

# CS 240 – Data Structures and Data Management

## Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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# Outline

- Lower bound for search
- Interpolation Search
- Tries
  - Standard Trie
  - Pruned Trie
  - Compressed Trie
  - Multiway Trie

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- Lower bound for search
- Interpolation Search
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# Dictionary ADT: Implementations Thus Far

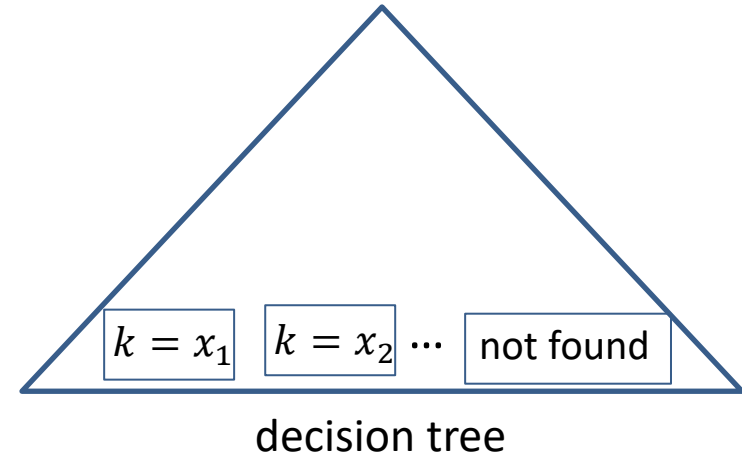
- Search is  $\Theta(\log n)$  in fastest implementations of dictionary ADT
  - $n$  is the number of items stored
- Search is  $\Omega(\log n)$  in all realizations of ADT we know
- **Question:** Can we do better than  $\Theta(\log n)$  search?
- **Answer:** *It depends on what we allow*
  - No: comparison-based searching lower bound is  $\Omega(\log n)$
  - Yes: non-comparison based searching can achieve  $o(\log n)$ 
    - keys have special properties
      1. interpolation search: keys have special distribution
      2. tries: keys are strings

# Lower Bound For Search

**Theorem:**  $\Omega(\log n)$  comparisons required for search in comparison based model

**Proof:**

- Let algorithm  $A$  search for key for  $k$  among  $n$  items  $x_1, x_2, \dots, x_n$
- There is a corresponding binary decision tree
- Chose a set of distinct keys  $S = \{x_1, x_2, \dots, x_n\}$
- Consider  $n + 1$  instances of search problem
  - search  $S$  for  $k = x_1$
  - search  $S$  for  $k = x_2$
  - ...
  - search  $S$  for  $k = x_n$
  - search  $S$  for  $k$  different from keys in  $S$
- Decision tree **must** have one leaf for each instance above
- Decision tree must have at least  $(n + 1)$  leaves
- Binary tree of height  $h$  has at most  $2^h$  leaves
- Thus  $2^h \geq n + 1$
- Taking log of both sides,  $h \geq \log(n + 1)$



# Outline

- Lower bound for search
- **Interpolation Search**
- Tries
  - Standard Trie
  - Pruned Trie
  - Compressed Trie
  - Multiway Trie

# Binary Search on Ordered Array

- insert and delete:  $\Theta(n)$ , search is  $\Theta(\log n)$

*Binary-search*( $A, n, k$ )

$A$ : Array of size  $n$ ,  $k$ : key

$l \leftarrow 0$

$r \leftarrow n - 1$

**while** ( $l \leq r$ )

$m \leftarrow \left\lfloor \frac{l+r}{2} \right\rfloor$

**if** ( $k = A[m]$ ) **return** “found at  $A[m]$ ”

**else if** ( $A[m] < k$ ) // key cannot be in the left part of  $A$

$l \leftarrow m + 1$

**else**  $r \leftarrow m - 1$  // key cannot be in the right part of  $A$

**return** “not found but would be between  $A[l - 1]$  and  $A[l]$ ”

# Interpolation Search: Motivation

- binary search looks at index

middle

$$\left\lfloor \frac{l+r}{2} \right\rfloor = l + \left\lfloor \frac{1}{2} (r-l-1) \right\rfloor$$

$l$		$r$
40		120



# Interpolation Search: Motivation

- binary search looks at index

middle

$$\left\lfloor \frac{l+r}{2} \right\rfloor = l + \left\lfloor \frac{1}{2} (r-l-1) \right\rfloor$$

$l$					$r$
40					120

- If keys are close to *evenly* distributed, where would key  $k = 100$  be?



