CS 240 – Data Structures and Data Management

Module 2: Priority Queues

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Based on lecture notes by many previous cs240 instructors

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Outline

- Priority Queues
  - Review: Abstract Data Types
  - ADT Priority Queue
  - Binary Heaps
  - Operations in Binary Heaps
  - PQ-Sort and Heapsort
  - Intro for the Selection Problem

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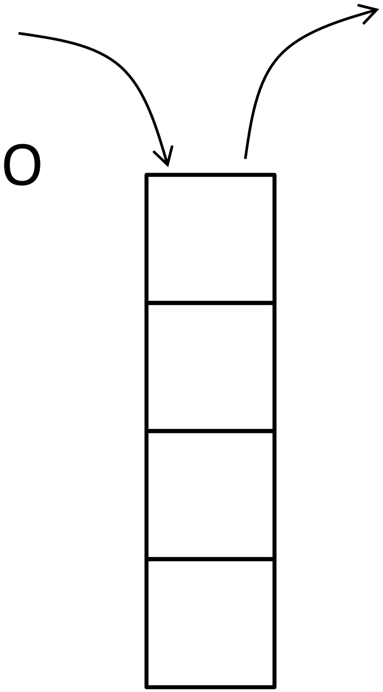
Abstract Data Type (ADT)

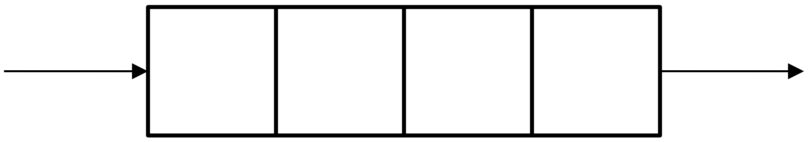
- A description of *information* and a collection of *operations* on that information
- The information accessed *only* through the operations
- ADT describes what is stored and what can be done with it, but not how it is implemented
- Can have various *realizations* of an ADT, which specify
  - how the information is stored (*data structure*)
  - how the operations are performed (*algorithms*)

Stack ADT (review)



- ADT consisting of a collection of items removed in LIFO (last in first out order)
- Operations
  - *push* insert an item
  - *pop* remove and return the most recently inserted item
- Extra operations
  - *size*, *isEmpty*, and *top*
- Applications
  - addresses of recently visited sites in a Web browser, procedure calls
- Realizations of Stack ADT
  - arrays
  - linked lists
    - both have constant time *push*/*pop*

Queue ADT



- ADT consisting of a collection of items removed in FIFO (first-in first-out) order
- Operations
  - *enqueue* insert an item
  - *dequeue* remove and return the least recently inserted item
- Extra operations
  - *size*, *isEmpty*, and *peek*
- Realizations of Queue ADT
  - (circular) arrays
  - linked lists
    - both have constant time *enqueue* /*dequeue*

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Priority Queue ADT

- Collection of items each having a *priority*
    - (*priority*, *other info*) or (*priority*, *value*)
    - priority is also called *key*
- Operations
    - *insert*: insert an item tagged with a priority
    - *deleteMax*: remove and return the item of highest priority
        - also called *extractMax*
- Definition is for a maximum-oriented priority queue
    - to define minimum-oriented priority queue, replace *deleteMax* by *deleteMin*
- Applications
    - typical "todo" list
    - sorting, etc.
- Question: How to simulate a stack/queue with a priority queue?

Using Priority Queue to Sort

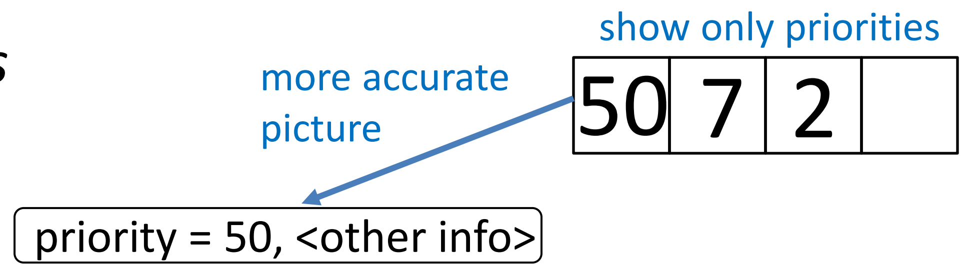
PQ-Sort$(A[0 \ldots n-1])$

1. initialize *PQ* to an empty priority queue
2. **for** $i \leftarrow 0$ **to** $n-1$ **do**
4. *PQ.insert*$(A[i])$
5. **for** $i \leftarrow n-1$ **downto** 0 **do**
6. $A[i] \leftarrow$ *PQ.deleteMax* ()

- $A[i]$ is item with priority $A[i]$
- Run-time $O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{deleteMax})$
  - depends on priority queue implementation

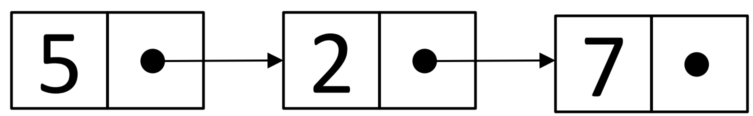
Realizations of Priority Queues

- **Attempt 1**: *unsorted arrays*



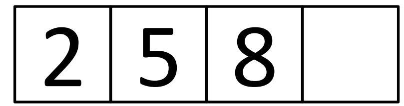
  - assume dynamic arrays
    - expand by doubling when needed
    - happens rarely, so amortized time over all insertions is $O(1)$
  - *insert*: $\Theta(1)$
  - *deleteMax*: $\Theta(n)$
  - PQ sort becomes $\Theta(n^2)$ in the worst and in the best cases
    - equivalent to *selection-sort*

- **Attempt 2**: *unsorted linked lists*
  - efficiency identical to Attempt 1



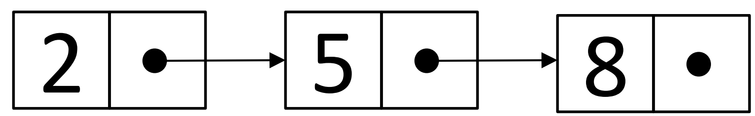
Realizations of Priority Queues

- **Attempt 3**: *sorted arrays*



  - store items in order of increasing priority
  - *deleteMax*: $\Theta(1)$
  - *insert*: $\Theta(n)$
    - in $O(1)$ in the best case (how?)
  - PQ-sort equivalent to insertion-sort
    - $\Theta(n^2)$ worst case
    - $\Theta(n)$ best case

- **Attempt 4**: *sorted linked-lists*



  - similar to Attempt 3

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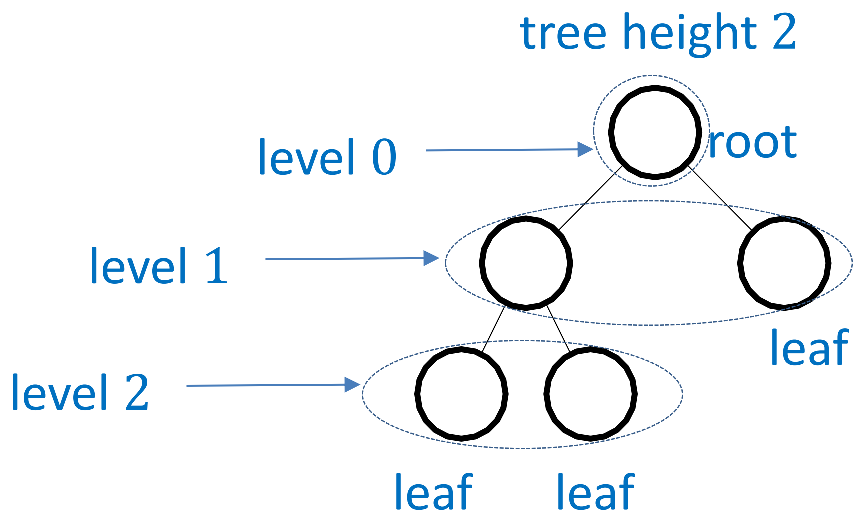
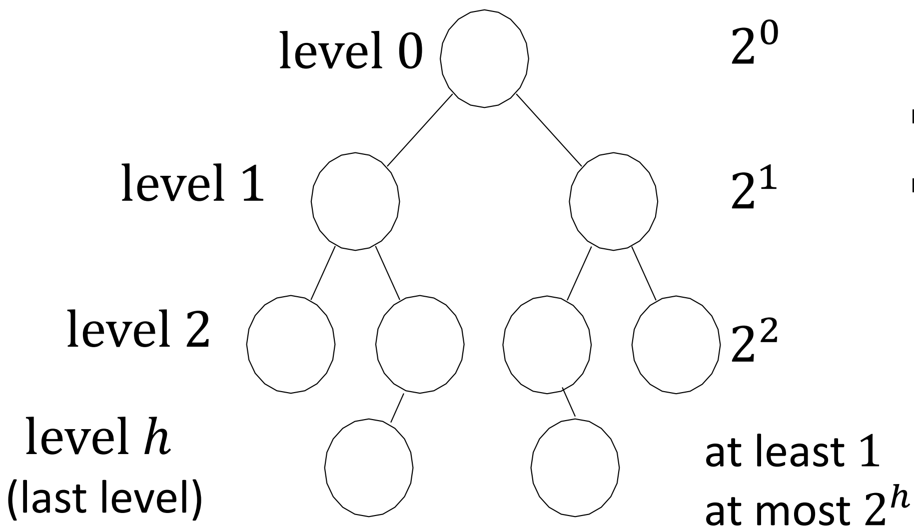
Binary Tree Review



- A *binary tree* is either
  - empty, or
  - consists of three parts
    - node
    - two binary trees
      - left subtree
      - right subtree
- Terminology
  - root, leaf, parent, child, level, sibling, ancestor, descendant
  - level l: all nodes with distance l from the root (root is on level 0)
  - height of the tree is the longest path in the tree

Lower Bound on Binary Tree Height

- Tree with n nodes has height $h \le n-1$
- Consider tree with n nodes of smallest possible height h
  - all levels must be as full as possible, except possibly the last level h



