CS 240 – Data Structures and Data Management

Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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Outline

1. Lower bound

2. Interpolation Search

3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries
Lower bound for search

The fastest realizations of \textit{ADT Dictionary} require $\Theta(\log n)$ time to search among $n$ items. Is this the best possible?
Lower bound for search

The fastest realizations of *ADT Dictionary* require $\Theta(\log n)$ time to search among $n$ items. Is this the best possible?

**Theorem:** In the comparison model (on the keys), $\Omega(\log n)$ comparisons are required to search a size-$n$ dictionary.

**Proof:** via decision tree

But can we beat the lower bound for special keys?
Remark: could also use
Claim: an algorithm (in the comparison model) to do search in a size-$n$ dictionary $\Rightarrow$ a decision tree with at least $n+1$ leaves.
\( x_0 < x_1. \)

if \( k \leq x_0 \)
  if \( x_0 \leq k \)
    return "found", 0
  else
    return "not found"
else
  if \( k \leq x_1 \)
    if \( x_1 \leq k \)
      return "found", 1
    else
      return "not found"
  else
    return "not found"
\( x_0 < a_1 \),

if \( k \leq x_0 \)
    if \( x_0 \leq k \)
        return "found", 0
    else
        return "not found"
else
    if \( k \leq a_1 \)
        if \( x_1 \leq k \)
            return "found" , 1
        else
            return "not found"
    else
        return "not found"
at least $3 = 2 + 1$ leaves

$0, 1 \quad NF$
Let $h$ be the worst-case # of comparisons that we do (for $n$ keys)

$\Rightarrow$ any possible input $k$ reaches a leaf after doing

at most $h$ comparisons

$\Rightarrow$ in the decision tree, there are at least $n+1$ leaves

of depth $\leq h$.

In any binary tree, the number of leaves of depth $\leq h$

is at most $2^h$. Proof: induction on $h$. 
\[ m \leq \text{# of leaves of } \leq 2^h \]

\[ \text{depth } \leq h \]

\[ \rightarrow m \leq 2^h \]

\[ \rightarrow \log(m) \leq h. \]
Binary Search

Recall the run-times in a *sorted array*:

- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

```
binary-search(A, n, k)
A: Sorted array of size n, k: key
1. $\ell \leftarrow 0$, $r \leftarrow n - 1$
2. while ($\ell \leq r$)
   3. $m \leftarrow \left\lfloor \frac{\ell + r}{2} \right\rfloor$
   4. if ($A[m] < k$) then $\ell = m + 1$
   5. else if ($k < A[m]$) then $r = m - 1$
   6. else return “found at $A[m]$”
7. return “not found, but would be between $A[\ell-1]$ and $A[\ell]$”
```
**Interpolation Search: Motivation**

```latex
\texttt{binary-search}(A[\ell, r], k): \text{ Compare at index } \left\lfloor \frac{\ell + r}{2} \right\rfloor = \ell + \left\lfloor \frac{1}{2}(r - \ell) \right\rfloor
```

![Diagram of binary search with indices \(\ell\) and \(r\)]
Interpolation Search: Motivation

\textit{binary-search}(A[\ell, r], k): Compare at index \([\frac{\ell + r}{2}] = \ell + \lfloor \frac{1}{2}(r - \ell) \rfloor\)

\begin{array}{c|c|c}
\ell & \downarrow & r \\
--- & --- & --- \\
40 & & 120 \\
\end{array}

\textbf{Question}: If keys are \textit{numbers}, where would you expect key \(k = 100\)?
Interpolation Search: Motivation

**binary-search**$(A[\ell, r], k)$: Compare at index $\left\lceil \frac{\ell + r}{2} \right\rceil = \ell + \left\lfloor \frac{1}{2}(r - \ell) \right\rfloor$

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>120</td>
</tr>
</tbody>
</table>

**Question**: If keys are **numbers**, where would you expect key $k = 100$?

**interpolation-search**$(A[\ell, r], k)$: Compare at index $\ell + \left\lfloor \frac{k - A[\ell]}{A[r] - A[\ell]}(r - \ell) \right\rfloor$
\[
\frac{A(r) - A(l)}{r-l} = \frac{k - A(l)}{m-l}
\]

\[
\frac{m-l}{r-l} = \frac{k - A(l)}{A(r) - A(l)} \quad \Rightarrow \quad m = l + \frac{k - A(l)}{A(r) - A(l)} (r-l).
\]
Interpolation Search Example

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
0 & 1 & 2 & 3 & 449 & 450 & 600 & 800 & 1000 & 1200 & 1500 \\
\end{array}
\]

\textit{interpolation-search}(A[0..10], 449):
Interpolation Search Example

\[\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
0 & 1 & 2 & 3 & 449 & 450 & 600 & 800 & 1000 & 1200 & 1500 \\
\ell & \uparrow & r
\end{array}\]