- 100 should be much further away from  $A[l] = 40$  than from  $A[r] = 120$

# Interpolation Search: Motivation

- binary search looks at index

$$\begin{array}{c} \text{middle} \\ \left\lfloor \frac{l+r}{2} \right\rfloor = l + \left\lfloor \frac{1}{2}(r-l-1) \right\rfloor \\ \begin{array}{|c|c|c|c|} \hline l & & & r \\ \hline 40 & & & 120 \\ \hline \end{array} \end{array}$$

- If keys are close to *evenly* distributed, where would key  $k = 100$  be?

$$\begin{array}{c} l \qquad \qquad \qquad r \\ \begin{array}{|c|c|c|c|} \hline 40 & \xleftarrow{k - A[l] = 60} & | & | & 120 \\ \hline \end{array} \\ \underbrace{\hspace{15em}}_{A[r] - A[l] = 80} \end{array}$$

- 100 should be much closer to  $A[r] = 120$  than to  $A[l] = 40$
- fractional distance:**  $\frac{k-A[l]}{A[r]-A[l]} = 60/80 = \frac{3}{4}$  of the way between  $l$  and  $r$

- Interpolation search looks at index  $l + \left\lfloor \frac{k-A[l]}{A[r]-A[l]}(r-l-1) \right\rfloor$

# Interpolation Search Example

$$m = l + \left\lceil \frac{k - A[l]}{A[r] - A[l]} (r - l - 1) \right\rceil$$

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	448	449	600	800	1000	1200	1500
$l$										$r$

- *Search*(449), iteration 1

$$l = 0, r = n - 1 = 10, \quad m = 0 + \left\lceil \frac{449 - 0}{1500 - 0} (10 - 0 - 1) \right\rceil = 3$$

# Interpolation Search Example

$$m = l + \left\lceil \frac{k - A[l]}{A[r] - A[l]} (r - l - 1) \right\rceil$$

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	448	449	600	800	1000	1200	1500
				$l$						$r$

- *Search*(449), iteration 2

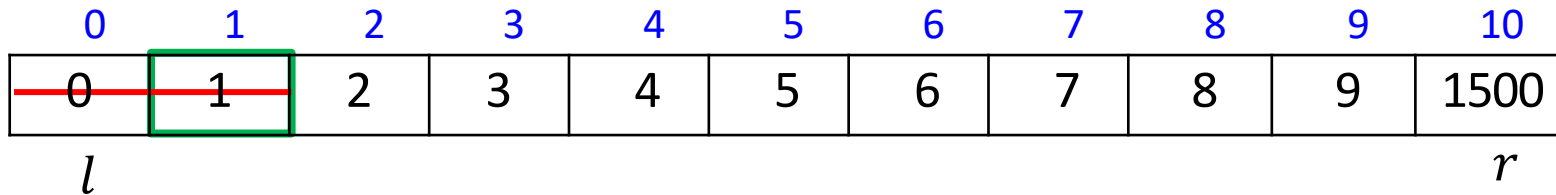
$$l = 4, r = 10,$$

$$m = 4 + \left\lceil \frac{449 - 498}{1500 - 498} (10 - 4 - 1) \right\rceil = 5$$

- Found!

# Interpolation Search

- Works well if keys are close to *evenly* distributed
- But worst case performance on unevenly distributed keys is  $\Theta(n)$
- Example: search(10)



- *Search*(10), iteration 1

$$l = 0, r = n - 1 = 10, \quad m = 0 + \left\lceil \frac{10 - 0}{1500 - 0} (10 - 0 - 1) \right\rceil = 1$$

# Interpolation Search

- Works well if keys are close to *evenly* distributed
- But worst case performance on unevenly distributed keys is  $\Theta(n)$
- Example: search(10)

0	1	2	3	4	5	6	7	8	9	10
0	1	<del>2</del>	<del>3</del>	4	5	6	7	8	9	1500
		$l$								$r$

- *Search*(10), iteration 2

$$l = 2, r = 10, \quad m = 2 + \left\lceil \frac{10 - 2}{1500 - 2} (10 - 2 - 1) \right\rceil = 3$$

# Interpolation Search

- Works well if keys are close to *evenly* distributed
- But worst case performance on unevenly distributed keys is  $\Theta(n)$
- Example: search(10)

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	1500
				$l$						$r$

- *Search*(10), iteration 3

$$l = 4, r = 10, \quad m = 4 + \left\lceil \frac{10 - 4}{1500 - 4} (10 - 4 - 1) \right\rceil = 5$$

- Will continue in steps of 2 at each iteration until reach the end of the array
  - $\Theta(n)$  runtime

# Interpolation Search

- Works well on *average*
  - can show (difficult):  $T^{avg}(n) \leq T^{avg}(\sqrt{n}) + \Theta(1)$ 
    - recurse into array of  $\sqrt{n}$  size, on average
  - resolves to  $T^{avg}(n) \in O(\log \log n)$
- Clever trick
  - use interpolation search for at most  $\log n$  steps
  - if key is still not found, switch to binary search
  - guarantees  $O(\log n)$  worst case, but could be  $O(\log \log n)$



