University of Waterloo
CS240 Winter 2024
Assignment 1

Due Date: Tuesday, January 23 at 5:00pm

Please read `https://student.cs.uwaterloo.ca/~cs240/w24/assignments.phtml#guidelines` for guidelines on submission. **Each question must be submitted individually to MarkUs as a PDF** with the corresponding file names: a1q1.pdf, a1q2.pdf, ... , a1q6.pdf . It is a good idea to submit questions as you go so you aren't trying to create several PDF files at the last minute.

Late Policy: Assignments are due at **5:00pm**, with the grace period until 11:59pm.

Notes:

- Logarithms are in base 2, if not mentioned otherwise.
- A positive function is a function that takes positive real values.

Question 1 [3+3+3+3+3=15 marks]

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation).

a) $10n^5 \log n + 42n^4 - 15n^2 \in O(n^5 \log n)$

b) $2n^4 - 10n^2 + 5n \in \Omega(n^4)$

c) $1 + (n^2 \mod 10) \in \Theta(1)$

d) $3^n \in o(3.1^n)$

e) $n^2 \in \omega(n^{1.9})$

Question 2 [2+4=6 marks]

Prove the following statements based on the definitions of the order notations. All functions take positive values.

a) If $f(n) \in \omega(1)$, then $f(n)h(n) \in \omega(h(n))$.

b) If $f(n) \in \Theta(g(n)), h(n) \in \Theta(r(n))$, and $g(n) \in o(r(n))$ then $f(n) \in o(h(n))$.

Question 3 [3+3+3=9 marks]

For each pair of the following functions, fill in the correct asymptotic notation among Θ, o, and ω in the statement $f(n) \in \sqcup(g(n))$. Prove the relationship using any relationship or technique described in class.

a) $f(n) = 5n + 7n^2 \log n$ versus $g(n) = n^2\sqrt{n}$

b) $f(n) = n!$ versus $g(n) = n^{100}$

c) $f(n) = 2^n$ versus $g(n) = 2^{\cos n}$

Question 4 [3+3+3+3=12 marks]

Analyze the following pieces of pseudocode and give a tight (Θ) bound on the running time as a function of n. Show your work. A formal proof is not required, but you should justify your answer (in all cases, n is assumed to be a positive integer).

a)
```
s := 0
for i=1 to n * n * n do
   for j=1 to i * i * i * i do
      s := s + 1
```

b)
```
p = 1
s = 0
for i = 1 to n do
   p = p * 2
   for j = 1 to p do
      s = s + 1
```

c)
```
s = 0
i = 1
while i <= n do
    j = n * n * n * n
    while j > 0 do
        s = s + 1
        j = j - i
    i = i * 2
```

d) For this question, you can derive O bound, but it must be as tight as possible.

```
foo(A[0...n-1]) # assume n>=1 is a power of two
   if n <= 1
       return
   print("x")
   foo(A[0...n/4-1]
   foo(A[n/4...n/2-1]
```

Question 5 [4+4=8 marks]

True or false? For each of the following assertions, indicate whether it is true or false. If true, prove it; if false, give a counter-example and briefly justify it.

a) $f(n)$ is $\Theta(f(n/2))$, where f takes positive values.

b) $1 + n + n^2 + \cdots + n^{k-1} \in \Theta(n^k)$.

Question 6 [1+4=5 marks]

For this problem, assume that we only deal with functions that are defined on non-negative integers and take positive real values (i.e. the domain is non-negative integers and the range is positive reals). Dr. I. M. Smart has recently invented a new class of functions, denoted $O'(f)$. A function $g(n)$ is in $O'(f)$ if there is a constant $c > 0$ such that $g(n) \leq cf(n)$ for all $n \geq 0$. Prove the following statements about relationship between $O(f)$ and $O'(f)$.

a) If $g(n) \in O'(f(n))$, then $g(n) \in O(f(n))$

b) If $g(n) \in O(f(n))$, then $g(n) \in O'(f(n))$