**interpolation-search**(A[0..10],449):
- Initially \( \ell = 0, \ r = n - 1 = 10, \ m = \ell + \left\lfloor \frac{449-0}{1500-0} (10 - 0) \right\rfloor = \ell + 2 = 2 \)
Interpolation Search Example

Interpolation-search(A[0..10], 449):

- Initially \( \ell = 0, \ r = n - 1 = 10, \ m = \ell + \lfloor \frac{449 - 0}{1500 - 0}(10 - 0) \rfloor = \ell + 2 = 2 \)
- \( \ell = 3, \ r = 10, \ m = \ell + \lfloor \frac{449 - 3}{1500 - 3}(10 - 3) \rfloor = \ell + 2 = 5 \)
Interpolation Search Example

<p>| | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>449</td>
<td>450</td>
<td>600</td>
<td>800</td>
<td>1000</td>
<td>1200</td>
<td>1500</td>
</tr>
</tbody>
</table>

\[\ell \quad \uparrow, r\]

\textit{interpolation-search}(A[0..10],449):

- Initially \(\ell = 0, r = n - 1 = 10\), \(m = \ell + \left\lfloor \frac{449-0}{1500-0} (10 - 0) \right\rfloor = \ell + 2 = 2\)
- \(\ell = 3, r = 10\), \(m = \ell + \left\lfloor \frac{449-3}{1500-3} (10 - 3) \right\rfloor = \ell + 2 = 5\)
- \(\ell = 3, r = 4\), \(m = \ell + \left\lfloor \frac{449-3}{449-3} (4 - 3) \right\rfloor = \ell + 1 = 4\), found at \(A[4]\)
Interpolation Search Example

\[\text{interpolation-search}(A[0..10], 449)\]:

- Initially \( \ell = 0, r = n - 1 = 10, m = \ell + \left\lfloor \frac{449 - 0}{1500 - 0} (10 - 0) \right\rfloor = \ell + 2 = 2 \)
- \( \ell = 3, r = 10, m = \ell + \left\lfloor \frac{449 - 3}{1500 - 3} (10 - 3) \right\rfloor = \ell + 2 = 5 \)
- \( \ell = 3, r = 4, m = \ell + \left\lfloor \frac{449 - 3}{449 - 3} (4 - 3) \right\rfloor = \ell + 1 = 4, \) found at \( A[4] \)

Works well if keys are \textit{uniformly} distributed:

- Can show: Recurrence relation is \( T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1) \)
- This resolves to \( T^{(\text{avg})}(n) \in \Theta(\log \log n) \)
- But: Worst case performance \( \Theta(n) \)
Interpolation Search Example

\[\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
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But: Worst case performance \( \Theta(n) \)

---

*Note: The image contains a table with numbers from 0 to 1500, and the text describes an interpolation search algorithm.*
\[ \pi(n) = T(\sqrt{n}) + c \]

\[
\begin{align*}
2 & \rightarrow 2^2 \\
& \rightarrow 2^4 \\
& \rightarrow 2^8 \\
& \rightarrow 2^{16} \\
& \rightarrow \ldots \\
& \rightarrow 2^{2^i} \\
\end{align*}
\]

\[ T(2^{2^i}) = T(2^{2^{i-2}}) + c \]

\[ = T(2^{2^{i-2}}) + 2c \]

\[ = T(2^{2^{i-3}}) + 3c \]

\[ \ldots = T(2^{2^{i-i}}) + ic = T(2) + ic \]

\[ \Rightarrow \text{for } n = 2^{2^i}, \quad T(n) = T(2) + c \cdot \log(\log(n)). \]
Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash during computation of \( m \).

\[
\text{interpolation-search}(A, n, k)
\]
\( A \): Sorted array of size \( n \), \( k \): key

1. \( \ell \leftarrow 0, \ r \leftarrow n - 1 \)
2. \( \text{while } (\ell \leq r) \)
3. \( \text{if } (k < A[\ell] \text{ or } k > A[r]) \text{ return } \text{“not found”} \)
4. \( \text{if } (A[\ell] = A[r]) \text{ then return } \text{“found at } A[\ell] \text{”} \)
5. \( m \leftarrow \ell + \left\lfloor \frac{k - A[\ell]}{A[r] - A[\ell]} \cdot (r - \ell) \right\rfloor \)
6. \( \text{if } (A[m] < k) \text{ then } \ell = m + 1 \)
7. \( \text{else if } (k < A[m]) \text{ then } r = m - 1 \)
8. \( \text{else return } \text{“found at } A[m] \text{”} \)
9. \( \text{// We always return from somewhere within while-loop} \)
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3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries
**Tries: Introduction**

**Trie** (also know as **radix tree**): A dictionary for bitstrings.
(Should know: string, word, $|w|$, alphabet, prefix, suffix, comparing words,...)