- level i has 2^i nodes
- level h has between 1 and 2^h nodes

$$
\begin{aligned}
2S &= \cancel{2^1} + \cancel{2^2} + \cdots \cancel{2^h} + 2^{h+1} \\
-\quad S &= 2^0 + \cancel{2^1} + \cancel{2^2} \ldots + \cancel{2^h} \\
\hline
S &= 2^{h+1} - 1
\end{aligned}
$$

$$n \le 2^0 + 2^1 + 2^2 + \cdots + 2^{h-1} + 2^h$$

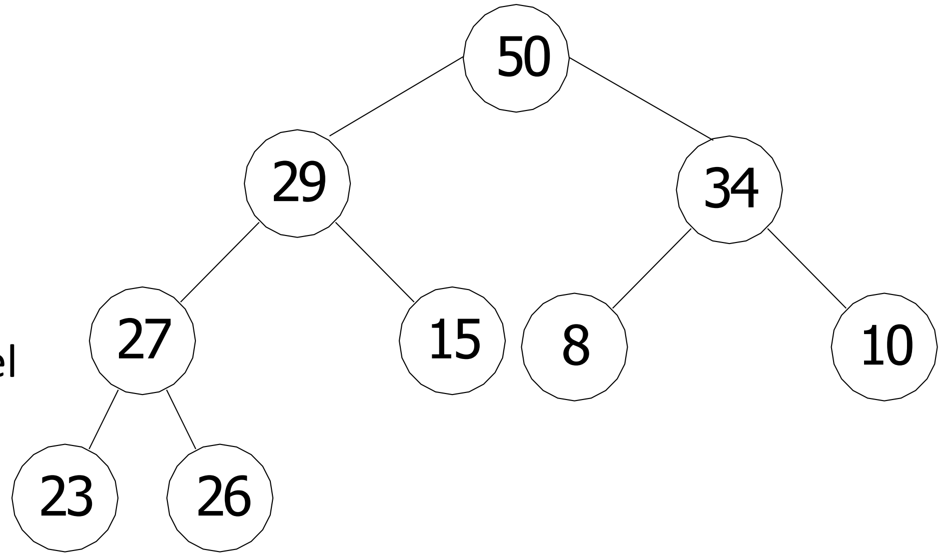
$$n \le 2^{h+1} - 1$$

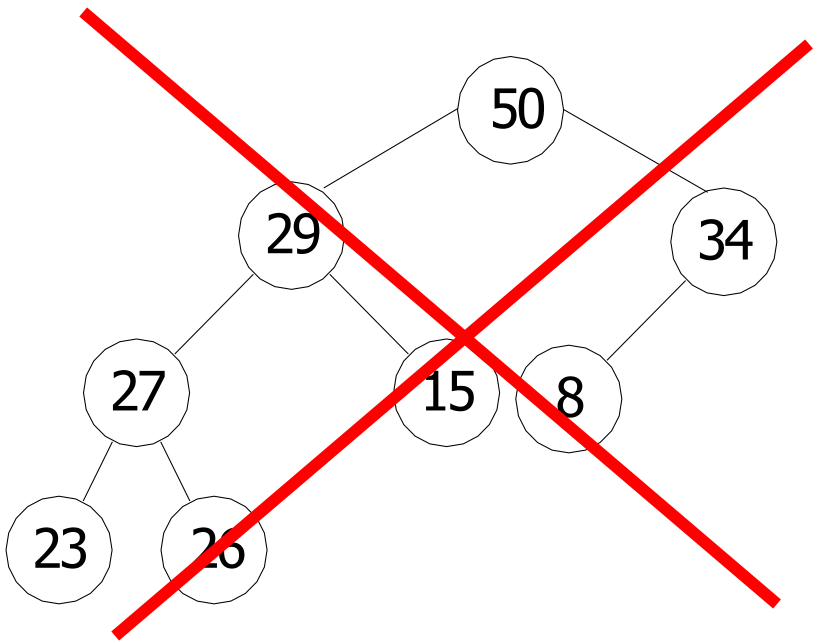
$$h \ge \log(n+1) - 1$$

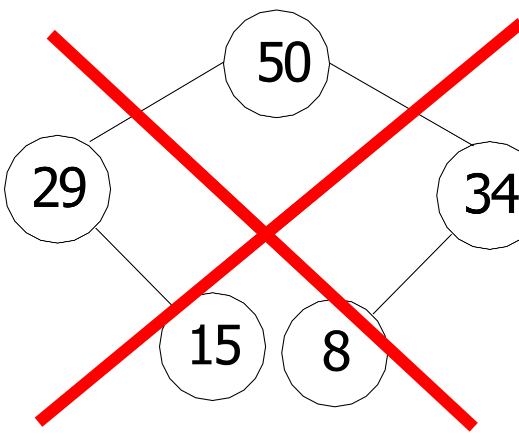
- Binary tree height is $\Omega(\log n)$
  - note use of asymptotic notation for function other than running time

Heaps: Definition

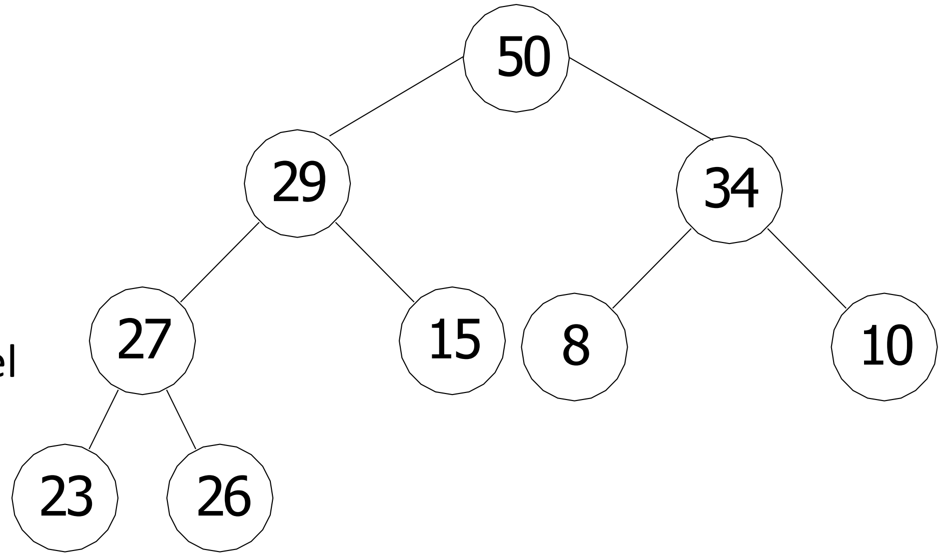
- A *max-oriented binary heap* is a binary tree with the following two properties
  1. Structural Property
     - all levels of a heap are completely filled, except (possibly) the last level
     - last level is *left-justified*







Heaps: Definition



- A *max-oriented binary heap* is a binary tree with the following two properties
  1. Structural Property
     - all levels of a heap are completely filled, except (possibly) the last level
     - last level is *left-justified*
  2. Heap-order Property
     - for any node i, key[parent of i] $\geq$ key[i]
- A *min-heap* is the same, but with opposite order property
- Heaps are ideal for implementing priority queues

Heap Height

Lemma: Height of a heap with n nodes is $\Theta(\log n)$

- heap is a binary tree $\Rightarrow$ height $h \in \Omega(\log n)$
- need to show $h \in O(\log n)$
- heap has all levels full except possibly level h
  - 2^i nodes at level $0 \leq i \leq h-1$
- Thus

at least last node at level h

$$n \geq 2^0 + 2^1 + 2^2 + \cdots + 2^{h-1} + 1$$

$$n \geq 2^h - 1 \quad + 1$$

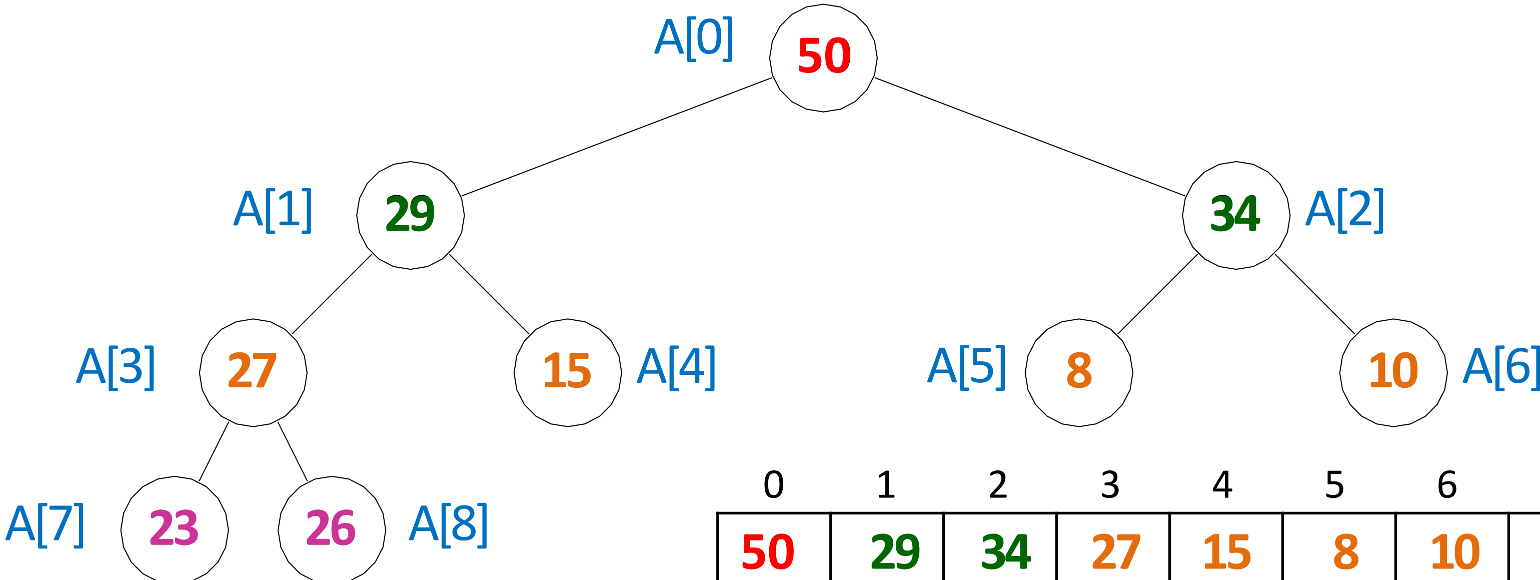
$$n \geq 2^h$$

$$h \leq \log n$$

- Thus $h \in O(\log n)$

Storing Heaps in Arrays

- Using linked structure for heaps wastes space
- Let H be a heap of n items and let A be an array of size n
  - store root in $A[0]$
  - continue storing *level-by-level* from top to bottom, in each level left-to-right

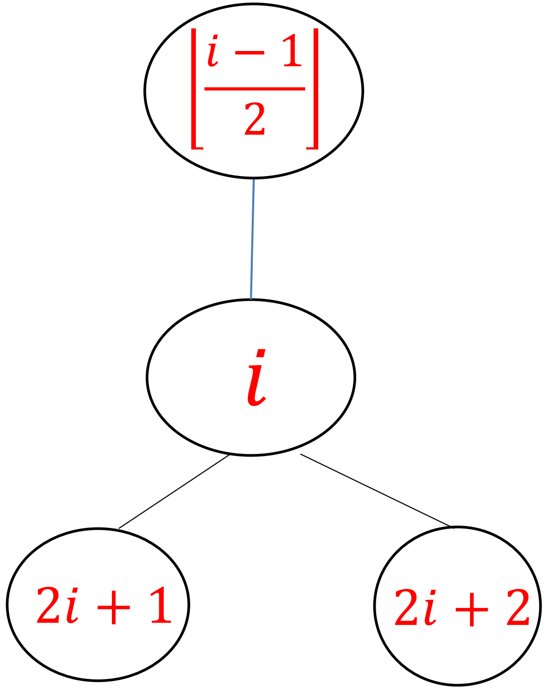


0	1	2	3	4	5	6	7	8
50	29	34	27	15	8	10	23	26
level 0	level 1		level 2				level 3	

- Use dynamic arrays
  - $A.\text{size}() = 9$
- Last heap node is in $A[n-1]$

Heaps in Arrays: Navigation

- Use node and index interchangeably
- Root is at index 0
- Last *node is* $n-1$
  - n is the size
- Left child of i, if exists, is $2i+1$
- Right child of i, if exists, is $2i+2$
- Parent of i, if exists, is $\left\lfloor \frac{i-1}{2} \right\rfloor$
- These nodes exist if index falls into range $\{0, \ldots, n-1\}$
- Hide implementation details using helper-function
  - functions *root*( ), *parent*(i), *left*(i), right(i), *last*()
    - some helper functions need to know n
      - *left*(i), right(i), *last*()
    - assume data structure stores n explicitly



