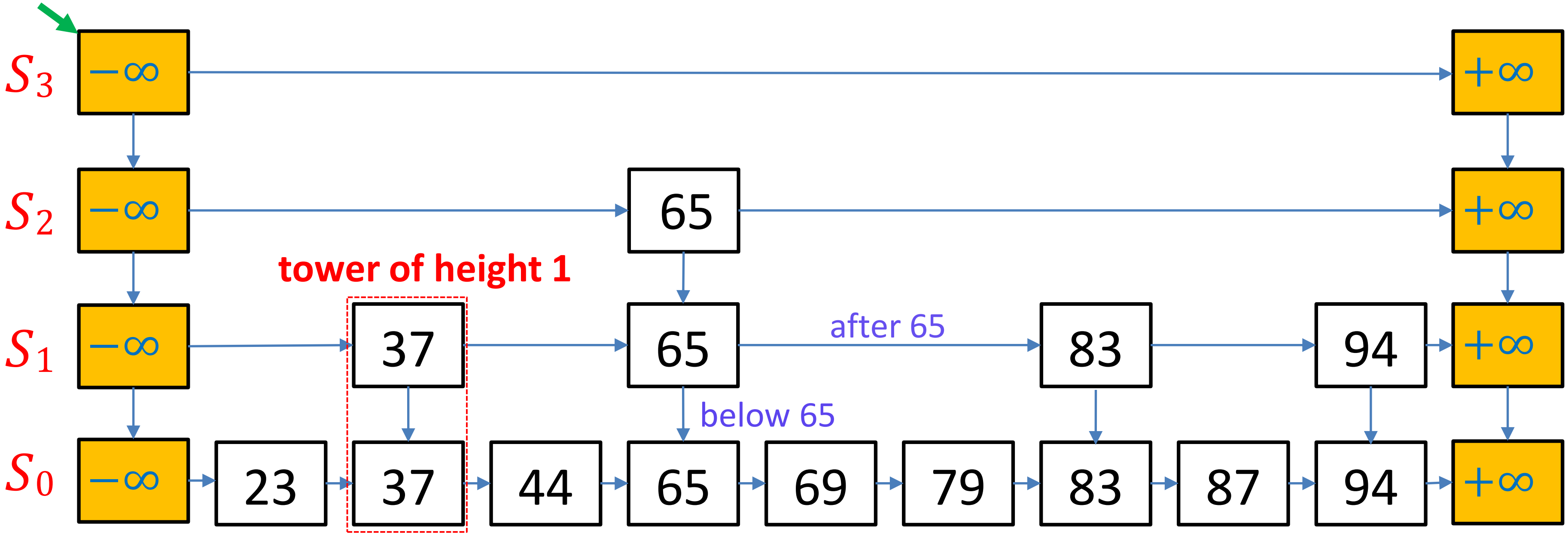
Skip Lists [Pugh'1989]

- A hierarchy S of ordered linked lists (*levels*) $S_0, S_1, \dots, S_h$
  - S_0 contains the KVPs of S in non-decreasing order
  - other lists store only keys, or links to nodes in S_0
  - each S_i contains special keys (sentinels) $-\infty$ and $+\infty$
  - each S_i is randomly generated subsequence of S_{i-1} i.e., $S_0 \supseteq S_1 \supseteq \dots \supseteq S_h$
  - S_h contains only sentinels, the left sentinel is the root



Skip Lists [Pugh'1989]

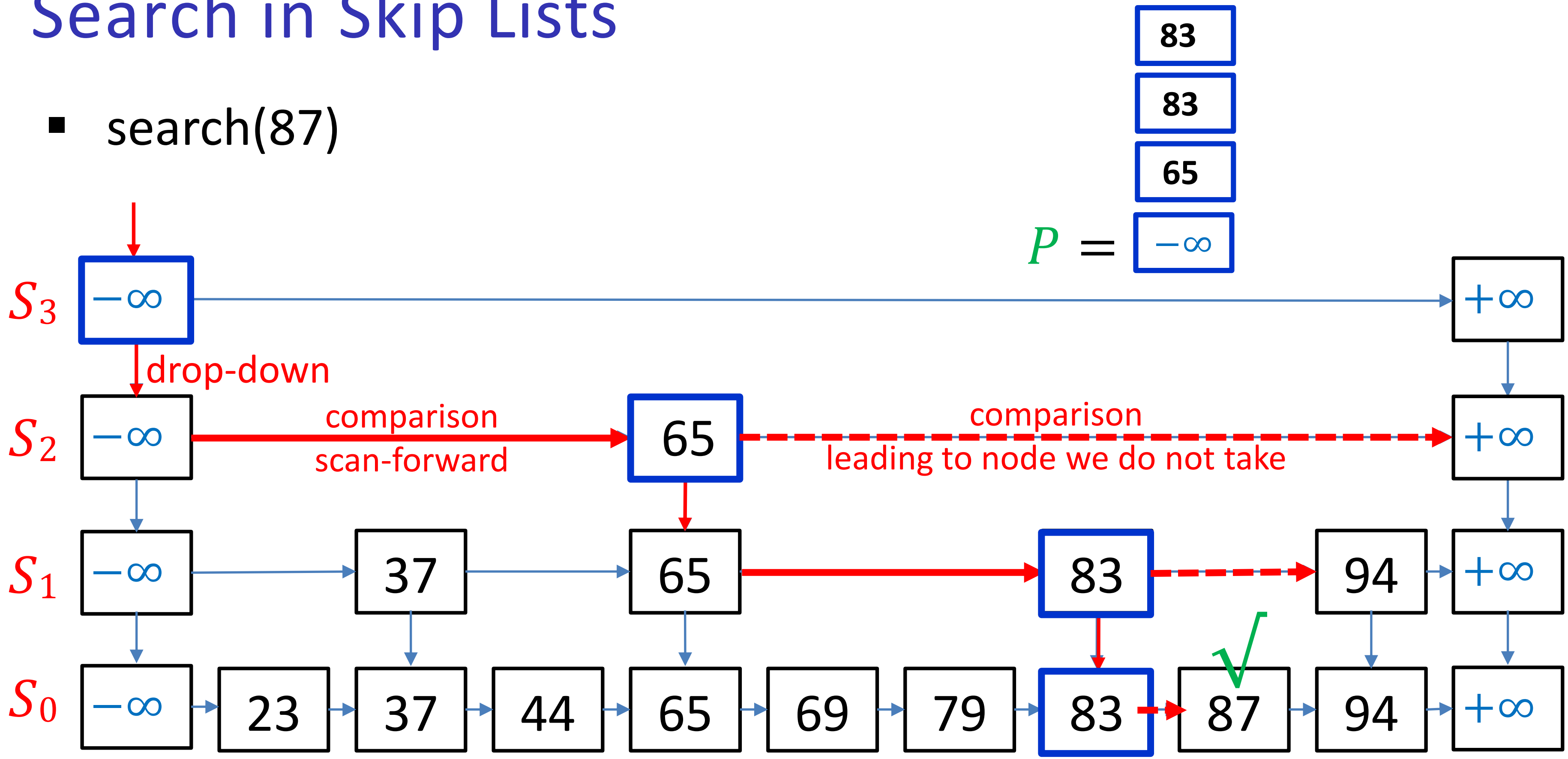
- Will show only keys from now on



- Each KVP belongs to a *tower* of nodes
- Height of the skip list is the maximum height of any tower
- Each node p has references to after(p) and below(p)
- There are (usually) more nodes than keys

Search in Skip Lists

- search(87)



- For each level, **predecessor** of key k is the node before node with key k, or, if key k is not present at that level, the node before where k would be
- P collects predecessors of key k at level S_0, S_1, …
  - these are needed for insert/delete
- k is in skip list if and only if $P.top().after$ has key k

Search in Skip Lists

getPredecessors(k)

$p \leftarrow$ *root*

$P \leftarrow$ stack of nodes, initially containing p

while $p.below \neq NIL$ **do** // keep dropping down until reach S_0

$p \leftarrow p.\text{below}$

while $p.after.key < k$ **do**

$p \leftarrow p.after$ //move to the right

$P.push(p)$ // this is next predecessor

return P

skipList::search(k)

$P \leftarrow$ *getPredecessors*(k)

$top \leftarrow P.top()$ //predecessor of k in S_0

if $top.after.key = k$ **return** $top.after$

else return 'not found, but would be after top'

Insert in Skip Lists

$S_3 \leftarrow$ if in S_2, then insert new item with probability ½

$S_2 \leftarrow$ if in S_1, then insert new item with probability ½

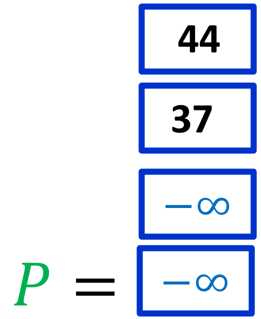
$S_1 \leftarrow$ insert new item with probability ½