- Comes from retrieval, but pronounced “try”
- A tree based on **bitwise comparisons**: Edge labelled with corresponding bit
- Similar to **radix sort**: use individual bits, not the whole key
More on tries

**Assumption:** Dictionary is **prefix-free**: no string is a prefix of another

- Assumption satisfied if all strings have the same length.
- Assumption satisfied if all strings end with ‘end-of-word’ character $.

**Example:** A trie for \{00$, 0001$, 0100$, 011$, 0110$, 110$, 1101$, 111$\}
More on tries

**Assumption:** Dictionary is **prefix-free:** no string is a prefix of another

- Assumption satisfied if all strings have the same length.
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**Example:** A trie for \{00\$, 0001\$, 0100\$, 011\$, 0110\$, 110\$, 1101\$, 111\$\}

Then items (keys) are stored *only* in the leaf nodes
Tries: Search

- start from the root and the most significant bit of $x$
- follow the link that corresponds to the current bit in $x$; return failure if the link is missing
- return success if we reach a leaf (it must store $x$)
- else recurse on the new node and the next bit of $x$

\[
\text{Trie::search}(v \leftarrow \text{root}, d \leftarrow 0, x)
\]

$v$: node of trie; $d$: level of $v$, $x$: word stored as array of chars
1. if $v$ is a leaf
2. return $v$
3. else
4. let $v'$ be child of $v$ labelled with $x[d]$
5. if there is no such child
6. return “not found”
7. else Trie::search($v'$, $d + 1$, $x$)
Tries: Search Example

Example: Trie::search(011$)

```
         0
        / \
       1   1
      /    /  \
     0    0    1
    /  \
   0   1
  /  \
 0   1
  /  \
$/  0
   /  \
 00$ 011$
   /  \
   $ 01001$
    /  \
    $ 01101$
   /  \
   110$ 111$
      /  \
     1101$
```
Tries: Search Example

Example: Trie::search(011$)

```
0
/  \
0   1
|   /
0  1 0
| /  $  \
0 1 0 1
| / /  $ \
0 1 0 1
$ $ $ $ $ \\
00$ 011$ 110$ 111$
```

0001$ 01001$ 01101$
Tries: Search Example

Example: Trie::search(011$)
Example: Trie::search(011$)
Example: Trie::search(011$) successful
Tries: Search Example

Example: Trie::search(0111$)
Example: Trie::search(0111$) unsuccessful

Tries: Search Example

Example: Trie::search(0111$) unsuccessful
Tries: Insert & Delete

- **Trie::insert(x)**
  - Search for x, this should be unsuccessful
  - Suppose we finish at a node v that is missing a suitable child.
    Note: x has extra bits left.
  - Expand the trie from the node v by adding necessary nodes that correspond to extra bits of x.

- **Trie::delete(x)**
  - Search for x
  - let v be the leaf where x is found
  - delete v and all ancestors of v until we reach an ancestor that has two children.

- **Time Complexity** of all operations: $\Theta(|x|)$
  $|x|$: length of binary string x, i.e., the number of bits in x
Tries: Insert Example

Example: \texttt{Trie::insert(0111$)}
Example: \textit{Trie::insert}(0111$)
Tries: Delete Example

Example: $Trie::delete(01001\$)$
Example: \textit{Trie::delete}(01001$)
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Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Internal nodes may now also represent keys. Use a *flag* to indicate such nodes.
- No need for end-of-word character $\$
- Now a trie of bitstrings is a binary tree. Can express 0-child and 1-child implicitly via left and right child.
- More space-efficient.

![Diagram of a trie with flags and arrows indicating space efficiency.]
Variations 3 of Tries

**Pruned Trie:** Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Note that now we *must* store the full keys (why?)
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)

This is in practice the most efficient version of tries, but the operations get a bit more complicated.
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Variation 4 of Tries

**Compressed Trie:** compress paths of nodes with only one child
- Each node stores an *index*, corresponding to the depth in the uncompressed trie.
  - This gives the next bit to be tested during a search
- A compressed trie with \( n \) keys has at most \( n - 1 \) internal nodes

Also known as **Patricia-Tries**:

*Practical Algorithm to Retrieve Information Coded in Alphanumeric*
Proof: by induction on the height.