# Interpolation Search

- Code similar to binary search, but compare at interpolated index
- Need extra test to avoid division by zero due to  $A[l] = A[r]$

*Interpolation-search*( $A, n, k$ )

*A*: Sorted array of size  $n$ ,  $k$ : key

$l \leftarrow 0, r \leftarrow n - 1$

**while** ( $l \leq r$ )

**if** ( $k < A[l]$  or  $k > A[r]$ ) **return** “not found”

**if** ( $k = A[r]$ ) **return** “found at  $A[r]$ ”

$m \leftarrow l + \left\lfloor \frac{k - A[l]}{A[r] - A[l]} (r - l - 1) \right\rfloor$

**if** ( $A[m] = k$ ) **return** “found at  $A[m]$ ”

**else if** ( $A[m] < k$ )

$l \leftarrow m + 1$

**elseif**  $r \leftarrow m - 1$

*// always return from inside the while loop*

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- **Tries**
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# Words: review

- **Scenario:** Keys in dictionary are words
- Words (=strings): sequence of characters over alphabet  $\Sigma$ 
  - $\{\text{be}, \text{bear}, \text{beer}\}$
- Typical alphabets:  $\{0,1\}$  (bitstrings), ASCII, etc.
- Stored in an array:  $w[i]$  gets  $i$ th character (for  $i = 0, 1, \dots$ )
- **Convention:** words have end-sentinel  $\$$  (sometimes not shown)
  - $\$$  is smaller than any other character and does not occur in  $\Sigma$
- $w.size = |w| =$  number of non-sentinel characters
  - $|\text{be}\$| = 2$
- Should know
  - prefix, suffix, substring
  - sorting of words lexicographically

$\text{be}\$ <_{\text{lex}} \text{bea}\mathbf{r}\$ \quad \text{bea}\mathbf{r}\$ <_{\text{lex}} \text{bee}\mathbf{r}\$$

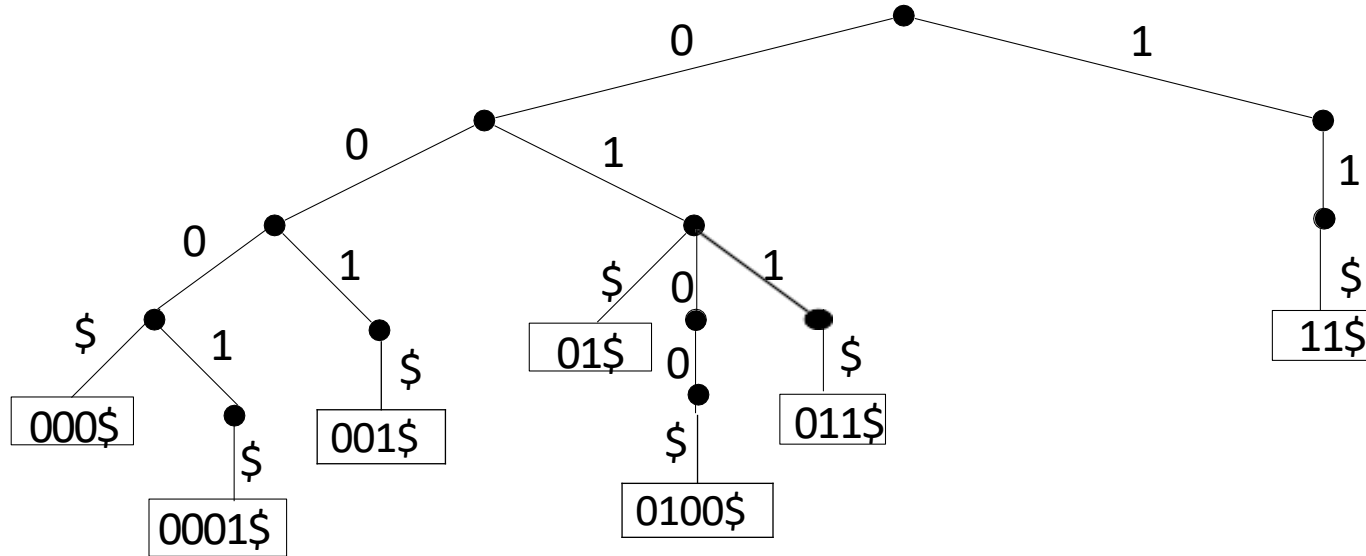
- this is different from sorting numbers