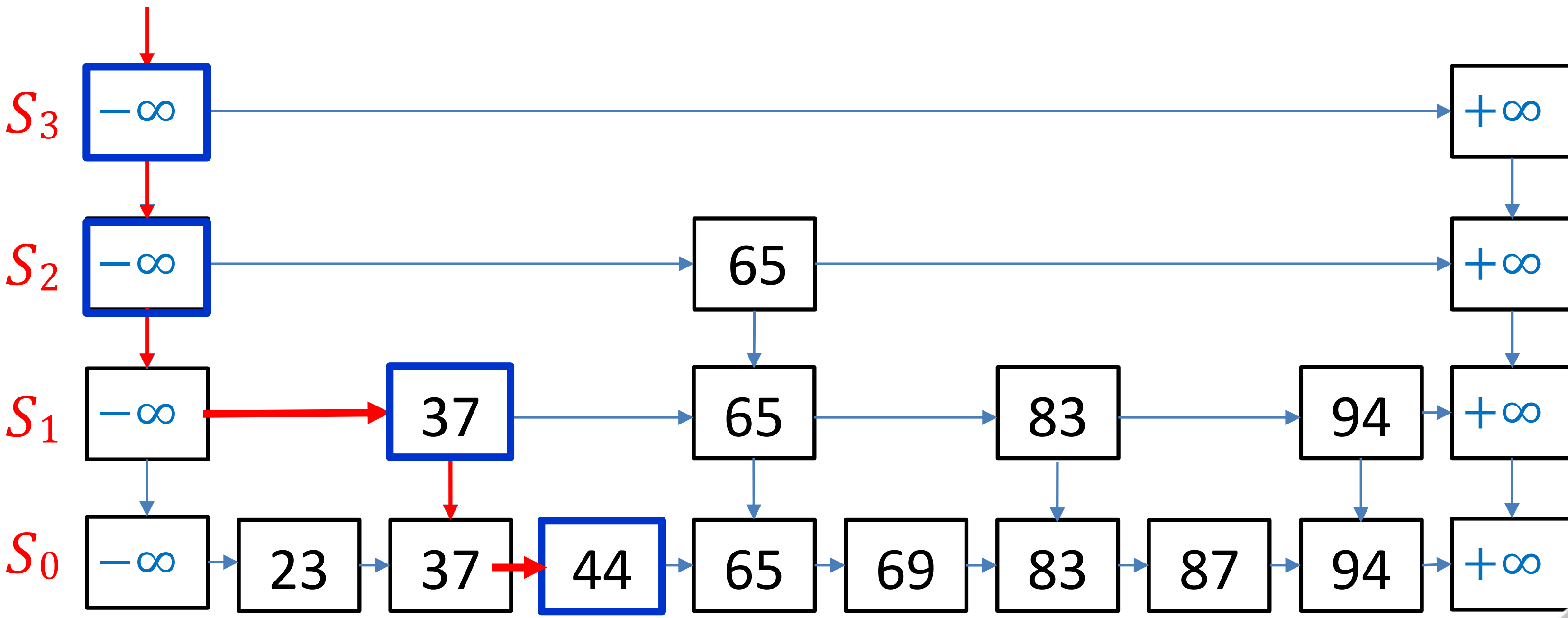
$S_0 \leftarrow$ insert new item

- Keep "tossing a coin" until T appears
- Insert into S_0 and as many other S_i as there are heads
- Examples
  - H, H, T (insert into S_0, S_1, S_2) $\Rightarrow$ will say $i = 2$
  - H, T (insert into S_0, S_1) $\Rightarrow$ will say $i = 1$
  - T (insert into S_0) $\Rightarrow$ will say $i = 0$

Insert in Skip Lists: Example 1

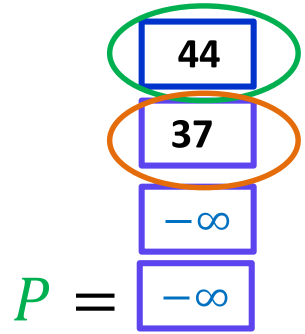
- *skipList::insert*$(52, v)$
- coin tosses: $H, T \Rightarrow i = 1$
- *getPredecessors*(52)

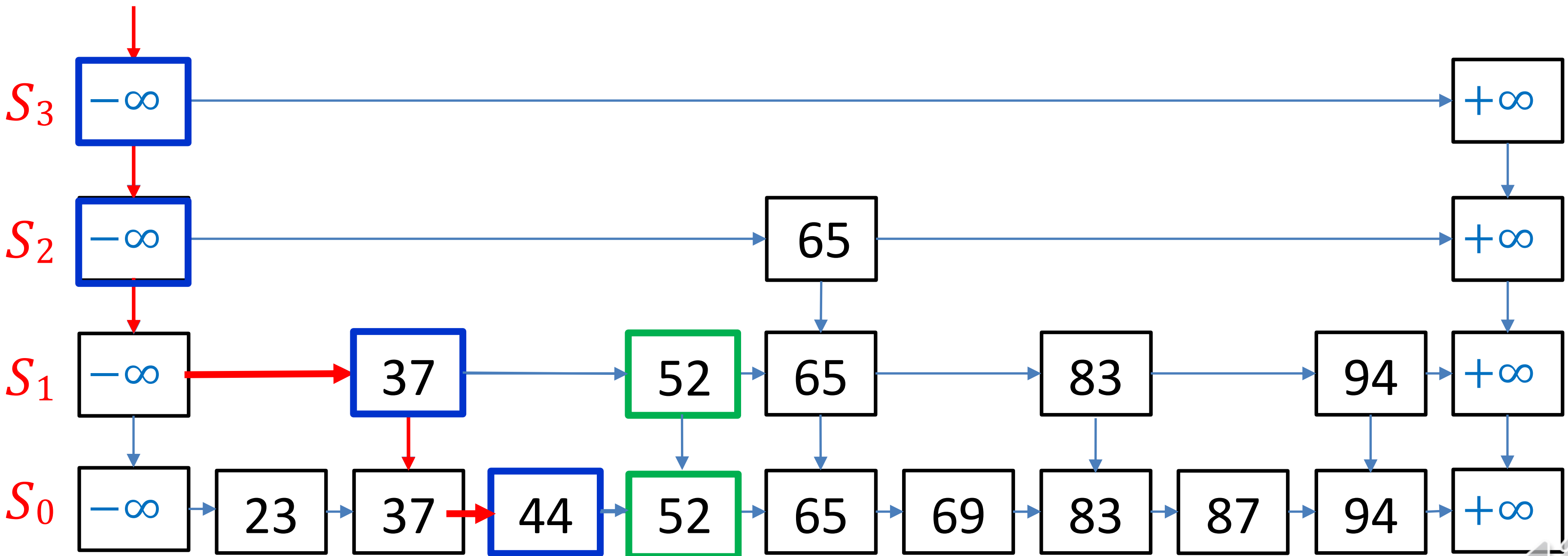




Insert in Skip Lists Example 1

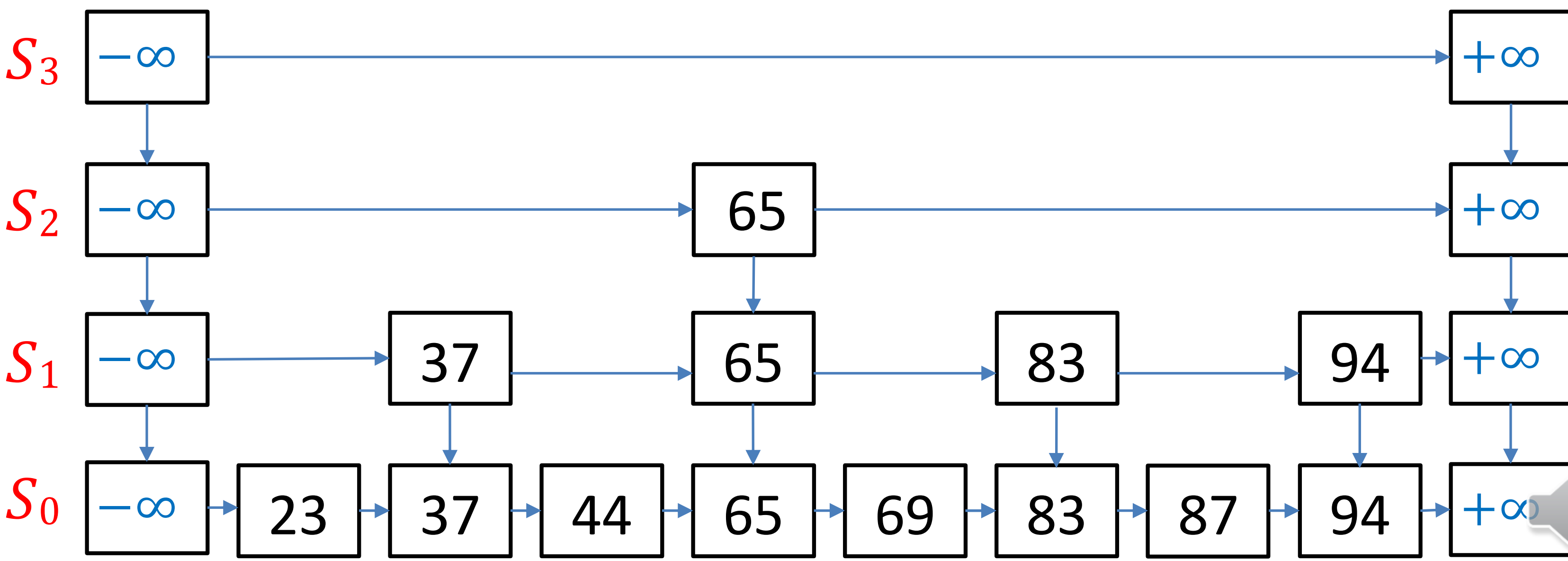
- *skipList::insert*$(52, v)$
- coin tosses: $H, T \Rightarrow i = 1$
- *getPredecessors*(52)
- now insert into S_0 and S_1