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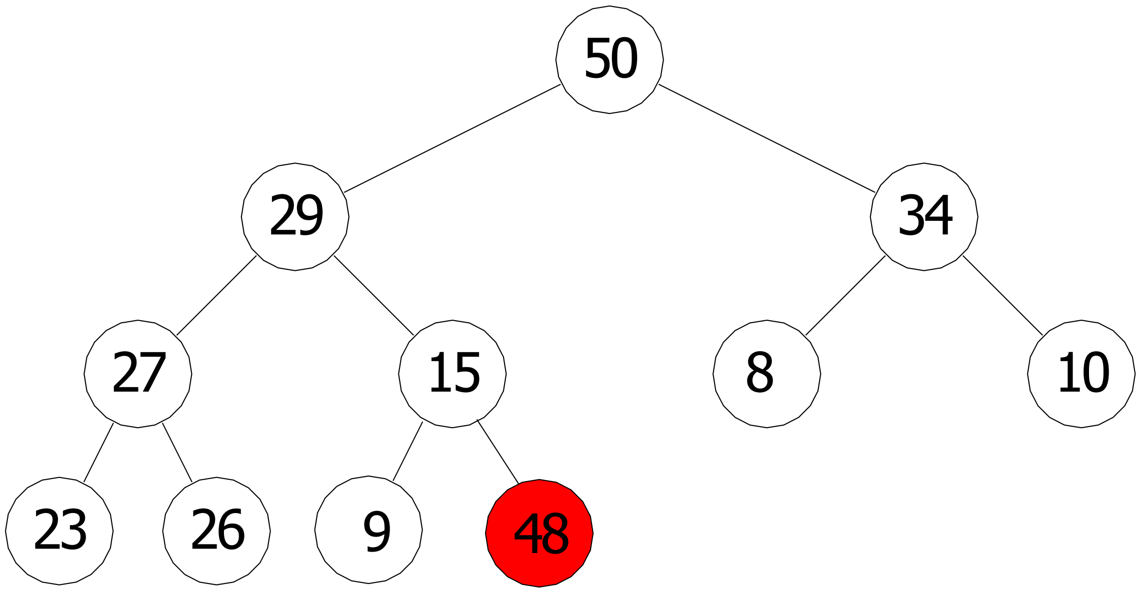
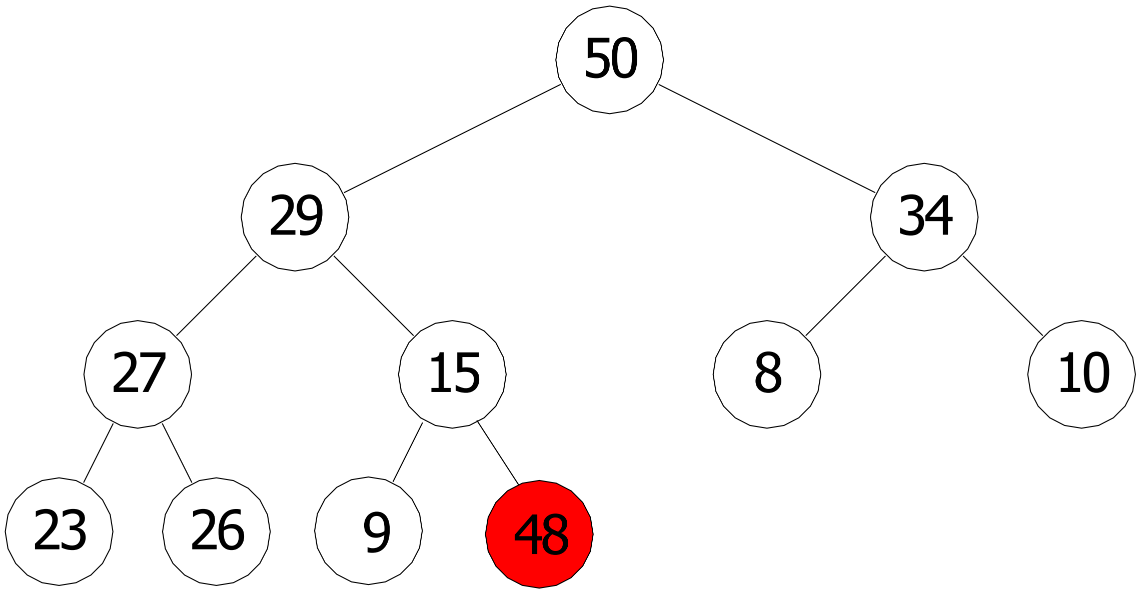
Insertion in Heaps

- Place new key at the first free leaf
- Heap-order property might be violated
- Perform a *fix-up*



fix-up example



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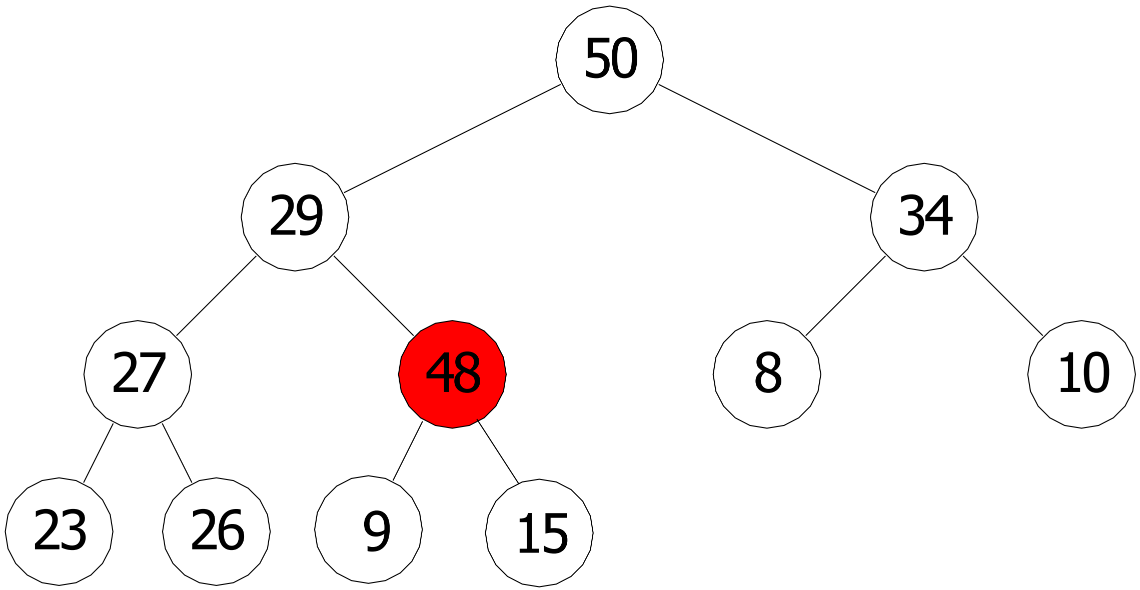

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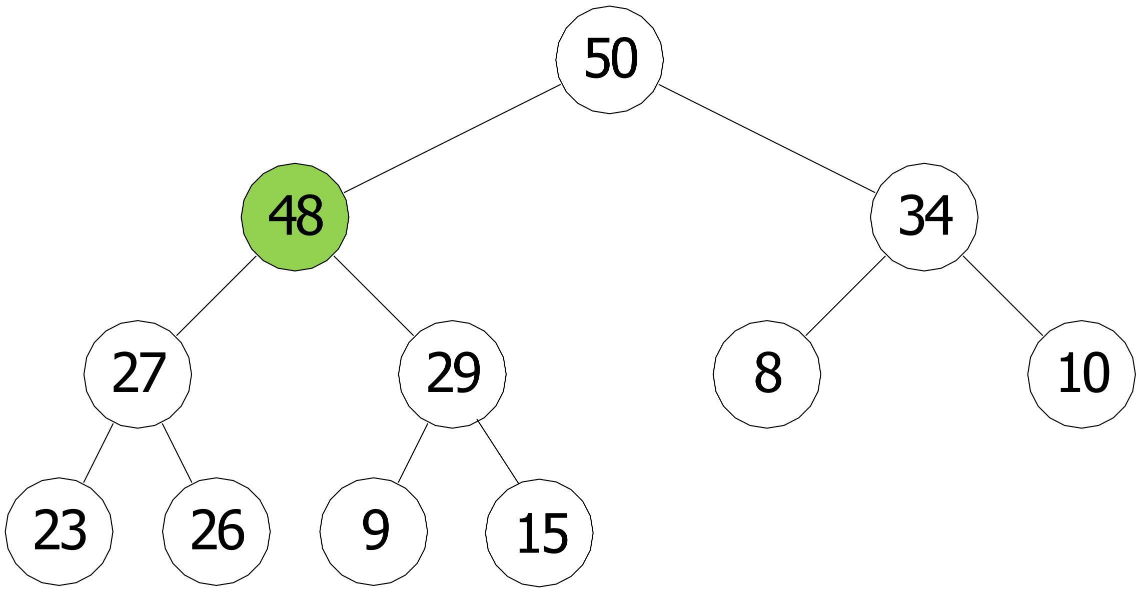


fix-up pseudocode

fix-up(A, i)
i: an index corresponding to heap node
while *parent*(i) exists **and** $A[parent(i)].key < A[i].key$ **do**
    swap $A[i]$ and $A[parent(i)]$
    $i \leftarrow parent(i)$ // move to one level up

- Time: $O(\text{heap height}) = O(\log n)$

Insert Pseudocode

$l = 4$

54	32	15	17	**44**

size = 5

fix up →

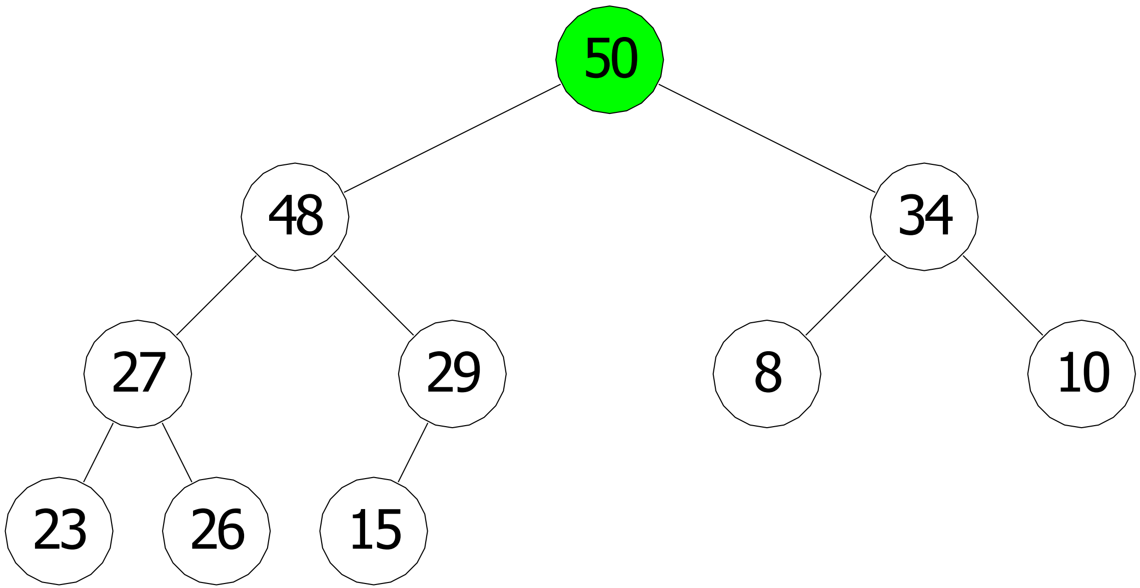
54	44	15	17	32

- Class for heap
  - variable *size* is a class variable to keep track of the number of items
- Store items in array A
- *insert* is $O(\log n)$

heap::insert(x)
    increase $size$
    $l \leftarrow last()$
    $A[l] \leftarrow x$
    fix-up (A, l)

deleteMax in Heaps

- The root has the maximum item
- Replace root by the last leaf and remove last leaf