$\mathbf{0}10\$ <_{\text{lex}} \mathbf{1}\$$

# Tries: Introduction

- **Trie** (also known as **radix tree**): a dictionary for bit strings
  - comes from word **retrieval**, but pronounced “try”
- Trie vs. AVL tree
  - let the number of strings in dictionary be  $n$
  - Trie: insert, find, delete is  $O(|w|)$  time
    - independent of  $n$
  - AVL tree: insert, find delete is  $O(|w|\log(n))$  time
    - $O(\log(n))$  nodes on a path,  $O(|w|)$  operations at each node
- Trie applications
  - auto-completion
    - smart phones, commands for operating systems
  - spell checking
  - DNA sequencing

# Tries: Introduction

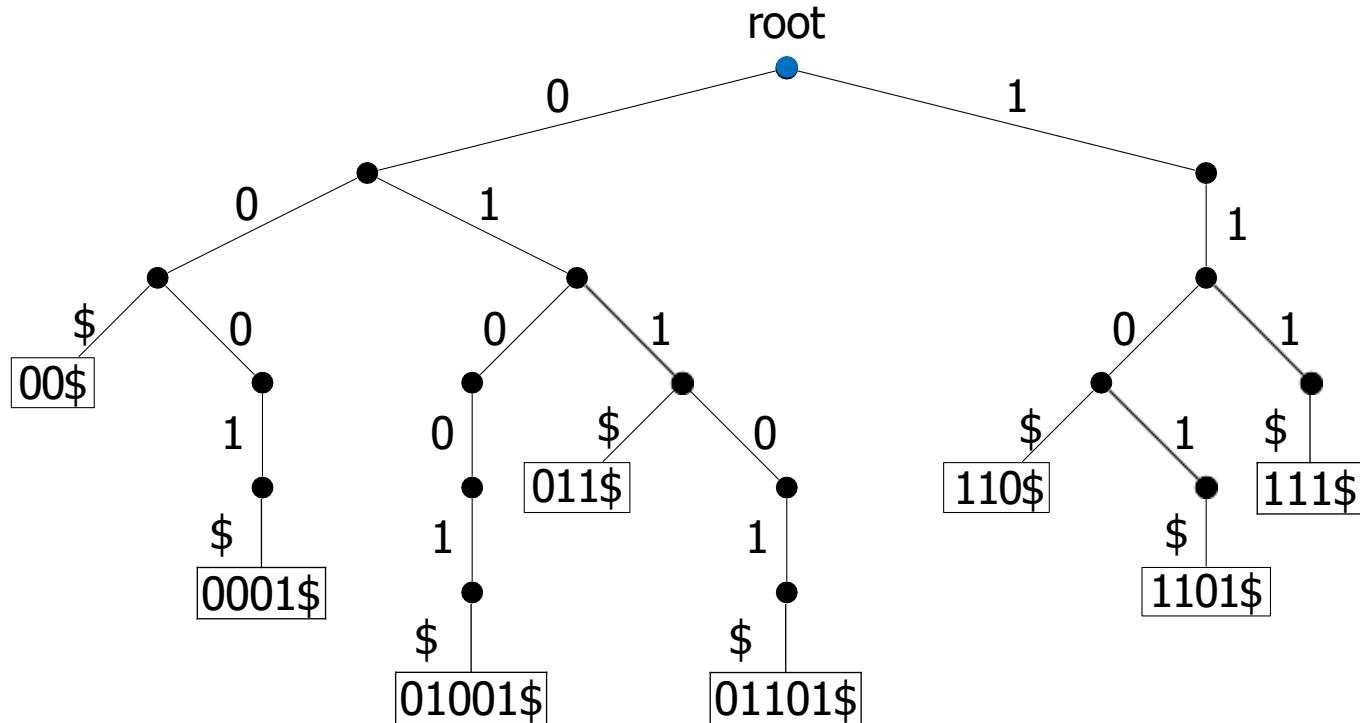


- Trie (radix tree): dictionary for bitstrings
  - tree based on bitwise comparisons
  - edges labelled with corresponding bit
  - store words by comparing edge labels and word bits
    - similar to radix sort: compare individual bits, not the whole key
  - due to end-sentinels \$, all key-value pairs are at leaves
  - $n$  is the number of words (strings) stored in the trie

# Tries: Search Example

Example: Search(011\$)