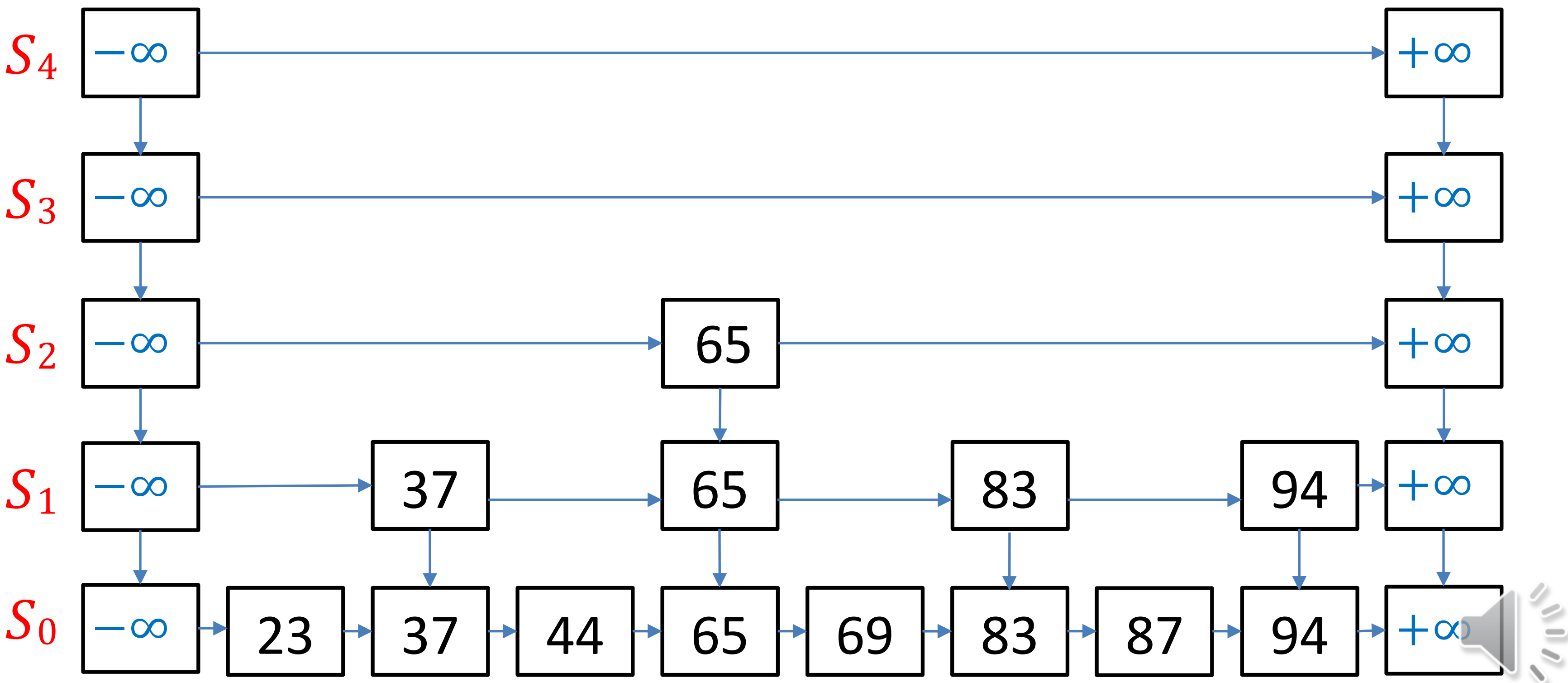
Insert in Skip Lists: Example 2

- *skipList::insert*$(100, v)$
- coin tosses: $H, H, H, T \Rightarrow i = 3$
- first increase height



Insert in Skip Lists: Example 2

- *skipList::insert*$(100, v)$
- coin tosses: $H, H, H, T \Rightarrow i = 3$
- first increase height
- next *getPredecessors* (100)



Insert in Skip Lists: Example 2

- *skipList::insert*$(100, v)$
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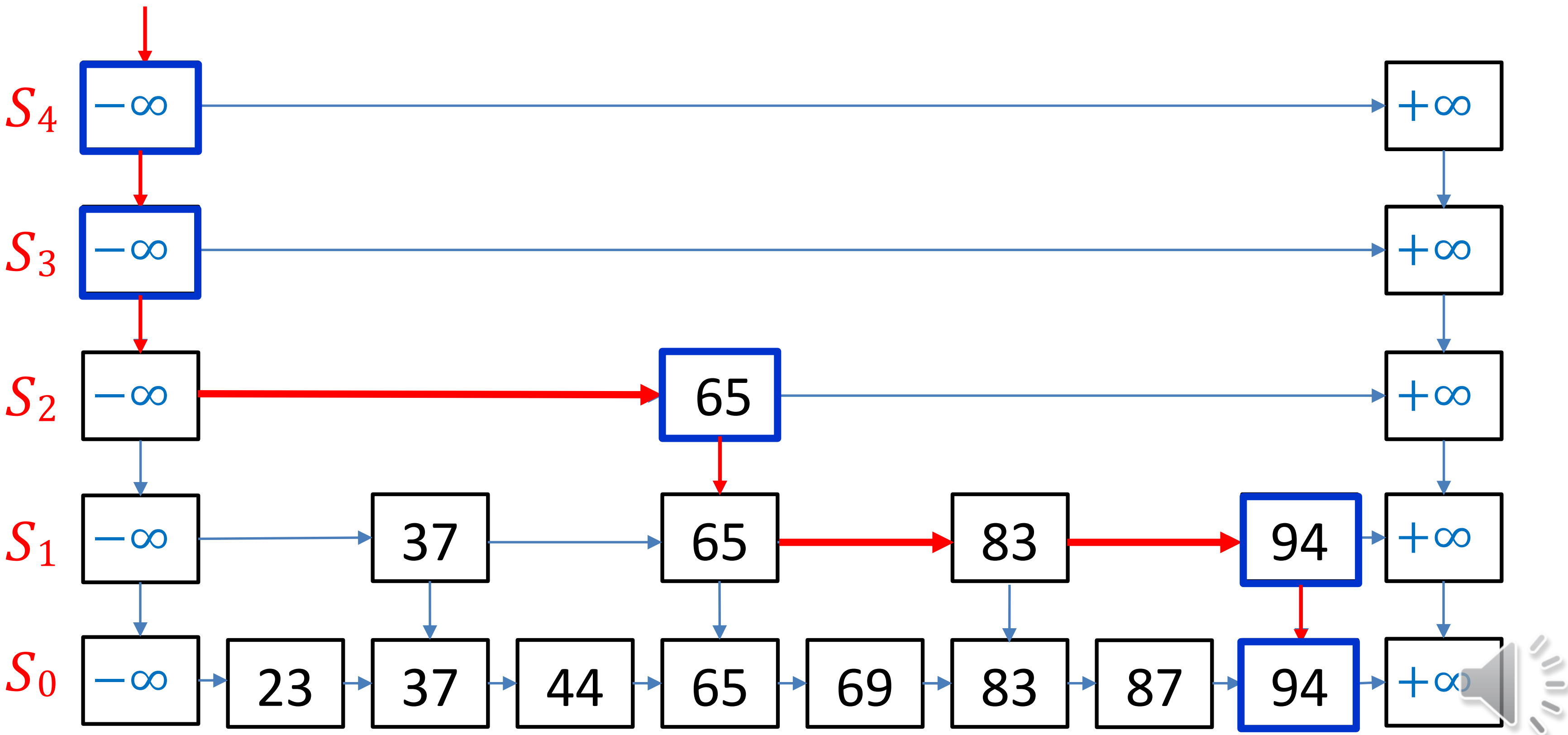
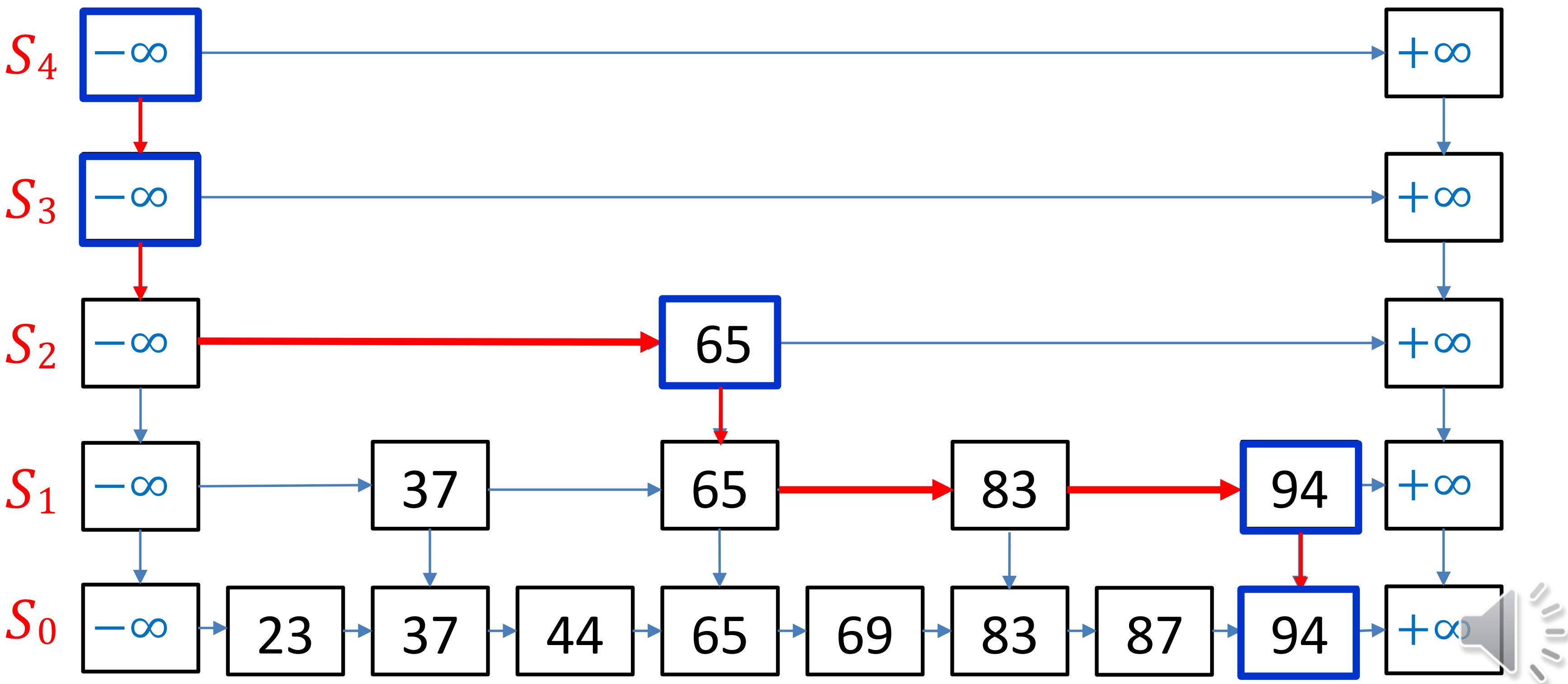