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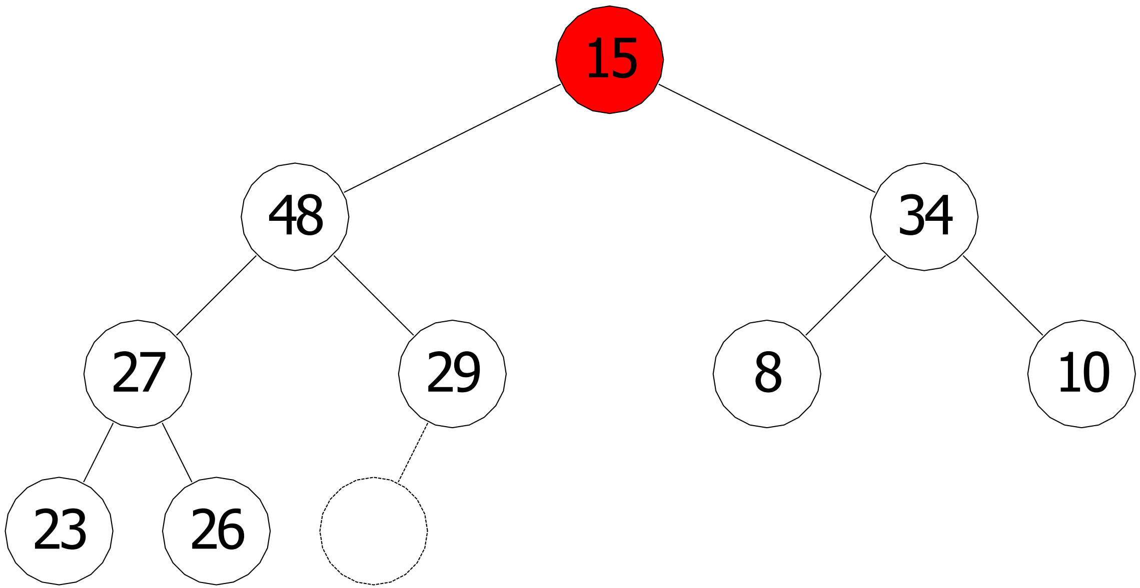

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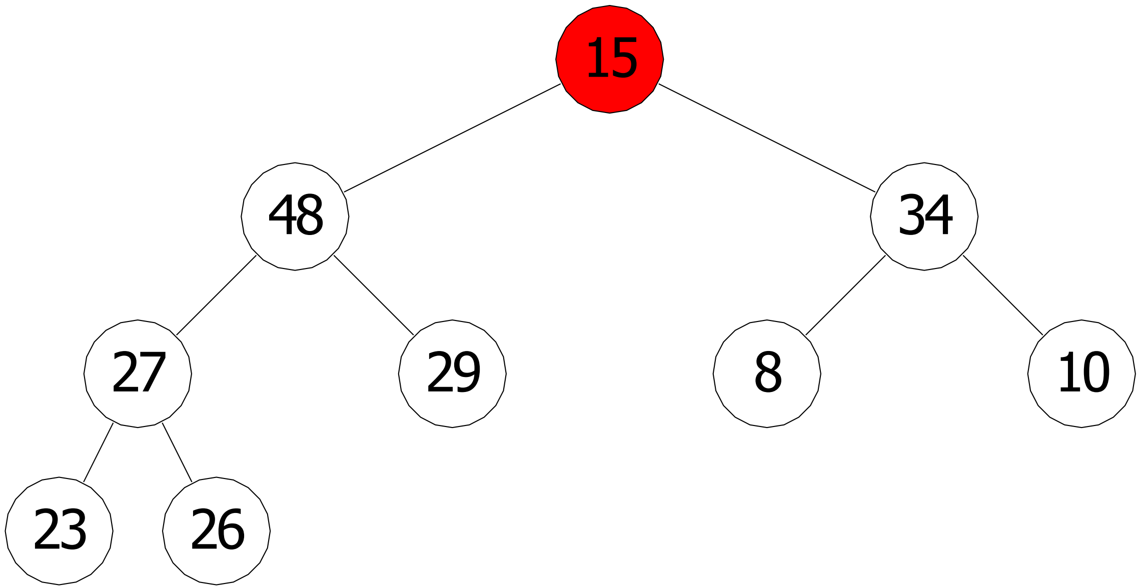
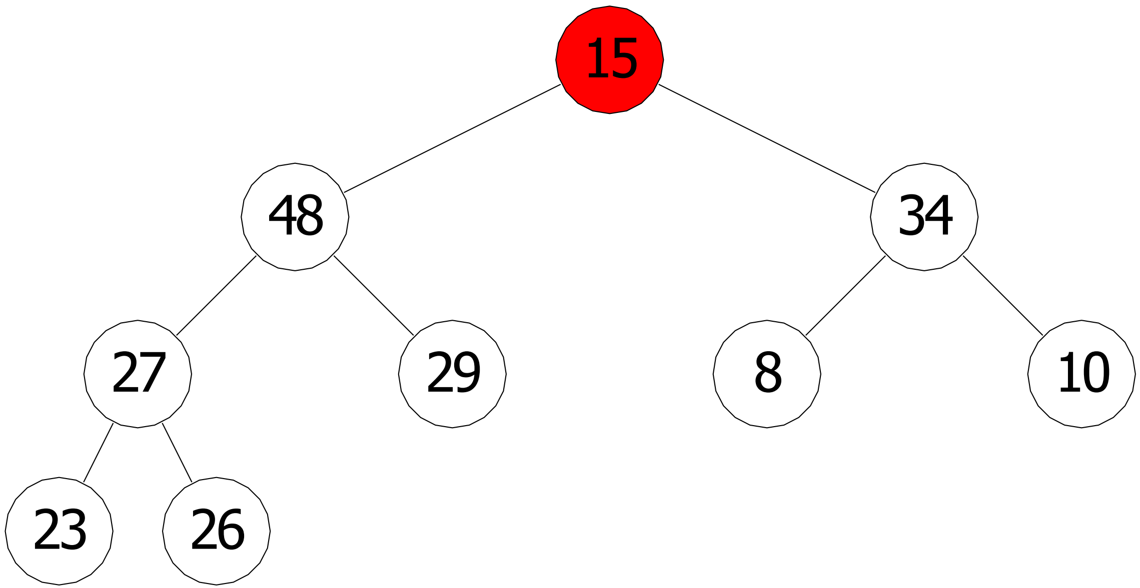


- The heap-order property might be violated
  - perform *fix-down*

fix-down example



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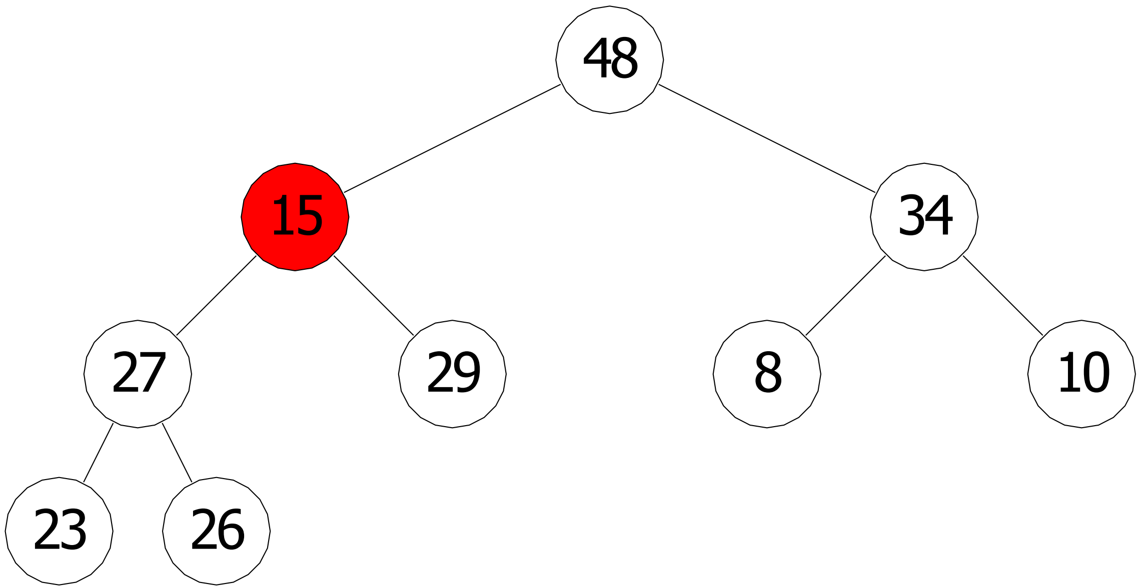

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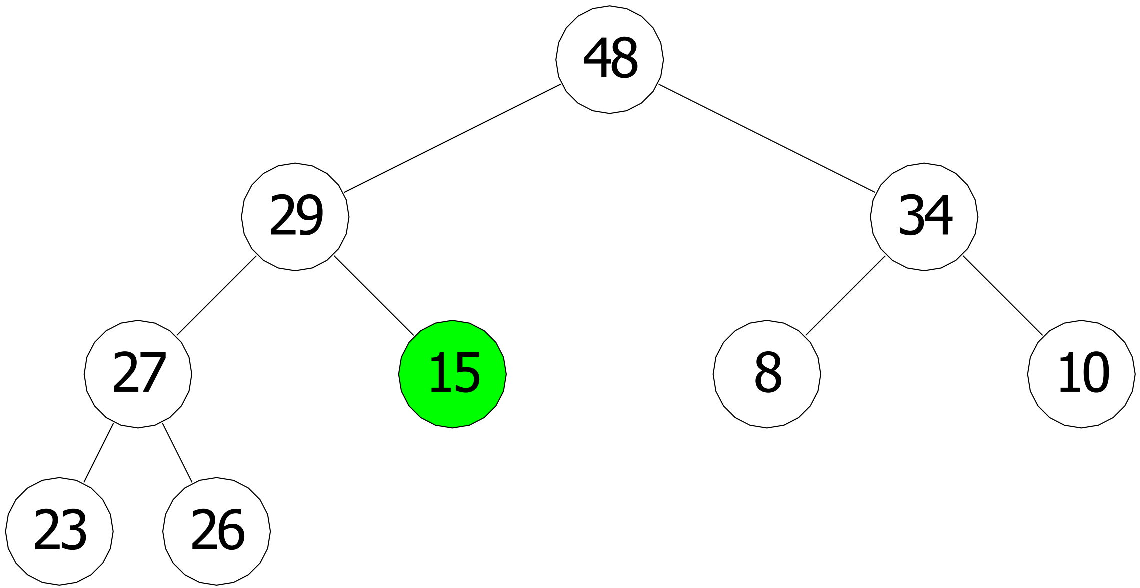