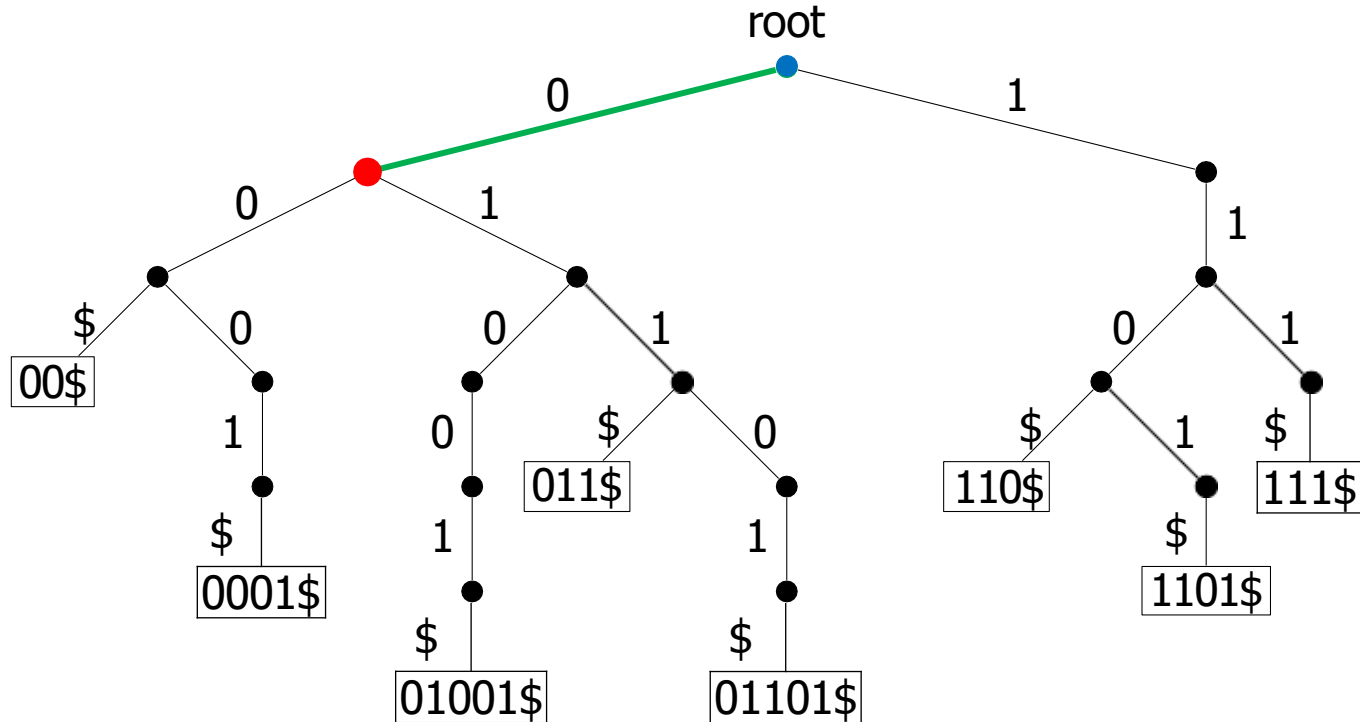
$P =$  ●



# Tries: Search Example

Example: Search(011\$)

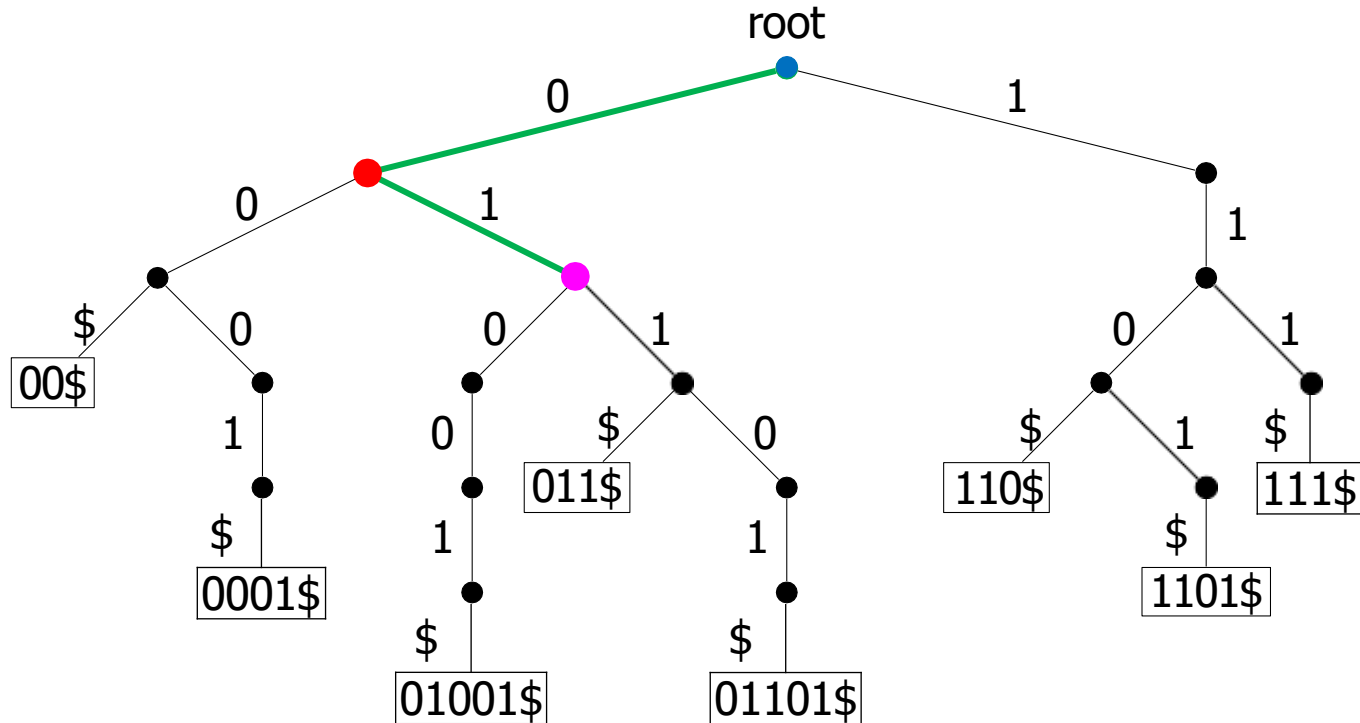
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●