1) if \( h = 0 \)

\[ n=1, \text{ no internal nodes} \Rightarrow \text{OK.} \]

2) suppose true for \( h - 1 \); prove it for a trie of height \( h \).

\[ n_1 \text{ leaves, at most } n_1 - 1 \text{ internal nodes} \]

\[ n_2 \text{ leaves, at most } n_2 - 1 \text{ internal nodes} \]
in the whole tree:

- \# leaves = n_1 + n_2

- \# internal nodes = \# internal nodes on the left + \# internal nodes on the right + 1

\[ \leq n_1 - 1 + n_2 - 1 + 1 = n_1 + n_2 - 1 \]
Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in \( x \);
  return failure if the link is missing
- if we reach a leaf, explicitly check whether word stored at leaf is \( x \)
- else recurse on the new node and the next bit of \( x \)

\begin{verbatim}
CompressedTrie::search(v ← root, x)
\end{verbatim}
\begin{align*}
v: & \text{ node of trie; } x: \text{ word} \\
1. & \textbf{if } v \text{ is a leaf} \\
2. & \textbf{return } \text{strcmp}(x, v.key) \\
3. & \textbf{if } x \text{ has at most } d \text{ bits} \\
4. & \textbf{return } \text{“not found”} \\
5. & \textbf{if there is no such child} \\
6. & \textbf{return } \text{“not found”} \\
7. & \textbf{return } \text{CompressedTrie::search}(v', x)
\end{align*}
Compressed Tries: Search Example

Example: CompressedTrie::search(10$)
Compressed Tries: Search Example

Example: `CompressedTrie::search(10$)` unsuccessful
Compressed Tries: Search Example

Example: `CompressedTrie::search(101$)`

```
0
  1             2
   0          1   0
      2      1      3
   $  0  1  $  0
00$ 0001$ 01001$ 110$ 1101$
   $  0
011$ 01101$
```
Compressed Tries: Search Example

Example: CompressedTrie::search(101$) unsuccessful
Compressed Tries: Search Example

Example: `CompressedTrie::search(1$)`
Compressed Tries: Search Example

Example: `CompressedTrie::search(1$)` unsuccessful

```
       0
      / \
     1   2
    / \ /  \
   2   2  3
  /   / \  /
00$ 0001$ 01001$ 110$ 1101$
  \  \   \  \  \
 $ 0 0 1 $  $ 1  $
  \   \  \   \  \\
011$ 01101$
```

"x too short"
Compressed Tries: Insert & Delete

- \textit{CompressedTrie::delete}(x):
  - Perform \textit{search}(x)
  - Remove the node \( v \) that stored \( x \)
  - Compress along path to \( v \) whenever possible.

- \textit{CompressedTrie::insert}(x):
  - Perform \textit{search}(x)
  - Let \( v \) be the node where the search ended.
  - Conceptually simplest approach:
    - Uncompress path from root to \( v \).
    - Insert \( x \) as in an uncompressed trie.
    - Compress paths from root to \( v \) and from root to \( x \).
  
  But it can also be done by only adding those nodes that are needed, see the textbook for details.

- All operations take \( O(|x|) \) time.
Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet* $\Sigma$
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character $\$)$
- Example: A trie holding strings $\{\text{bear}\$, $\text{ben}\$, $\text{be}\$, $\text{soul}\$, $\text{soup}\$\}
Compressed Multiway Tries

- **Variation:** Compressed multi-way tries: compress paths as before
- **Example:** A compressed trie holding strings \{ bear\$, ben\$, be\$, soul\$, soup\$ \}
Multiway Tries: Summary

- Operations $\text{search}(x)$, $\text{insert}(x)$ and $\text{delete}(x)$ are exactly as for tries for bitstrings.
- Run-time $O(|x| \cdot \text{(time to find the appropriate child)})$
Multiway Tries: Summary

- Operations $search(x)$, $insert(x)$ and $delete(x)$ are exactly as for tries for bitstrings.

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Each node now has up to $|\Sigma| + 1$ children. How should they be stored?
Multiway Tries: Summary

- Operations \( \text{search}(x) \), \( \text{insert}(x) \) and \( \text{delete}(x) \) are exactly as for tries for bitstrings.
- Run-time \( O(|x| \cdot \text{(time to find the appropriate child)}) \)

Each node now has up to \( |\Sigma| + 1 \) children. How should they be stored?

**Solution 1:** Array of size \( |\Sigma| + 1 \) for each node.
Complexity: \( O(1) \) time to find child, \( O(|\Sigma|) \) space per node.

**Solution 2:** List of children for each node.
Complexity: \( O(|\Sigma|) \) time to find child, \( O(\#\text{children}) \) space.

**Solution 3:** Dictionary (AVL-tree?) of children for each node.
Complexity: \( O(\log(\#\text{children})) \) time, \( O(\#\text{children}) \) space.
Best in theory, but not worth it in practice unless \( |\Sigma| \) is huge.

In practice, use **hashing** (keys are in (typically small) range \( \Sigma \)).
Multiway Tries: Summary

- Operations $\text{search}(x)$, $\text{insert}(x)$ and $\text{delete}(x)$ are exactly as for tries for bitstrings.
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Each node now has up to $|\Sigma| + 1$ children. How should they be stored?

**Solution 1:** Array of size $|\Sigma| + 1$ for each node.
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