Fix-Down

fix-down(A, i, n)
A: array that stores a heap of size n in locations $0 \ldots n-1$
i: index corresponding to a heap node,
while i is not a leaf **do**
  $j \leftarrow$ left child of i
  if i has right child **and** A[right child of i]$.key > A[\,j\,].key$ **then**
    $j \leftarrow$ right child of i // right child has larger key
  if $A[i]$.key $\geq A[\,j\,].key$
    break   // order is fixed, done
  swap $A[i]$ **and** $A[j]$
  $i \leftarrow j$ // move to one level down

- Time: $O(\text{heap height}) = O(\log n)$
- Pass n because for some usages of *fix-down*, A stores heap only in the front part

54	44	15	17	32	99	100

$\underbrace{54 \quad 44 \quad 15 \quad 17 \quad 32}_{\text{heap}} \quad \underbrace{99 \quad 100}_{\text{not heap}}$

Pseudocode for deleteMax

deleteMax ( )

  $l \leftarrow last()$

  $toReturn = A[root()]$

  $A[root()] = A[l]$

  decrease $size$

  $fix\text{-}down(A, root(\,), size)$

  return *toReturn*

$l = 3$

54	32	15	17

size = 4

toReturn = 54

Pseudocode for deleteMax

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17	32	15

fix down

32	17	15

size = 3

toReturn = 54

- *deleteMax* is $O(\log n)$

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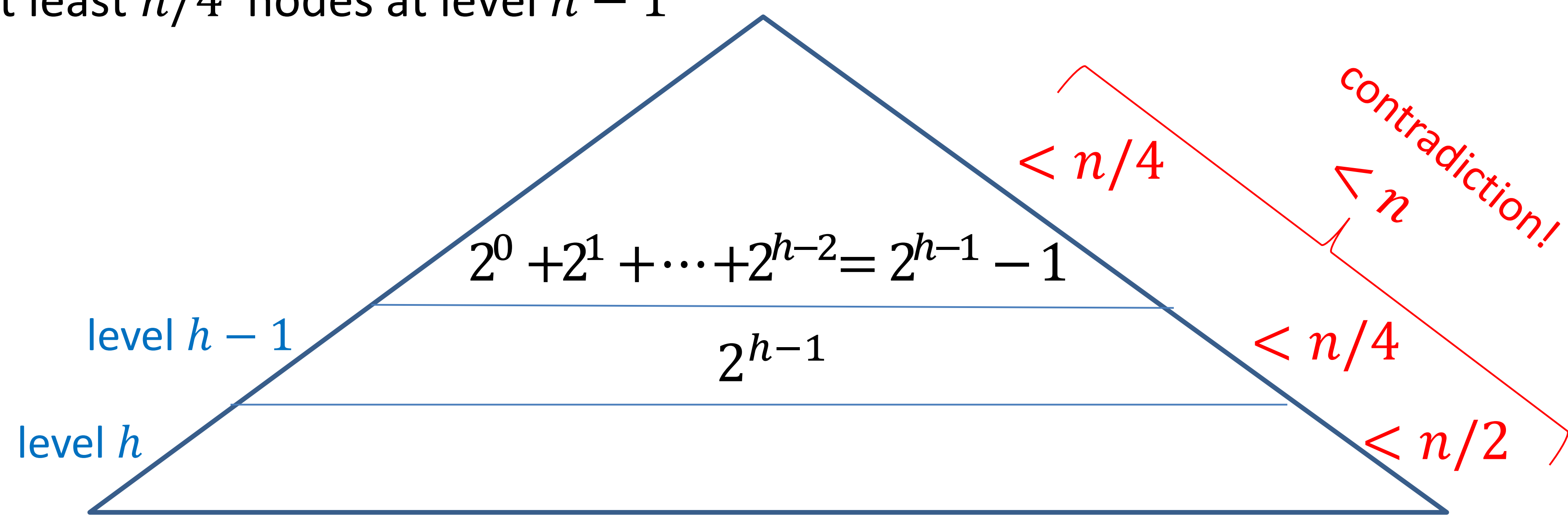
Sorting using Heaps

- Priority queue sort is $O(init + n \cdot insert + n \cdot deleteMax)$ time
- **Blue** part of the algorithm
  - simple heap building
  - additional array of size n for heap H
  - insert uses *fix-up* which is $O(\log n)$
  - worst-case: A in increasing order

PQSortWithHeaps(A)
$H \leftarrow$ empty heap
for $i \leftarrow 0$ **to** $n-1$ **do**
$H.$*insert*$(A[i])$
for $k \leftarrow n-1$ **downto** 0 **do**
$A[i] \leftarrow H.$*deleteMax*$()$

- Fact: heap with n nodes and height h has at least $n/4$ nodes at level $h-1$



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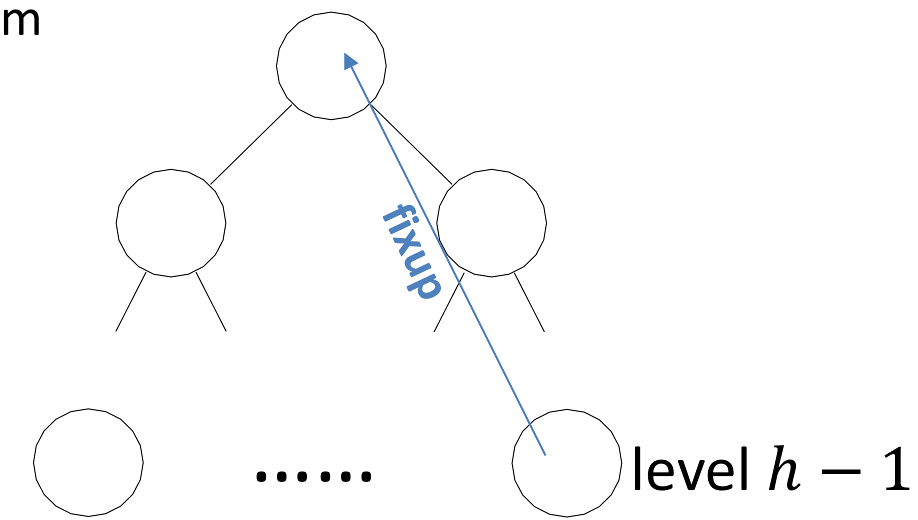
  - insert uses *fix-up* which is $O(\log n)$
  - worst-case: A in increasing order
    - level $h-1$ has at least $n/4$ nodes
    - fix-up is $c \log n$ for each of them
    - total time $cn/4 \log n$
    - $\Theta(n \log n)$

```
PQSortWithHeaps(A)
        H ← empty heap
        for i ← 0 to n − 1 do
            H.insert(A[i])
        for k ← n − 1 downto 0 do
            A[i] ← H.deleteMax( )
```



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- *PQ-Sort* with heap is $\Theta(n \log n)$ and not in place
  - need $\Theta(n)$ auxiliary space for heap array H
- Heapsort: improvement to *PQ-Sort* with two added tricks
  1. use the input array A to store the heap!
  2. heap can be built in linear time if know all items in advance
  - heapsort is in-place and $O(1)$ auxiliary space

Building Heap Directly In Input Array

A

17	32	15	54	2	25	3

A

54	25	32	17	2	15	3

Problem statement: build a heap from n items in $A[0, \dots, n-1]$ without using additional space

- i.e. put items in $A[0, \dots, n-1]$ in heap-order

Building Heap Directly In Input Array

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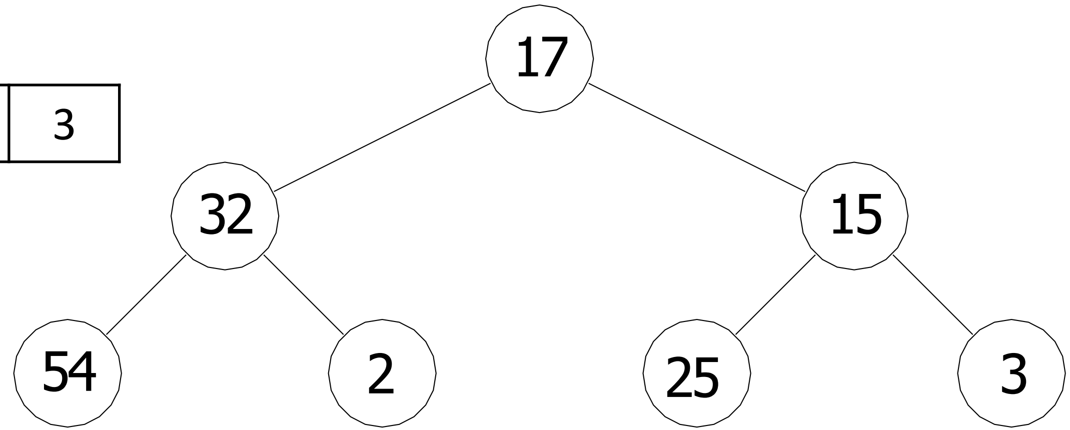
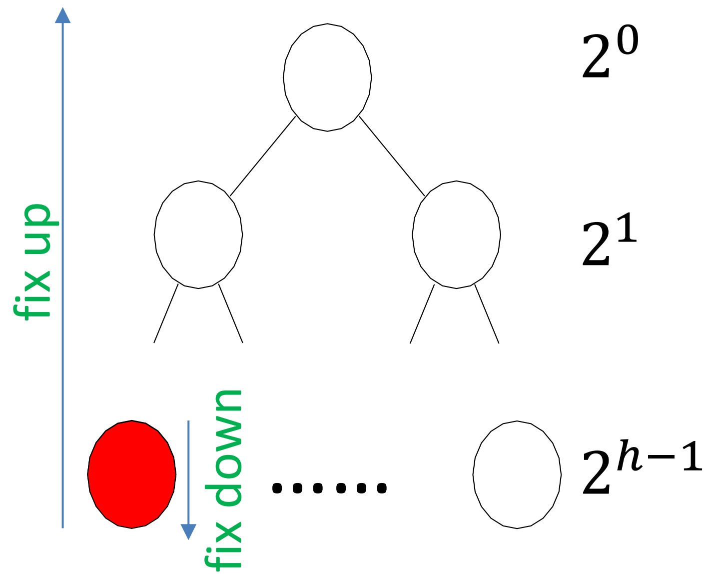


Problem statement: build a heap from n items in $A[0, \ldots, n-1]$ without using additional space

- i.e. put items in $A[0, \ldots, n-1]$ in heap-order
- Look at array A as a binary tree
- Heap-order (most likely) does not hold
- To create heap-order, can either
  1. run *fix-up* for each node
  2. run *fix-down* for each node
     - turns out to be more efficient