# Tries: Search Example

Example: Search(011\$)

$P =$  ●  
●  
●

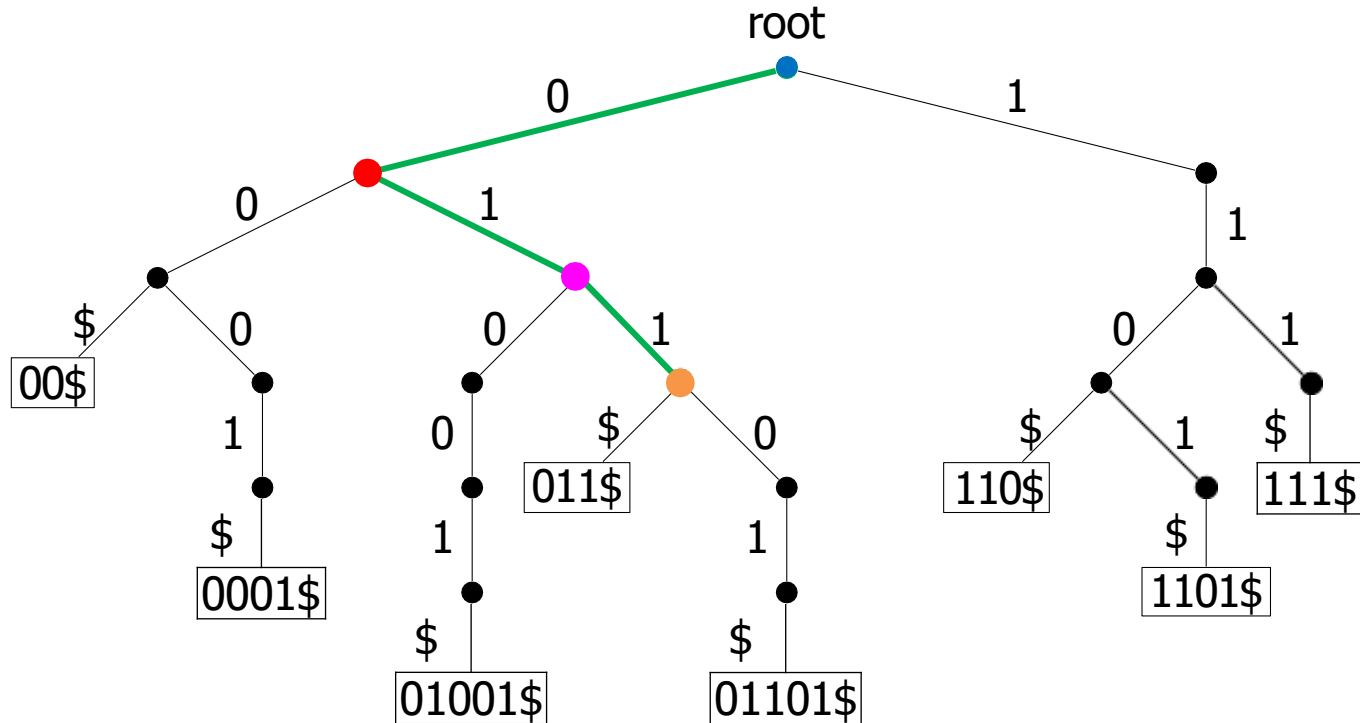




# Tries: Search Example

Example: Search(011\$)