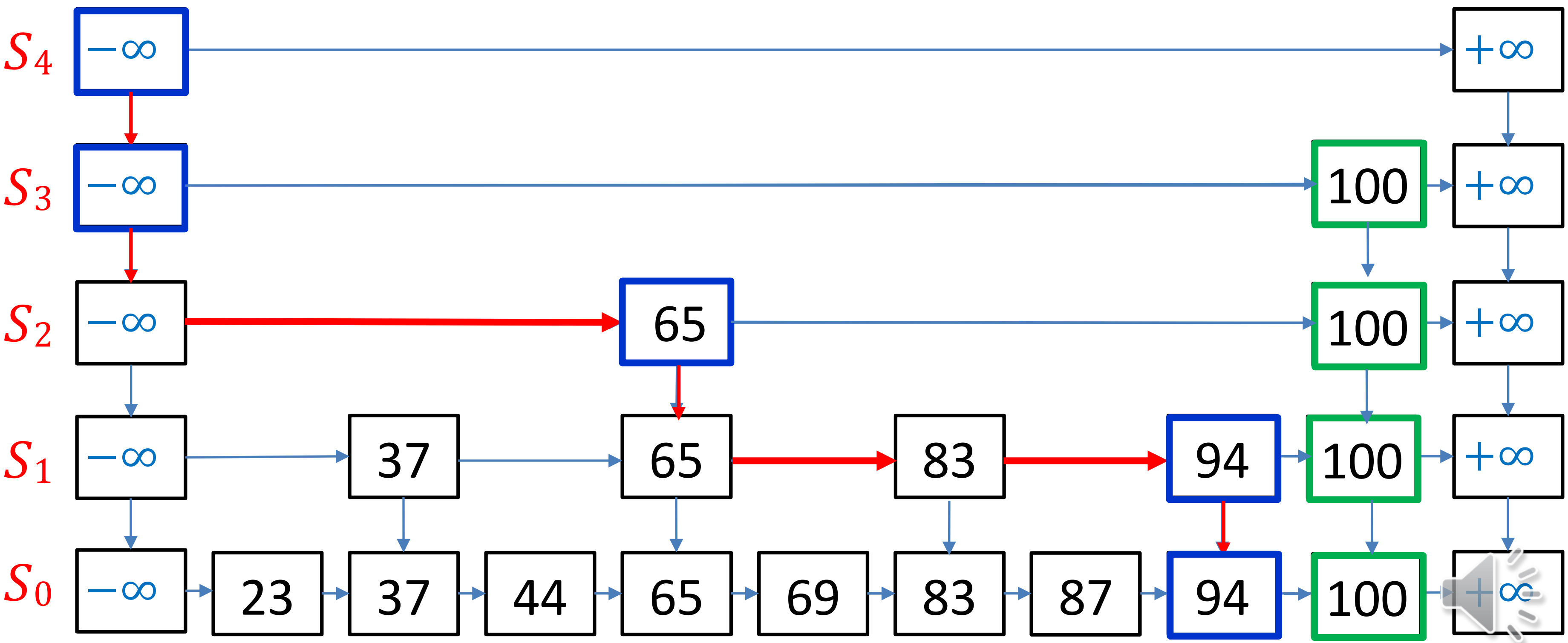
Insert in Skip Lists: Example 2

- *skipList::insert*$(100, v)$
- coin tosses: $H, H, H, T \Rightarrow i = 3$
- first increase height
- next *getPredecessors* (100)
- insert new key



Insert in Skip Lists: Example 2

- *skipList::insert*$(100, v\ )$
- coin tosses: $H, H, H, T \ \Rightarrow \ i \ = 3$
- first increase height
- next *getPredecessors* (100)
- insert new key

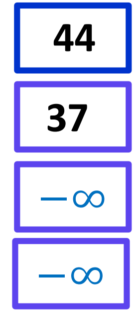


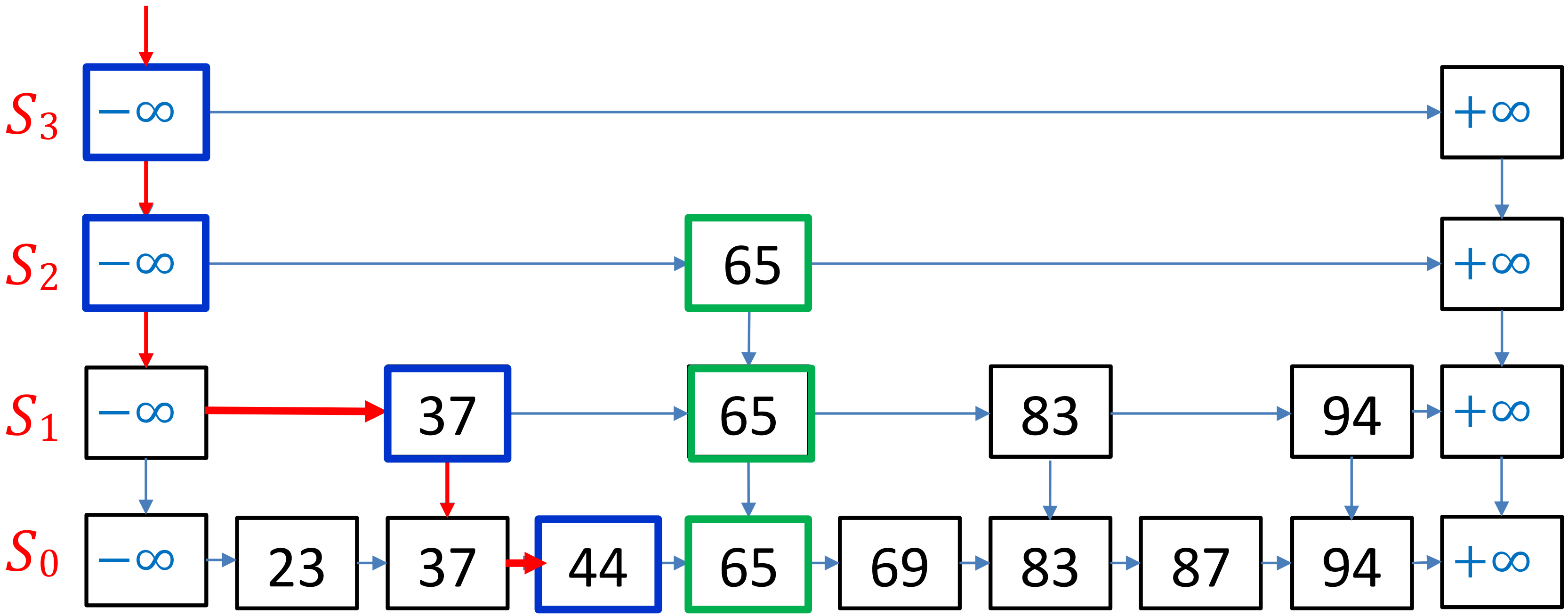
Insert in Skip Lists

```
skipList::insert(k, v)
        P ← getPredecessors(k)
        for (i ← 0; random(2) = 1; i ← i + 1) {}   // random tower height
        while i ≥ P.size()                          // increase skip-list height?
            root ← new sentinel-only list linked in appropriately
            P.append(left sentinel of root)
        p ← P.pop()                                 // insert (k, v) in S_0
        zBellow ← new node with (k, v) inserted after p
        while i > 0                                 // insert k in S_1 S_2,..., S_i
            p ← P.pop()
            z ← new node with k added after p
            z.below ← zBellow
            zBellow ← z
            i ← i − 1
```