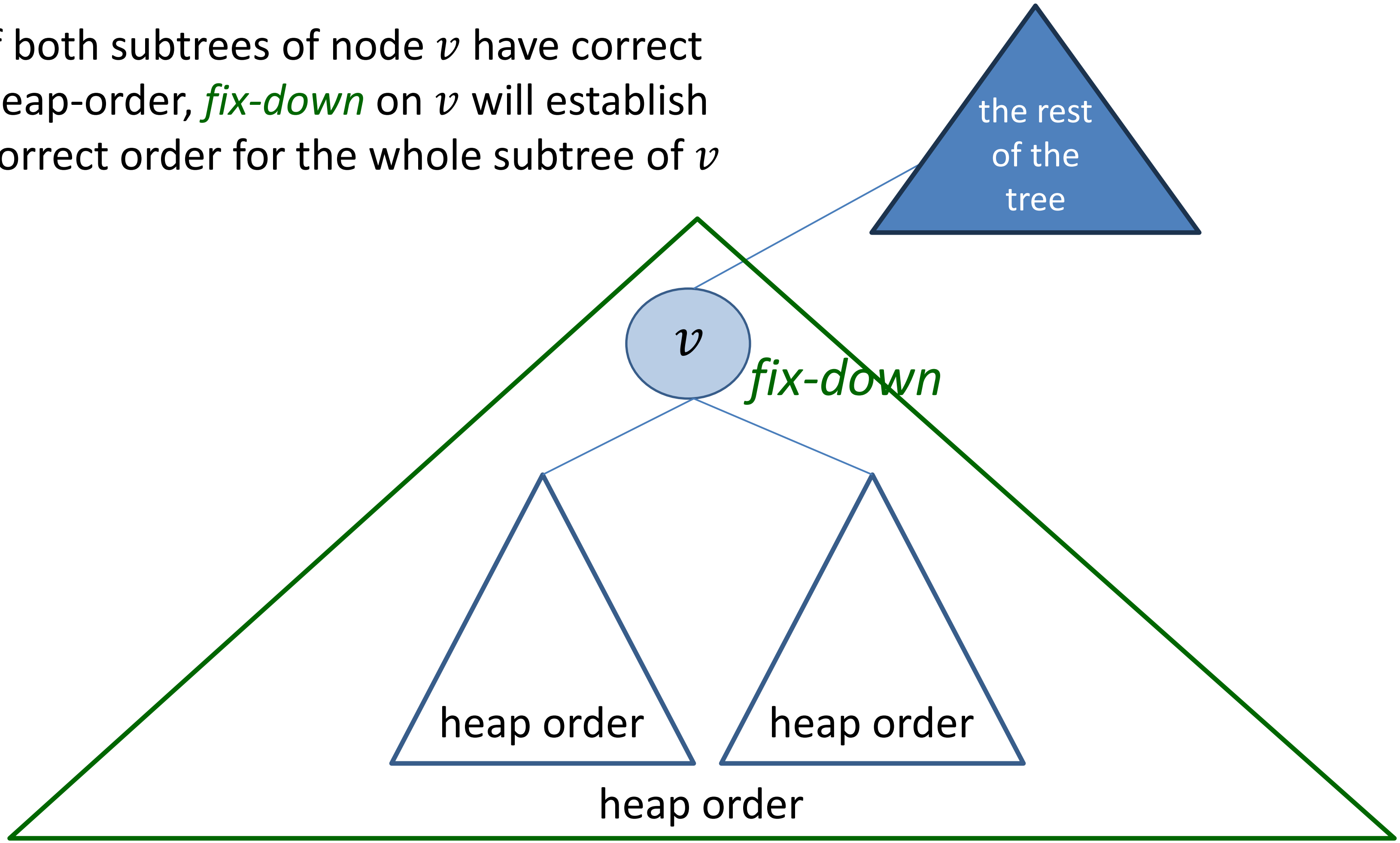
Building Heap Directly In Input Array: Fix-Up vs. Fix-Down

- Level $h-1$ has at least $n/4$ nodes
- For each such node
    - fix-up takes $O(\log n)$ time
    - fix-down takes $O(1)$ time

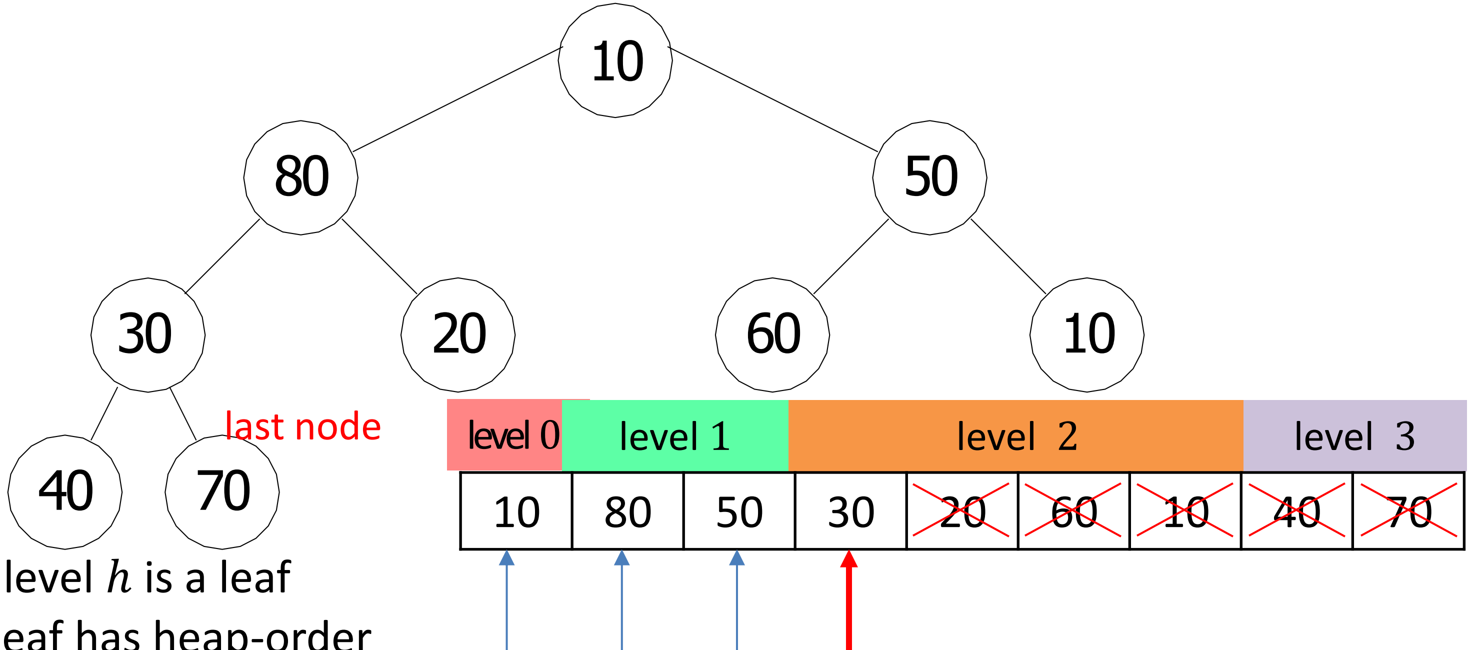


Establishing Heap Order with *fix-down*s

- If both subtrees of node v have correct heap-order, *fix-down* on v will establish correct order for the whole subtree of v



Establishing Heap Order with *fix-downs*

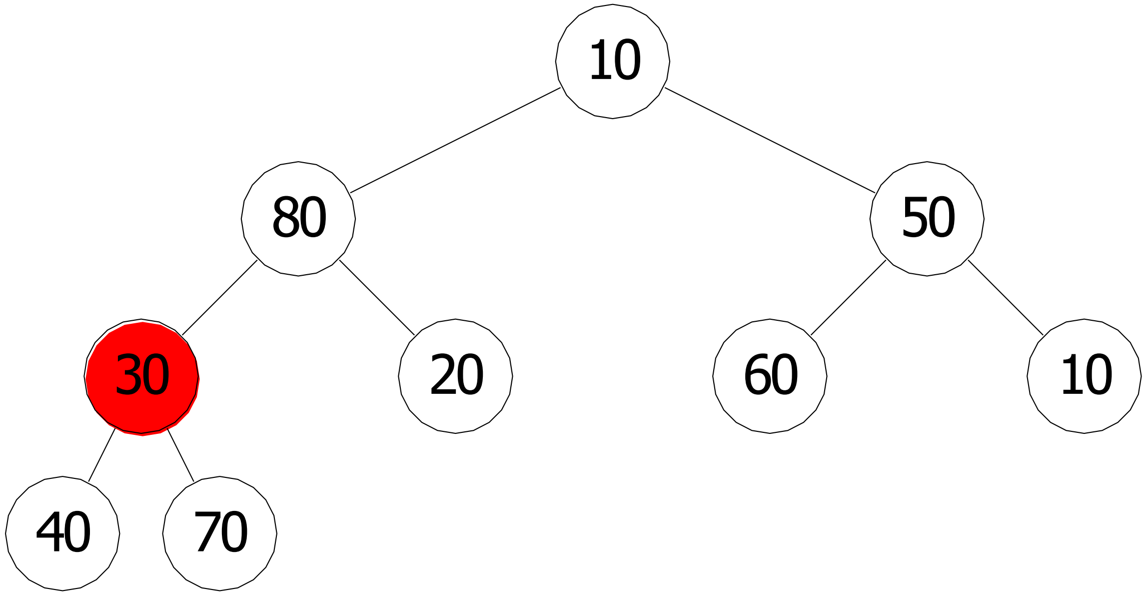


- Every node at level h is a leaf
  - any leaf has heap-order
- *fix-down* on a leaf does not do anything, so start with the rightmost non-leaf node
  - this is parent of the *last*() node
- Run *fix-down* for level $h-1$ nodes (starting with the first non-leaf node)
  - subtree of any level $h-1$ node has heap order
- Run *fix-down* for level $h-2$ nodes
  - subtree of any level $h-2$ node has heap order
- .....
- Run *fix-down* for level 0 node
  - the whole tree has heap-order