$P =$  ●  
●  
●  
●



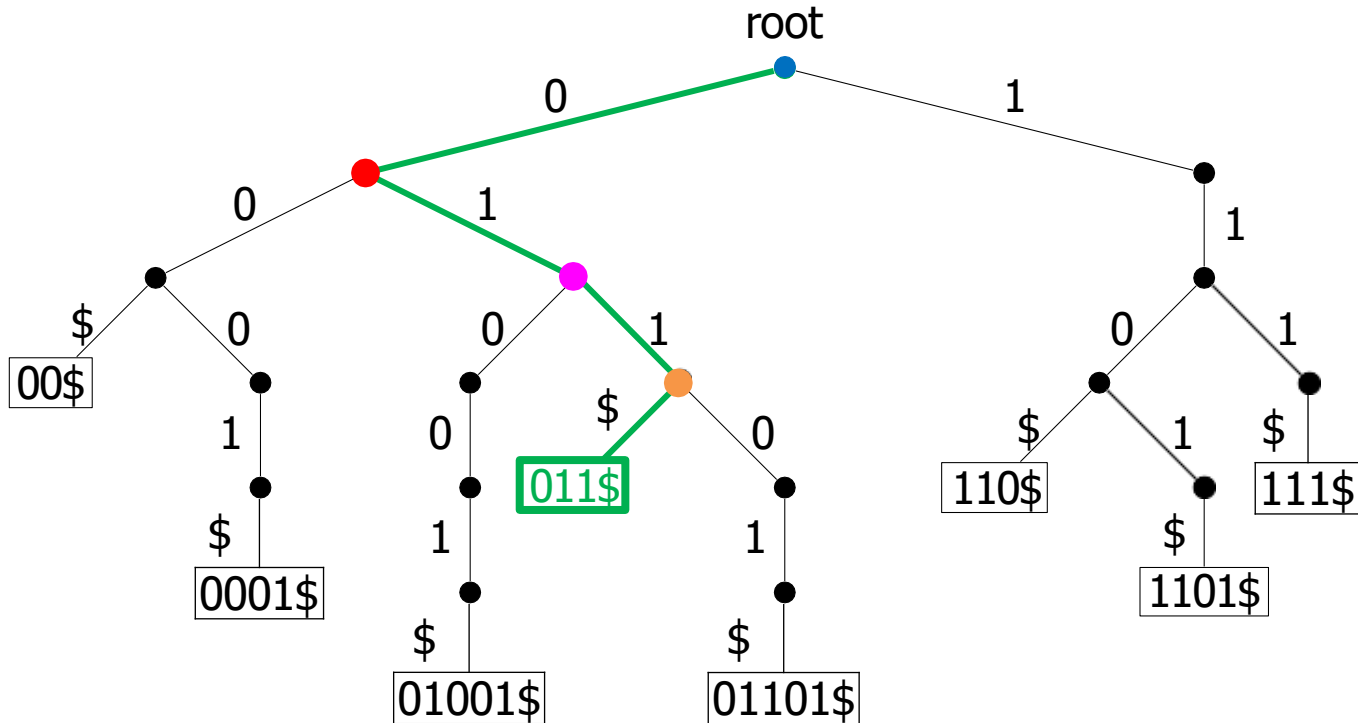
# Tries: Search Example

011\$

Example: Search(011\$) successful

$P =$

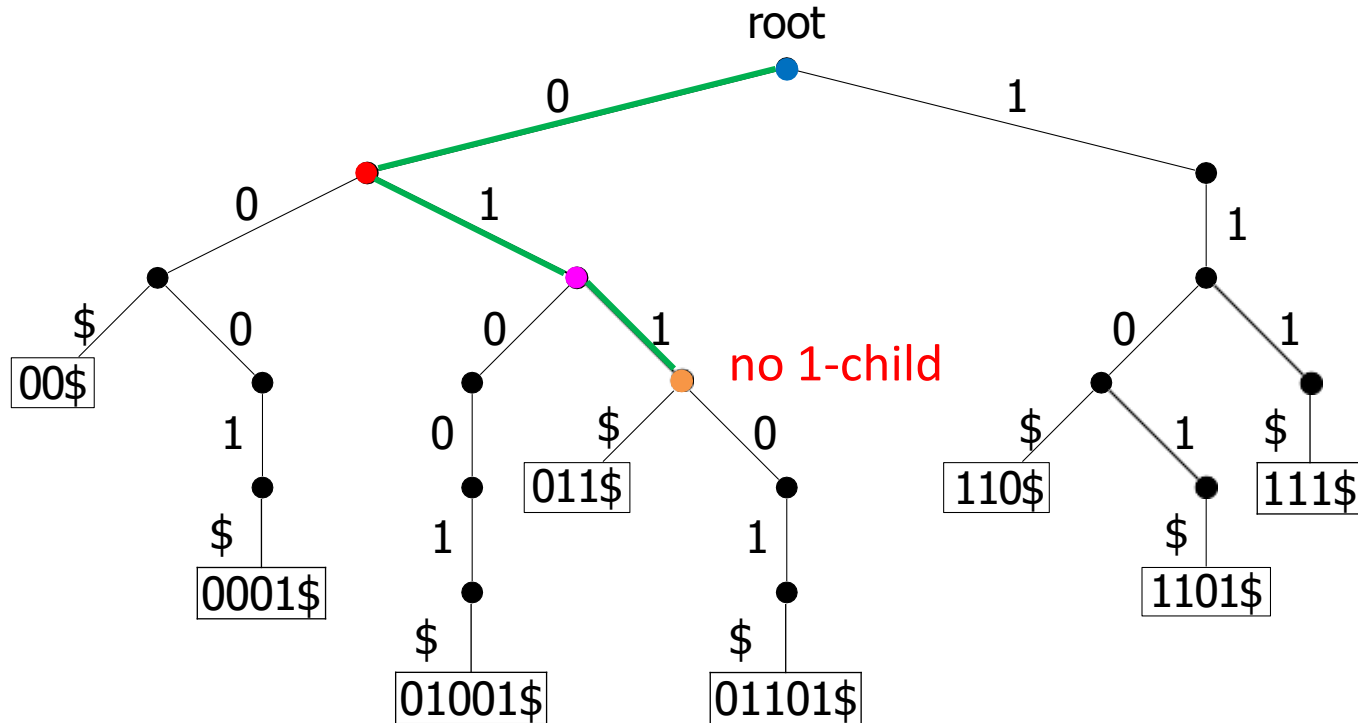
- 
- 
- 
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# Tries: Search Example