Example: Delete in Skip Lists

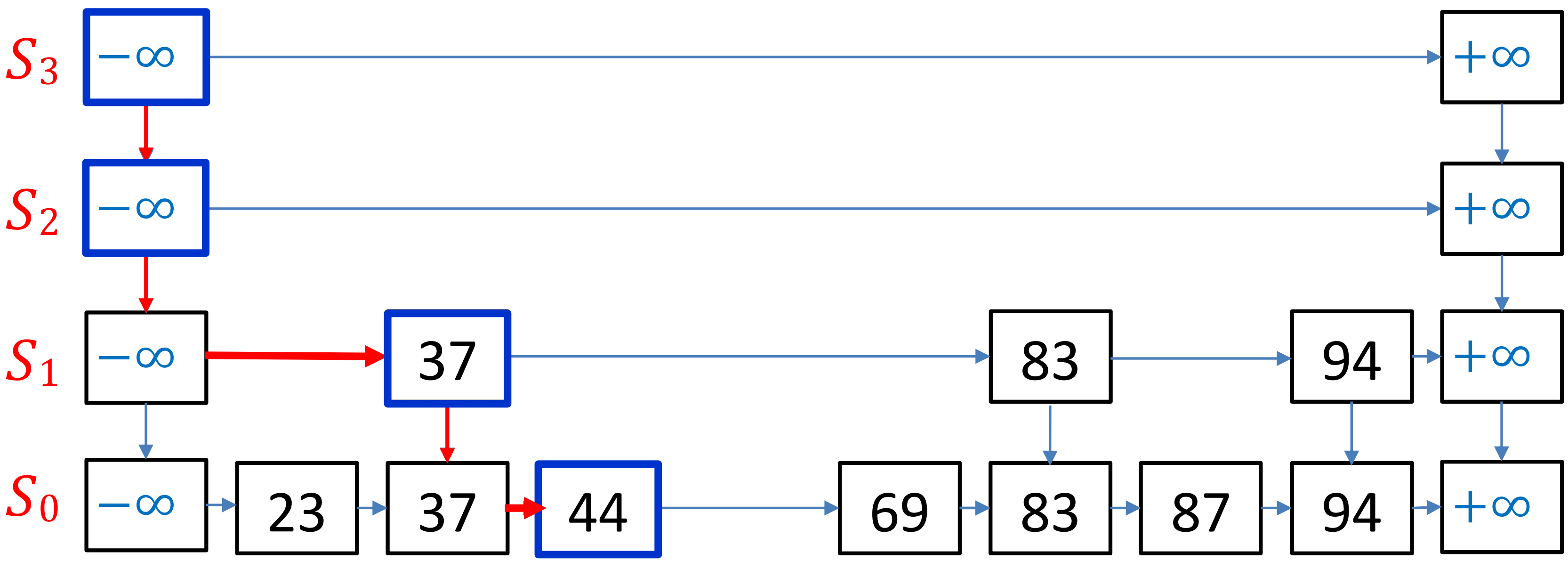
- *skipList::delete*(65)
  - first *getPredecessors*$(S, 65)$
  - then delete key 65 from all S_i
    - P has predecessor of each node to be deleted

$P =$ 



Example: Delete in Skip Lists

- *skipList::delete*(65)
  - first *getPredecessors*$(S, 65)$
  - then delete key 65 from all S_i
    - P has predecessor of each node to be deleted
  - height decrease: delete all unnecessary S_i, if any



Example: Delete in Skip Lists

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  - first *getPredecessors*$(S, 65)$
  - then delete key 65 from all S_i
    - P has predecessor of each node to be deleted
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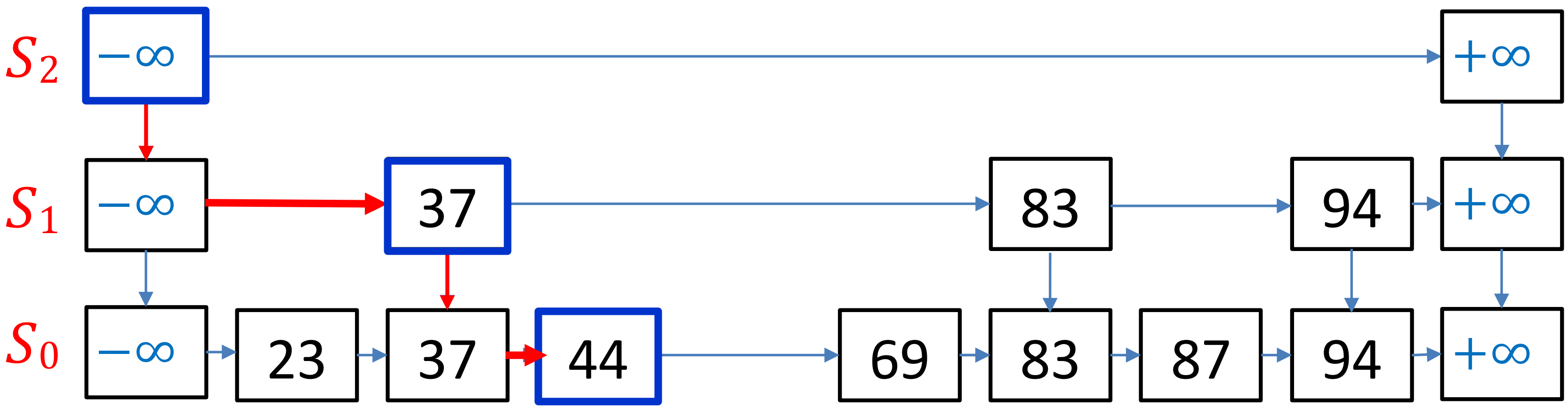


Delete in Skip Lists

skipList::delete(k)

$P \leftarrow$ *getPredecessors*(k)

while P is non-empty

$p \leftarrow P.pop()$ // predecessor of k in some layer

if $p.after.key = k$

$p.after \leftarrow p.after.after$

else break // no more copies of k

$p \leftarrow$ left sentinel of the root-list

while $p.below.after$ is the ∞ sentinel

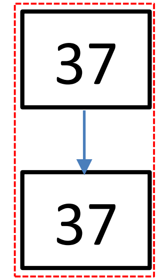
// the two top lists are both only sentinels, remove one

$p.below \leftarrow p.below.below$ // removes the second empty list

$p.after.below \leftarrow p.after.below.below$

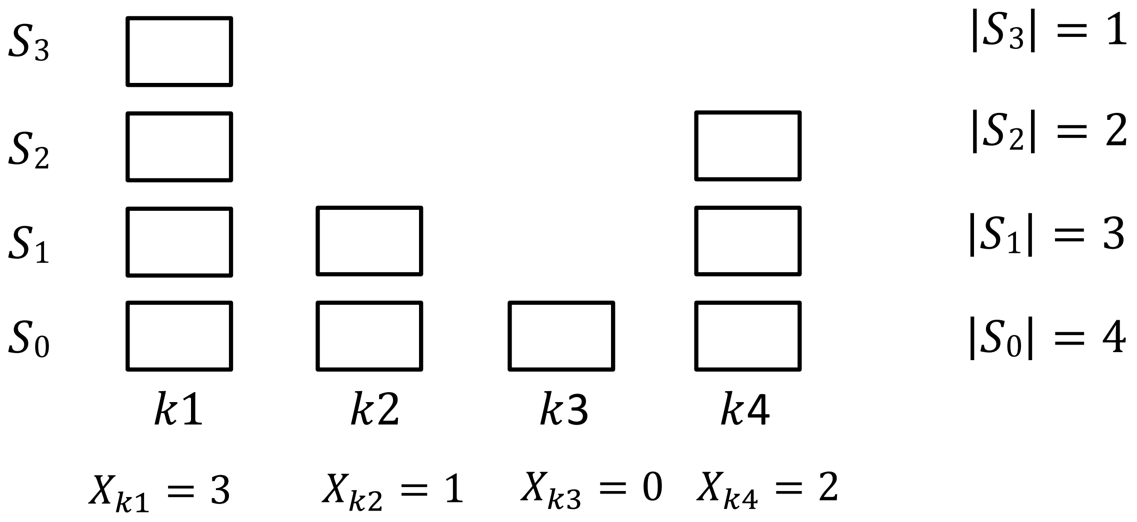
Skip List Analysis

tower of height 1



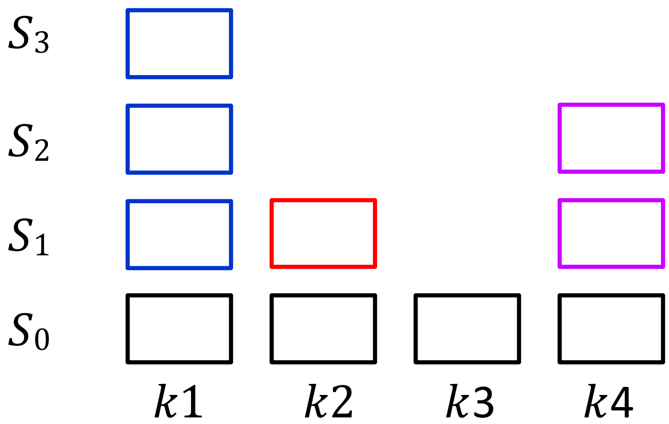
- Let X_k be the height of tower for key k
  - $P(X_k \geq 1) = \frac{1}{2}$, $P(X_k \geq 2) = \frac{1}{2} \cdot \frac{1}{2}$, $P(X_k \geq 3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$
- In general $P(X_k \geq i) = P(\underbrace{H\ H\ \dots\ H}_{i \text{ times}}) = \left(\frac{1}{2}\right)^i$
- In the worst case, the height of a tower could be arbitrary large
  - no bound on height in terms of n
- Therefore operations could be arbitrarily slow, and space requirements arbitrarily large
- But this is exceedingly unlikely
- Therefore we analyse *expected* run-time and space-usage