Heapify Example

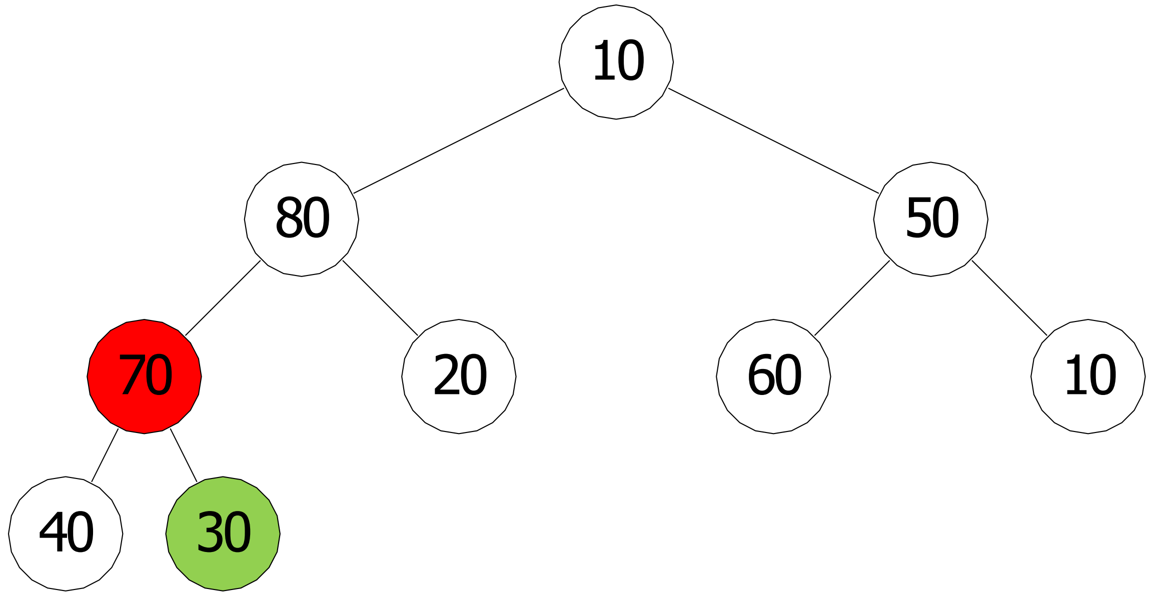
A

10	80	50	30	20	60	10	40	70



Heapify Example

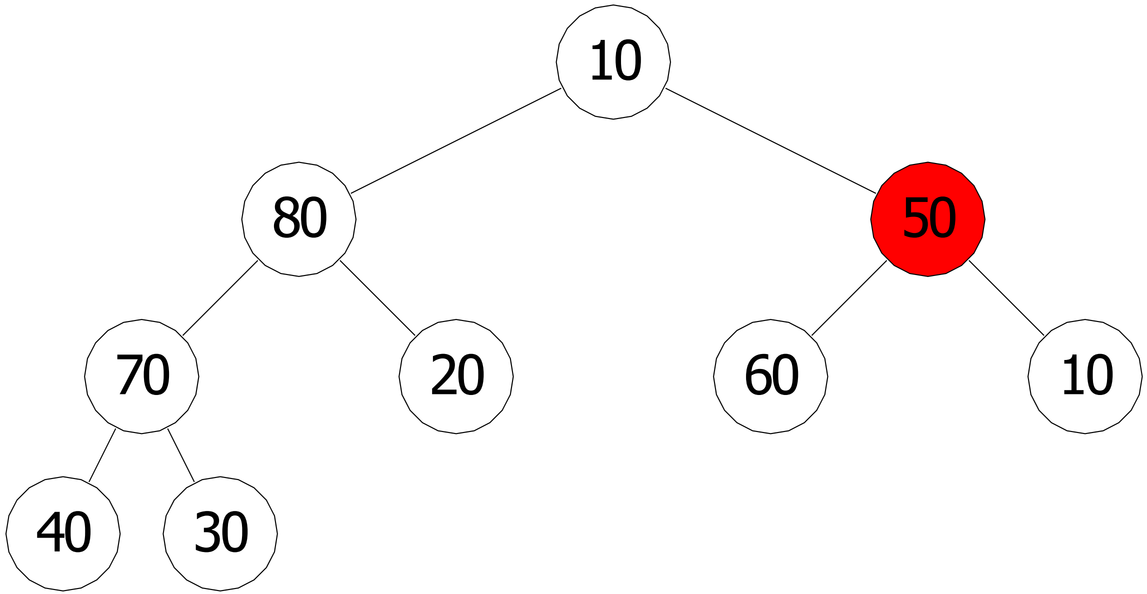
A									
	10	80	50	70	20	60	10	40	30



Heapify Example

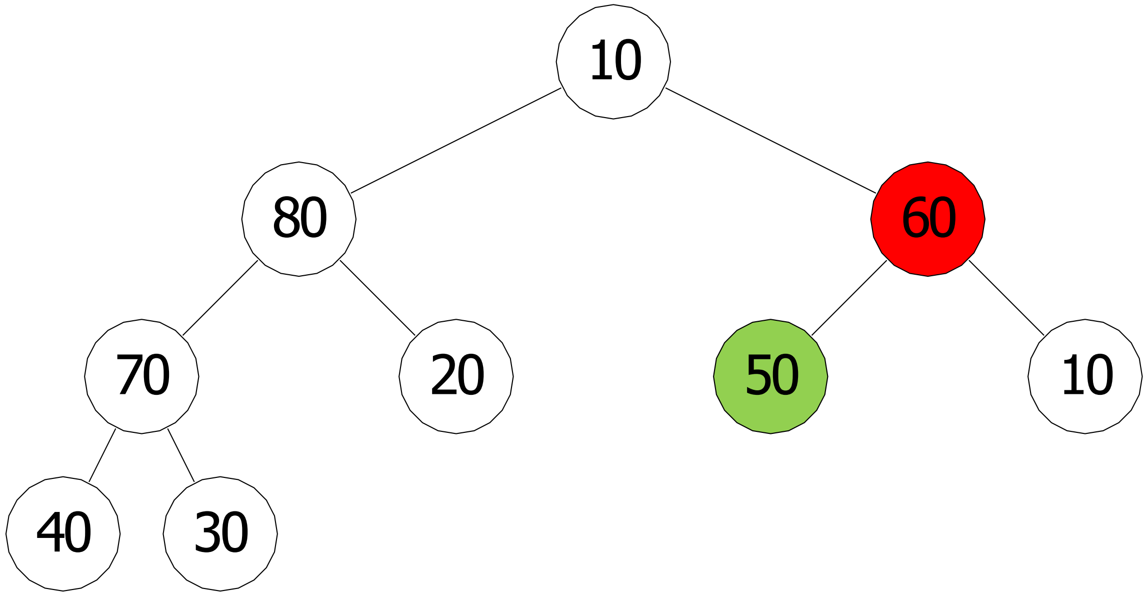
A

10	80	50	70	20	60	10	40	30



Heapify Example

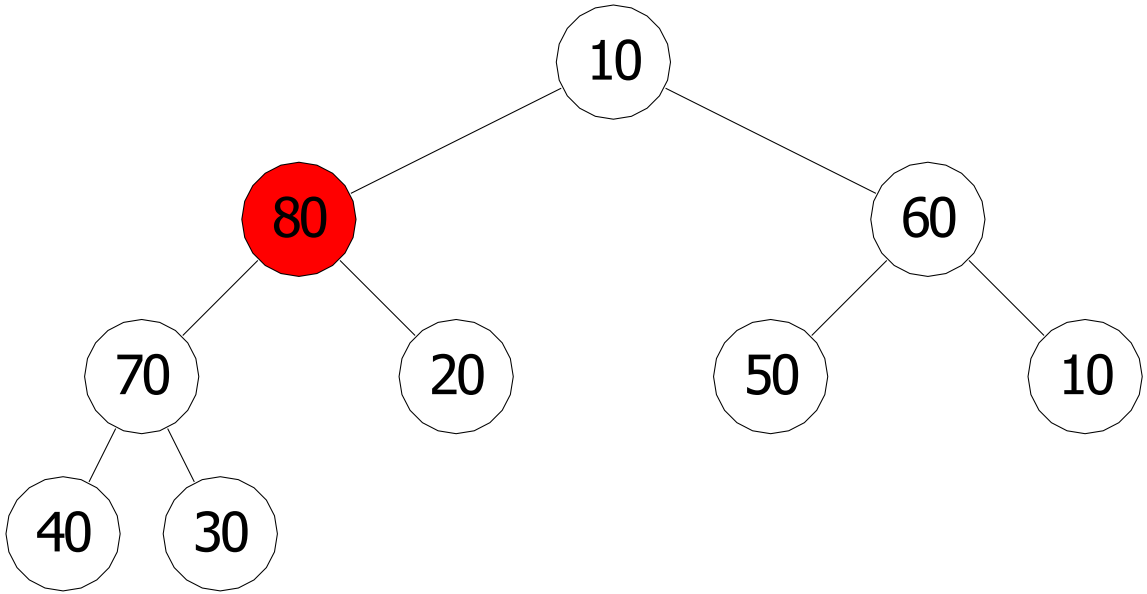
A								
10	80	60	70	20	50	10	40	30



Heapify Example

A

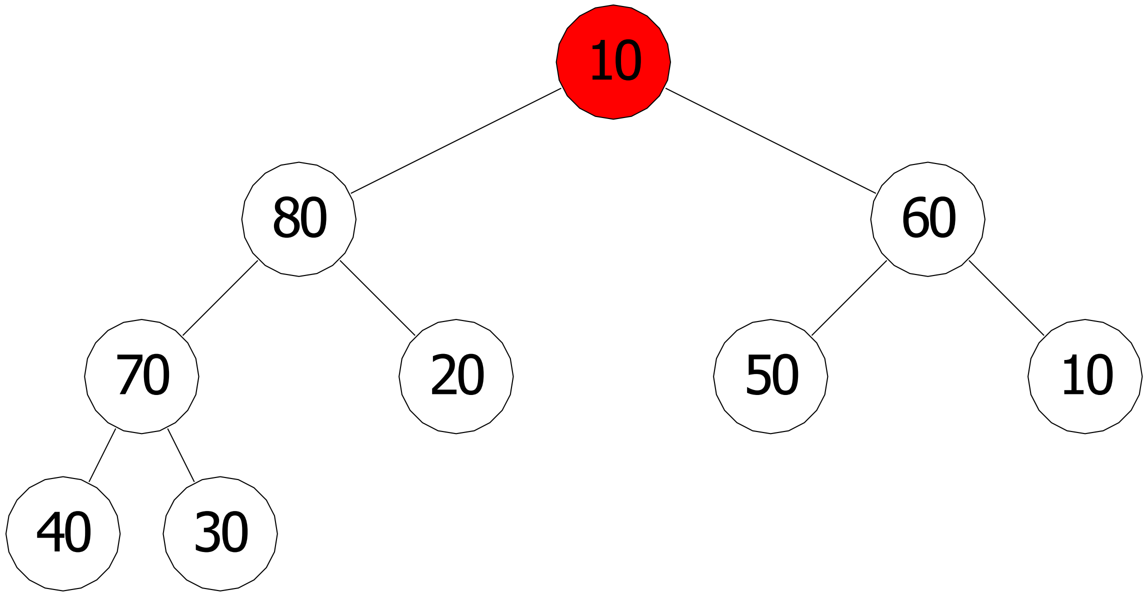
10	80	60	70	20	50	10	40	30



no need to do anything

Heapify Example

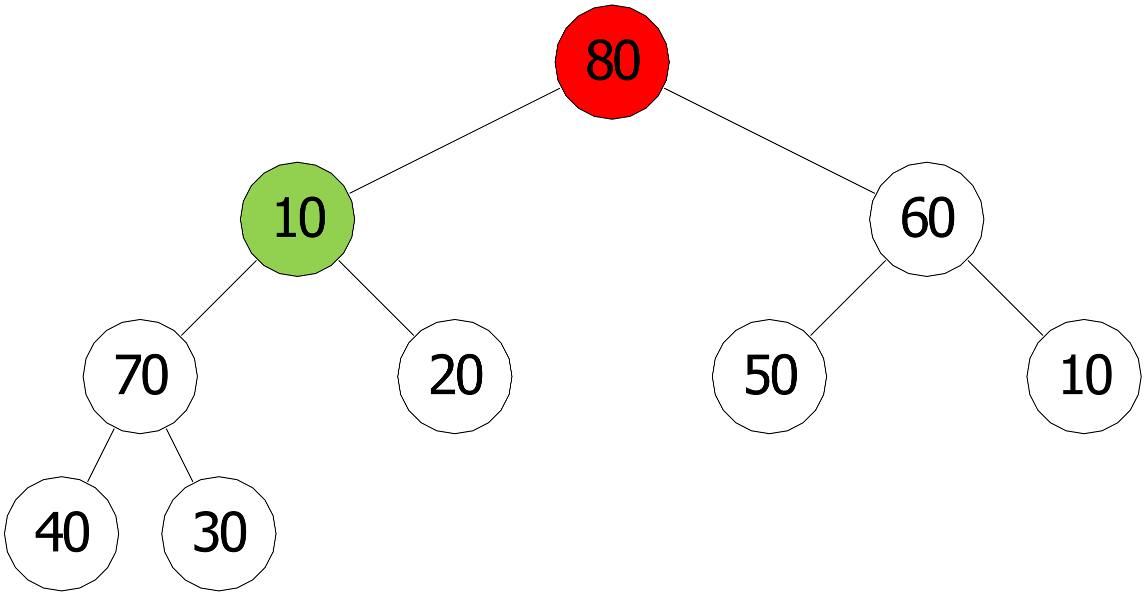
A	10	80	60	70	20	50	10	40	30



Heapify Example

A

80	10	60	70	20	50	10	40	30



Heapify Example

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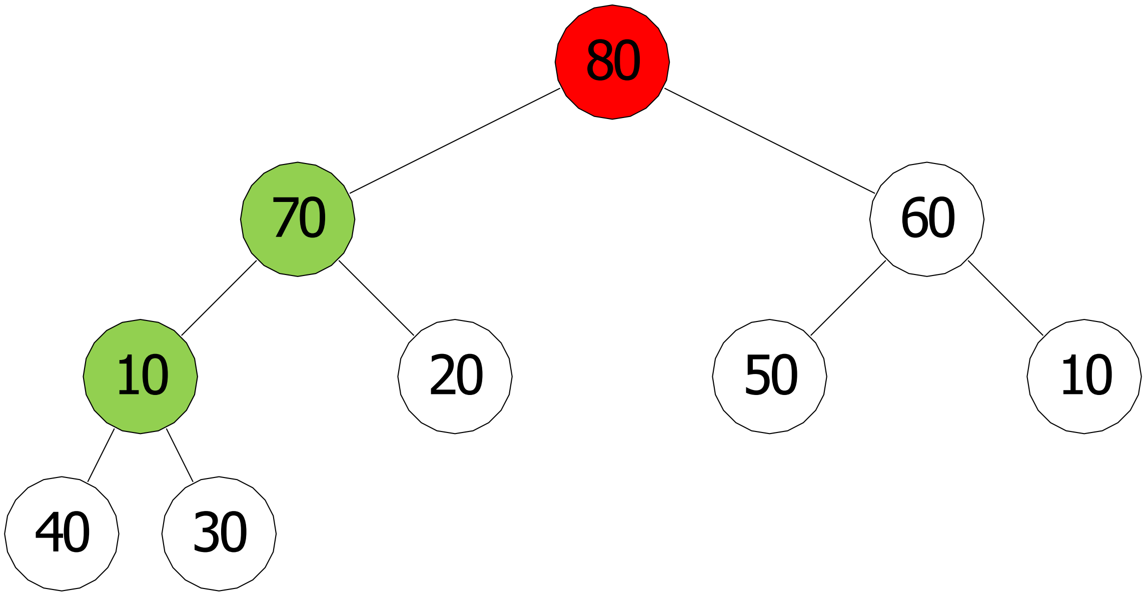
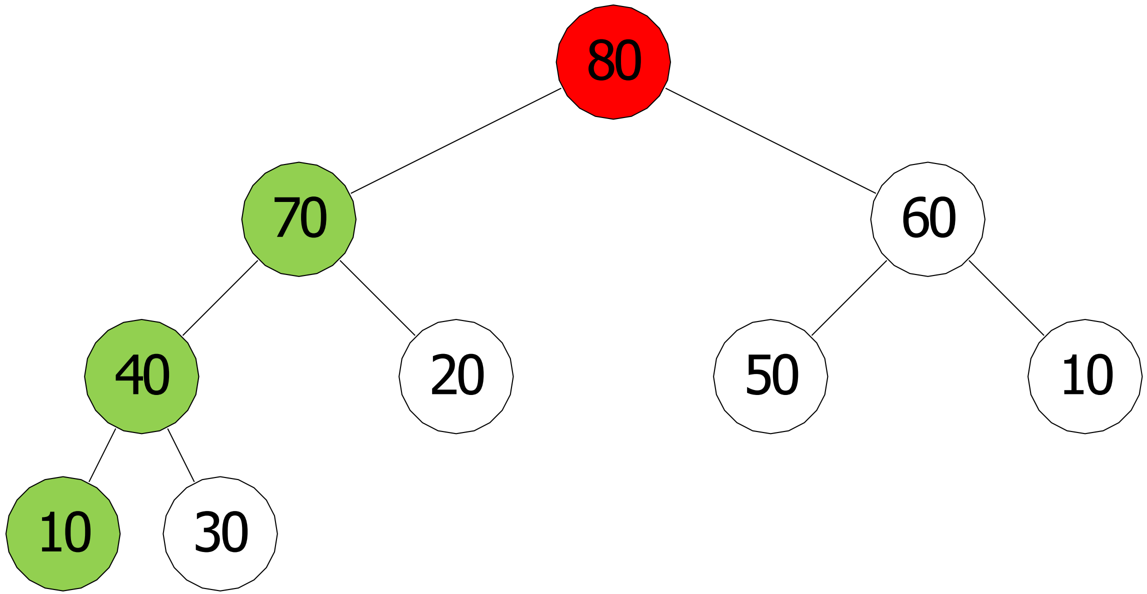


Heapify Example

A									
	80	70	60	40	20	50	10	10	30



done!

Heapify Pseudocode

heapify (A)

A : an array

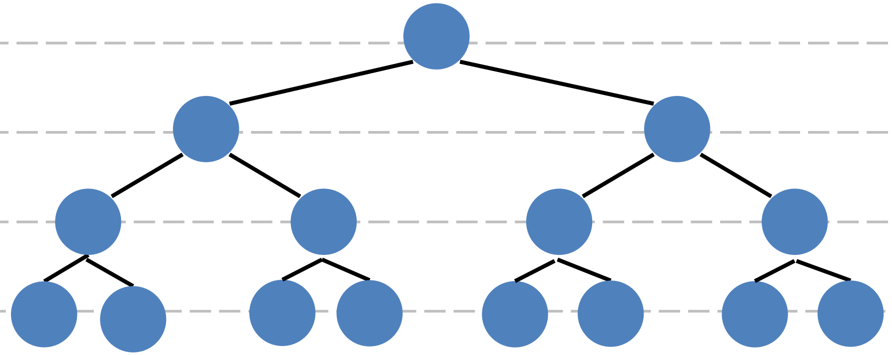
    for $i \leftarrow$ *parent* $($*last*$())$ **downto** 0 **do**

        fix-down (A, i, n)

- Straightforward analysis yields complexity $O(n \log n)$
- Careful analysis yields complexity $\Theta(n)$
- A heap can be built in linear time if we know all items in advance

Heapify Analysis

depth	nodes	work per node
0	2^0	h
1	2^1	$h-1$
i	2^i	$h-i$
$h-1$	2^{h-1}	1



$$\sum_{i=0}^{h-1} 2^i(h-i) = 2^h \sum_{i=0}^{h-1} \frac{2^i(h-i)}{2^h} = 2^h \sum_{i=0}^{h-1} \frac{(h-i)}{2^{h-i}}$$

$$= 2^h\left(\frac{h}{2^h} + \frac{h-1}{2^{h-1}} + \cdots + \frac{1}{2^1}\right)$$

$h \le \log n$

$$= 2^h \underbrace{\sum_{i=1}^{h} \frac{i}{2^i}}_{\text{convergent series}} \le 2^h c \le 2^{\log n} c = cn$$

$$\lim_{i \to \infty} \frac{2^i(i+1)}{i2^{i+1}} = \frac{1}{2}$$

HeapSort

	30	54	15	17	5	32	6

heapify

$n = 7$	54	30	32	17	5	15	6

swap root and heap end

$n = 7$	6	30	32	17	5	15	54

decrease n

$n = 6$	6	30	32	17	5	15	54

fix-down(root)

$n = 6$	32	30	15	17	5	6	54

HeapSort

$n = 6$	32	30	15	17	5	6	54
$n = 5$	30	17	15	6	5	32	54
$n = 4$	17	6	15	5	30	32	54
$n = 3$	15	6	5	17	30	32	54
$n = 2$	6	5	15	17	30	32	54
$n = 1$	5	6	15	17	30	32	54

swap root and heap end, decrease n and fix-down(root)

swap root and heap end, decrease n and fix-down(root)