Skip List Analysis



- Let X_k be the height of tower for key k, we know $P(X_k \geq i) = \frac{1}{2^i}$
- If $X_k \geq i$ then list S_i includes key k
- Let $|S_i|$ be the number of keys in list S_i
  - sentinels do not count towards the length
  - S_0 always contains all n keys

Skip List Analysis



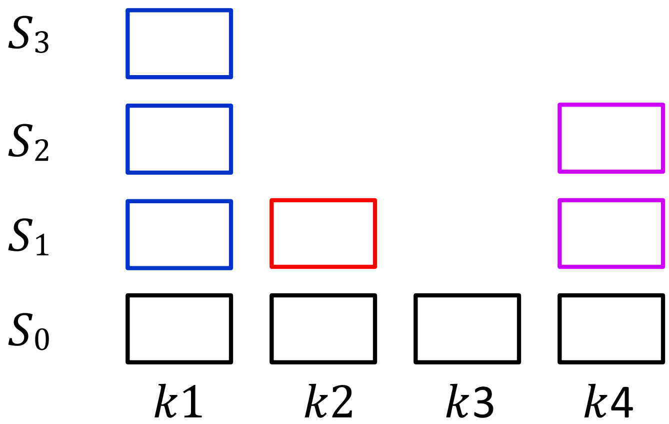
$I_{3,\,k1} = 1$ $\quad I_{3,\,k2} = 0$ $\quad I_{3,\,k3} = 0$ $\quad I_{3,\,k4} = 0$

$I_{2,\,k1} = 1$ $\quad I_{2,\,k2} = 0$ $\quad I_{2,\,k3} = 0$ $\quad I_{2,\,k4} = 1$

$I_{1,\,k1} = 1$ $\quad I_{1,\,k2} = 1$ $\quad I_{1,\,k3} = 0$ $\quad I_{1,\,k4} = 1$

- Let X_k be the height of tower for key k, we know $P(X_k \geq i) = \frac{1}{2^i}$
- If $X_k \geq i$ then list S_i includes key k
- Let $|S_i|$ be the number of keys in list S_i
  - sentinels do not count towards the length
- Let $I_{i,\,k} = \begin{cases} 0 & \text{if } X_k < i \\ 1 & \text{if } X_k \geq i \end{cases}$
- $|S_i| = \sum_{key\ k} I_{i,\,k}$

Skip List Analysis



$I_{3,\,k1} = 1$ $\quad I_{3,\,k2} = 0$ $\quad I_{3,\,k3} = 0$ $\quad I_{3,\,k4} = 0$

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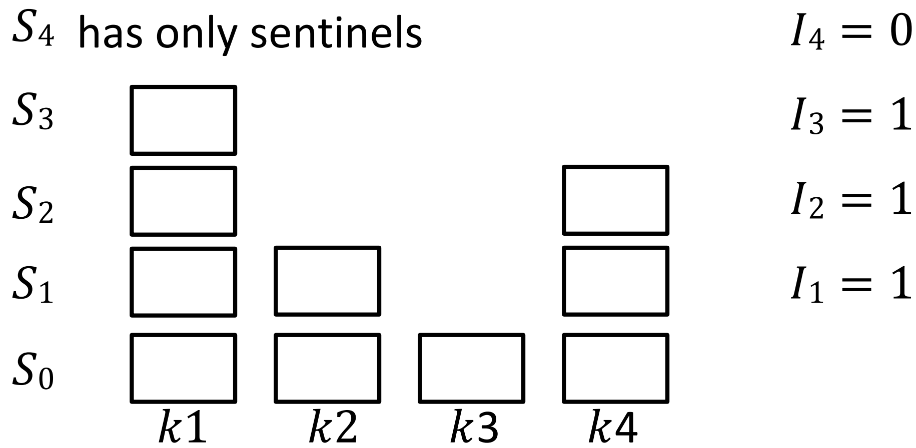
$I_{1,\,k1} = 1$ $\quad I_{1,\,k2} = 1$ $\quad I_{1,\,k3} = 0$ $\quad I_{1,\,k4} = 1$

- Let X_k be the height of tower for key k, we know $P(X_k \geq i) = \frac{1}{2^i}$
- Let $|S_i|$ be the number of keys in list S_i
- Let $I_{i,\,k} = \begin{cases} 0 & \text{if } \; X_k < i \\ 1 & \text{if } \; X_k \geq i \end{cases}$
- $|S_i| = \sum_{key\ k} I_{i,\,k}$
- $E[|S_i|] = E\left[\sum_{key\ k} I_{i,\,k}\right] = \sum_{key\ k} E[I_{i,\,k}] = \sum_{key\ k} P(I_{i,\,k} = 1) = \sum_{key\ k} P\,(X_k \geq i) = \frac{n}{2^i}$
- The expected length of list S_i is $\frac{n}{2^i}$

Skip List Analysis



- $|S_i|$ is number of keys in list S_i
  - $E[|S_i|] = \frac{n}{2^i}$
- Let $I_i = \begin{cases} 0 & \text{if } |S_i| = 0 \\ 1 & \text{if } |S_i| \geq 1 \end{cases}$
- $h = 1 + \sum_{i \geq 1} I_i$ (here +1 is for the sentinel-only level)
- Since $I_i \leq 1$ we have that $E[I_i] \leq 1$
- Since $I_i \leq |S_i|$ we have that $E[I_i] \leq E[|S_i|] = \frac{n}{2^i}$
- For ease of derivation, assume n is a power of 2
- $$E[h] = E\left[1 + \sum_{i \geq 1} I_i\right] = 1 + \sum_{i \geq 1} E[I_i] = 1 + \sum_{i=1}^{\log n} E[I_i] + \sum_{i=1+\log n}^{\infty} E[I_i]$$
  $$\leq 1 + \sum_{i=1}^{\log n} 1 + \sum_{i=1+\log n}^{\infty} \frac{n}{2^i}$$
  $$\leq 1 + \log n + \sum_{i=0}^{\infty} \frac{n}{2^{i+1+\log n}}$$

Skip List Analysis

S_4 has only sentinels $\qquad I_4 = 0$

$S_3 \qquad I_3 = 1$

- $|S_i|$ is number of keys in list S_i
  - $E[|S_i|] = \frac{n}{2^i}$
- Let $I_i = \begin{cases} 0 & \text{if } |S_i| = 0 \\ 1 & \text{if } |S_i| \geq \end{cases}$
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$$\sum_{i=0}^{\infty} \frac{n}{2^{i+1+\log n}} = \frac{1}{2}\sum_{i=0}^{\infty} \frac{n}{2^i 2^{\log n}}$$

$$= \frac{1}{2}\sum_{i=0}^{\infty} \frac{n}{2^i n}$$

$$= \frac{1}{2}\sum_{i=0}^{\infty} \frac{1}{2^i} = 1$$

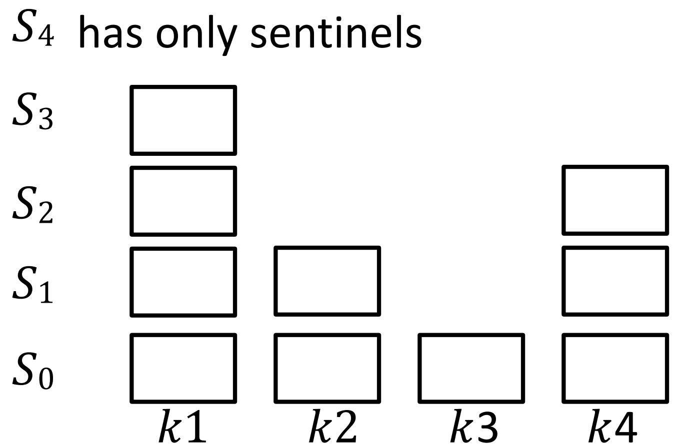
$$S = \sum_{i=0}^{\infty} \frac{1}{2^i} \qquad 2S = \sum_{i=0}^{\infty} \frac{1}{2^{i-1}} = 2 + \sum_{i=0}^{\infty} \frac{1}{2^i}$$

$$S = 2S - S = 2$$

$$\ldots{+}\log n \; 2^i$$

$$+ \sum_{i=0}^{\infty} \frac{n}{2^{i+1+\log n}}$$

Skip List Analysis



$I_4 = 0$

$I_3 = 1$

$I_2 = 1$

$I_1 = 1$

- $|S_i|$ is number of keys in list S_i
  - $E[|S_i|] = \frac{n}{2^i}$
- Let $I_i = \begin{cases} 0 & \text{if } |S_i| = 0 \\ 1 & \text{if } |S_i| \geq 1 \end{cases}$
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- Since $I_i \leq 1$ we have that $E[I_i] \leq 1$
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- For ease of derivation, assume n is a power of 2
- $E[h] = E\left[1 + \sum_{i\geq 1} I_i\right] = 1 + \sum_{i\geq 1} E[I_i] = 1 + \sum_{i=1}^{\log n} E[I_i] + \sum_{i=1+\log n}^{\infty} E[I_i]$

$$\leq 1 + \sum_{i=1}^{\log n} 1 + \sum_{i=1+\log n}^{\infty} \frac{n}{2^i}$$

$$\leq 1 + \log n + 1$$

- Expected height of skip list is at most $2 + \log n$

Skip List Analysis: Expected Space

- We need space for nodes storing sentinels and nodes storing keys

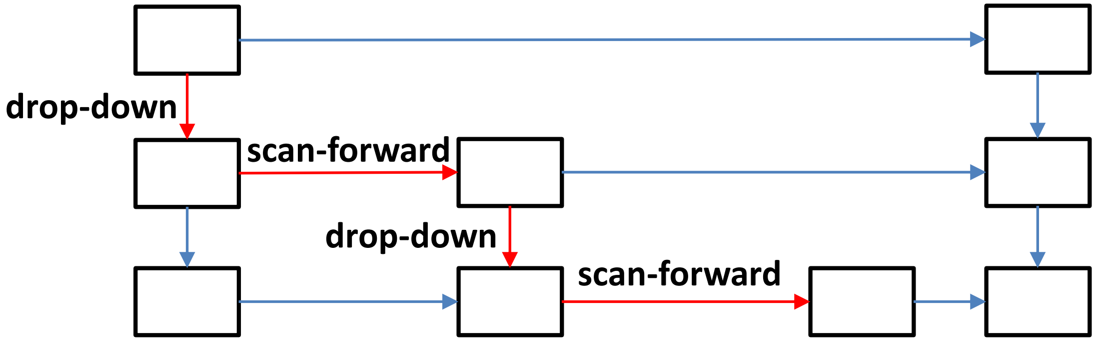
1. Space for nodes storing sentinels
   - there are $2h + 2$ sentinels, where h be the skip list height
   - $E[h] \leq 2 + \log n$
   - expected space for sentinels is at most

$$E[2h + 2] = 2E[h] + 2 \leq 6 + 2\log n$$

2. Space for nodes storing keys
   - Let $|S_i|$ be the number of keys in list S_i
     - $E[|S_i|] = \frac{n}{2^i}$
   - expected space for keys is $E\left[\sum_{i \geq 0} |S_i|\right] = \sum_{i \geq 0} \frac{n}{2^i} = 2n$

- Total expected space is $\Theta(n)$

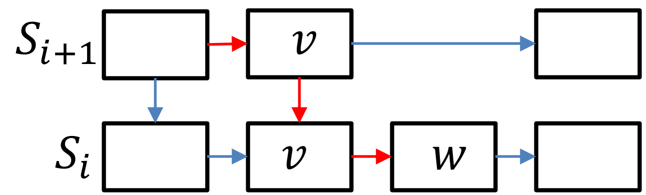
Skip List Analysis: Expected Running Time



- *search*, *insert*, and *delete* are dominated by the running time of *getPredecessors*
- So let us analyze the expected time of *getPredecessors*
- In *getPredecessors*, running time is proportional to the number of 'drop-down' and 'scan-forward'
- We can 'drop-down' at most h times, where h is skip list height
    - expected height h is $O(\log n)$
    - total expected time spent on 'drop-down' operations is $O(\log n)$
- Will show on the next slide that the expected number of 'scan-forward' is also $O(\log n)$
- So the expected running time is $O(\log n)$

Skip List Analysis: Expected Running Time

- What about ‘scan-forward’?
  - assume $i < h$ (if $i = h$, then we are at the top list and do not scan forward at all)
  - let v be leftmost key in S_i we visit during search
    - we v reached by dropping down from S_{i+1}
  - let w be the key right after v
    - height of tower of w in this case is at least i



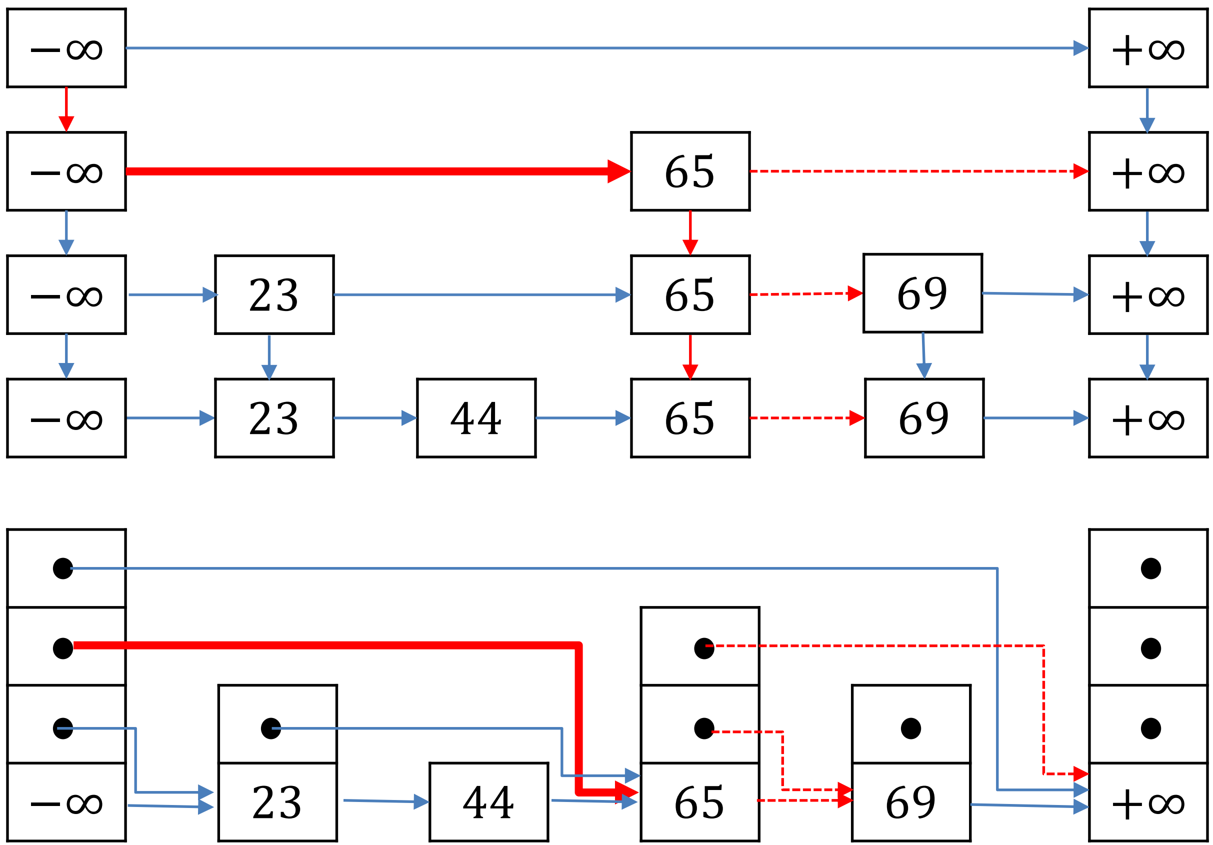
  - What is the probability of scanning from v to w?
  - If we do scan forward from v to w, then w did not exist in S_{i+1}
    - otherwise, we would scan forward from v to w in S_{i+1}
    - in other words, we always enter the tower of any node ‘at the top’
  - Thus if we do scan forward from v to w, then the tower of w has height i
    - P(tower of w has height i | tower of w has height at least i) $=$ ½
    - we scan forward from v to w with probability at most ½
      - ‘at most’ because we could scan-down down if $key < w$
    - repeating the argument, the probability of scan-forward l times is at most $(1/2)^l$

$$E[\text{ number of scans}] = \sum_{l\geq 1} l \cdot \mathrm{P}(\text{scans} = l) = \sum_{l\geq 1} \mathrm{P}(\text{scans} \geq l) \leq \sum_{l\geq 1} \frac{1}{2^l} = 1$$

- Expected number of scan-forwards at any level is 1, over all levels h, which is $\mathrm{O}(\log n)$

Arrays Instead of Linked Lists

- As described now, they are no faster than randomized binary search trees
- Can save links by implementing each tower as an array
    - this not only saves space, but gives better running time in practice
    - when ‘scan-forward’, we know the correct array location to look at (level i)
- Search(67)