swap root and heap end, decrease n and fix-down(root)

swap root and heap end, decrease n and fix-down(root)

swap root and heap end, decrease n and fix-down(root)

Sorted!

HeapSort

HeapSort(A)
$n \leftarrow A$.size()
for $i \leftarrow$ *parent* (*last*()) **downto** 0 **do**
fix-down (A, i, n)
while $n > 1$
swap items $A[root()]$ and $A[last()]$
decrease n
fix-down$(A, root(), n)$

heapify
$\Theta(n)$

$\Theta(n \log n)$

- Total time is $\Theta(n \log n)$
- Similar to *PQ-Sort* with heaps, but uses input array A for storing heap
- In-place, i.e. only $O(1)$ extra space

Heap Summary

- Binary heap: binary tree that satisfies structural property and heap order property
- Heaps are one possible realization of ADT PriorityQueue
    - *insert* takes $O(\log n)$ time
    - *deleteMax* takes $O(\log n)$ time
    - also supports *findMax* in $O(1)$ time
- A binary heap can be built in linear time, if all elements are known beforehand
- With binary heaps have an in-place sorting algorithm with $O(n \log n)$ worst case time
- We have seen max-oriented version of heaps
- There exists a symmetric min-oriented version supporting *insert* and *deleteMin* with same run times

Outline

- Priority Queues
  - Abstract Data Types
  - ADT Priority Queue
  - Binary Heaps
  - Operations in Binary Heaps
  - PQ-Sort and Heapsort
  - Intro for the Selection Problem

Selection

	0	1	2	3	4	5	6
	3	6	10	0	5	4	9
sorted	0	3	4	5	6	9	10

- **Select(k) problem** find item that would be in $A[k]$ if A was sorted nondecreasing
    - example: select(3) = 5
- **Solution 1**
    - make $k+1$ passes through A, deleting minimum each time
    - $\Theta(kn)$ time
    - $k = n/2$, time complexity is $\Theta(n^2)$
        - efficient solution is harder to obtain if k is a median
- **Solution 2**
    - sort A and return $A[k]$
    - $\Theta(n \log n)$
    - time does not depend on k

Selection

0	1	2	3	4	5	6	7	8	9
3	6	10	0	5	4	9	2	1	7

- **Solution 3**
  - make A into a min-heap by calling *heapify*(A)
    - $\Theta(n)$ time
  - call *deleteMin*(A) $k+1$ times
  - $\Theta(n + k\log n)$
  - if $k = n/2$, this solution is $\Theta(n\log n)$
    - can we do better?