Summary of Skip Lists

- For a skip list with n items
  - expected space usage is $O(n)$
  - expected running time for search, insert, delete is $O(\log n)$
- Two efficiency improvements
  - use arrays for key towers for more efficient implementation
  - can show: a biased coin-flip to determine tower-height gives smaller expected run-times
  - with arrays and biased coin-flip skip lists are fast in practice and easy to implement

Outline

- Dictionaries with Lists Revisited
  - Dictionary ADT
    - implementations so far
  - Skip Lists
  - Re-ordering items

Re-ordering Items

- Unordered arrays (or lists) are among simplest data structures to implement
- But for Dictionary ADT
    - *search*: $\Theta(n)$, *insert*: $\Theta(1)$, *delete*: $\Theta(1)$ (after a search)
- Lists/arrays are a very simple a popular implementation
- Can we make search in unordered arrays (or lists) more effective in practice?
    - No: if items are accessed equally likely
    - Yes: otherwise
        - intuition: frequently accessed items should be in the front
- Two cases
    - know the access distribution beforehand
    - do not know access distribution beforehand
- For short lists or extremely unbalanced distributions this may be faster than AVL trees or Skip Lists, and easier to implement

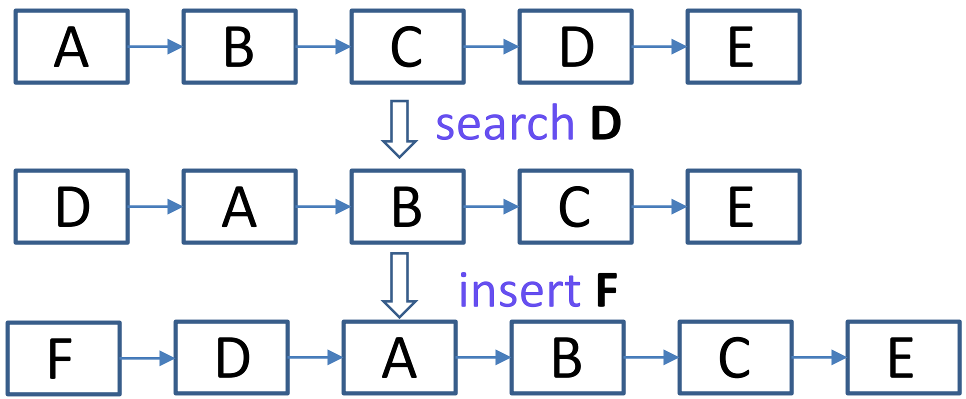
Optimal Static Ordering

key	A	B	C	D	E
frequency of access	2	8	1	10	5
access probability	$\frac{2}{26}$	$\frac{8}{26}$	$\frac{1}{26}$	$\frac{10}{26}$	$\frac{5}{26}$

- Order C A B D E has expected cost
  $$\frac{1}{26}\cdot 1+\frac{2}{26}\cdot 2+\frac{8}{26}\cdot 3+\frac{10}{26}\cdot 4+\frac{5}{26}\cdot 5 \approx 3.61$$
- Order D B E A C has expected cost
  $$\frac{10}{26}\cdot 1+\frac{8}{26}\cdot 2+\frac{5}{26}\cdot 3+\frac{2}{26}\cdot 4+\frac{1}{26}\cdot 5 \approx 2.54$$
- Claim: ordering items by non-increasing access-probability minimizes expected access cost, i.e. best *static* ordering
- Proof Idea: for any other ordering, exchanging two items that are out-of-order according to access probabilities makes total cost decrease

Dynamic Ordering

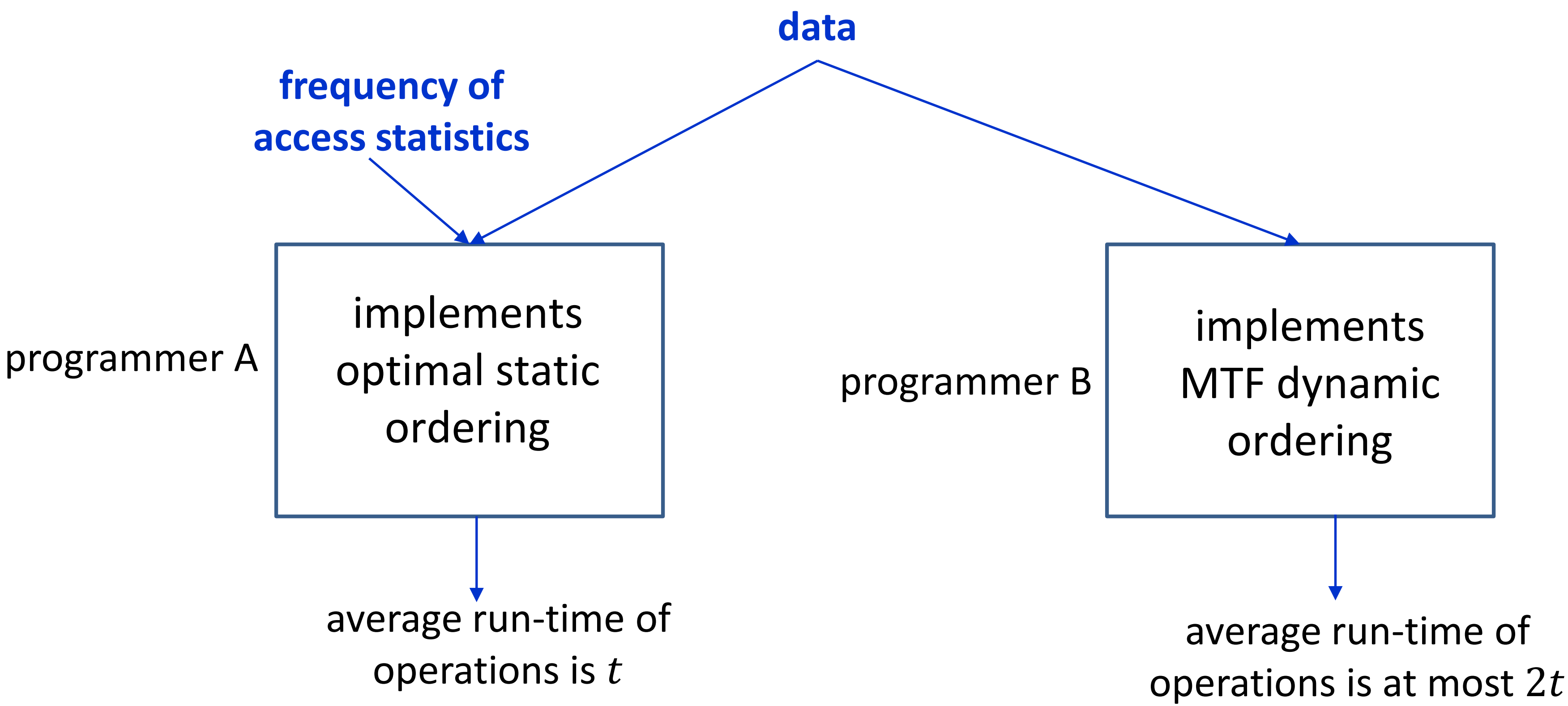
- What if we do *not know the access probabilities* ahead of time?
- Rule of thumb (*temporal locality*)
    - recently accessed item is likely to be accessed soon again
- In list: always insert at the front
- Move-To-Front heuristic (MTF): after search, move the accessed item to the front



- We can also do MTF on an array
    - but should then insert and search from the back so that we have room to grow

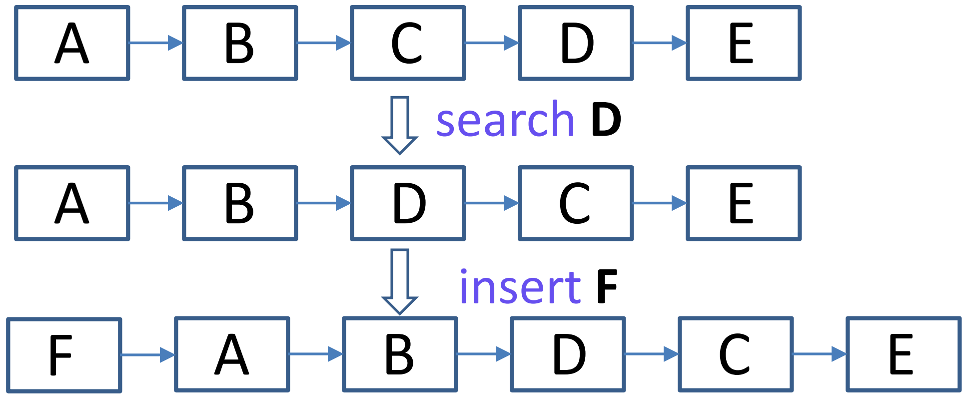
Dynamic Ordering: MTF

- Can show: MTF is "2-competitive"
  - no more than twice as bad as the optimal "offline" ordering



Dynamic Ordering: Transpose

- Transpose heuristic: Upon a successful search, swap accessed item with the immediately preceding item



- Avoids drastic changes MTF might do, while still adapting to access patterns
- Worst case is $\Theta(n)$ for both transpose and MTF