$P =$  ●  
●  
●  
●

Example: Search(0111\$) **unsuccessful**



# Tries: Search

- Follow links that correspond to current bits in  $w$
- Repeat until  $w$  is found or no such link

*Trie::get-path-to*( $w$ )

Output:  $w$  in the trie: stack holds path leading to and including  $w$

$w$  not in the trie: stack holds ancestors of where  $w$  would have been

$P \leftarrow$  empty stack;  $z \leftarrow$  root;  $d \leftarrow 0$ ;  $P.push(z)$

**while**  $d \leq |w|$

**if**  $z$  has a child-link labelled with  $w[d]$

$z \leftarrow$  child at this link;  $d++$ ;  $P.push(z)$

**else break**

**return**  $P$

*Trie::search*( $w$ )

$P \leftarrow$  *get-path-to*( $w$ );  $z \leftarrow P.top()$

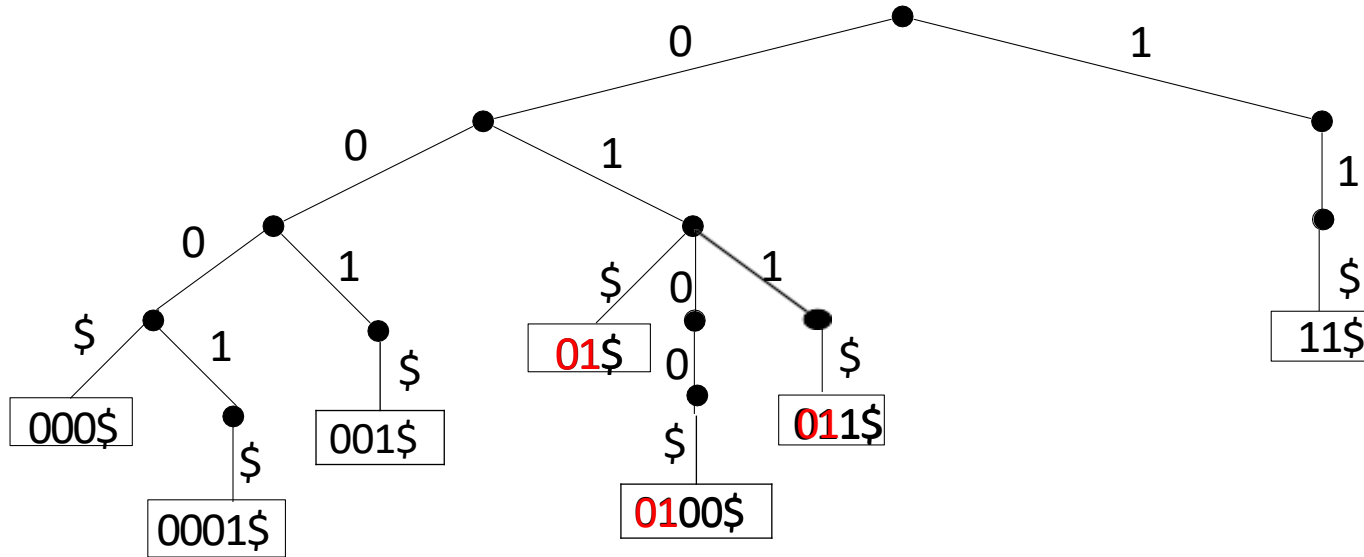
**if**  $z$  is not a leaf **then**

**return** “not found, would be in sub-trie of  $z$ ”

**return** key-value pair at  $z$

# Tries: Leaf-References

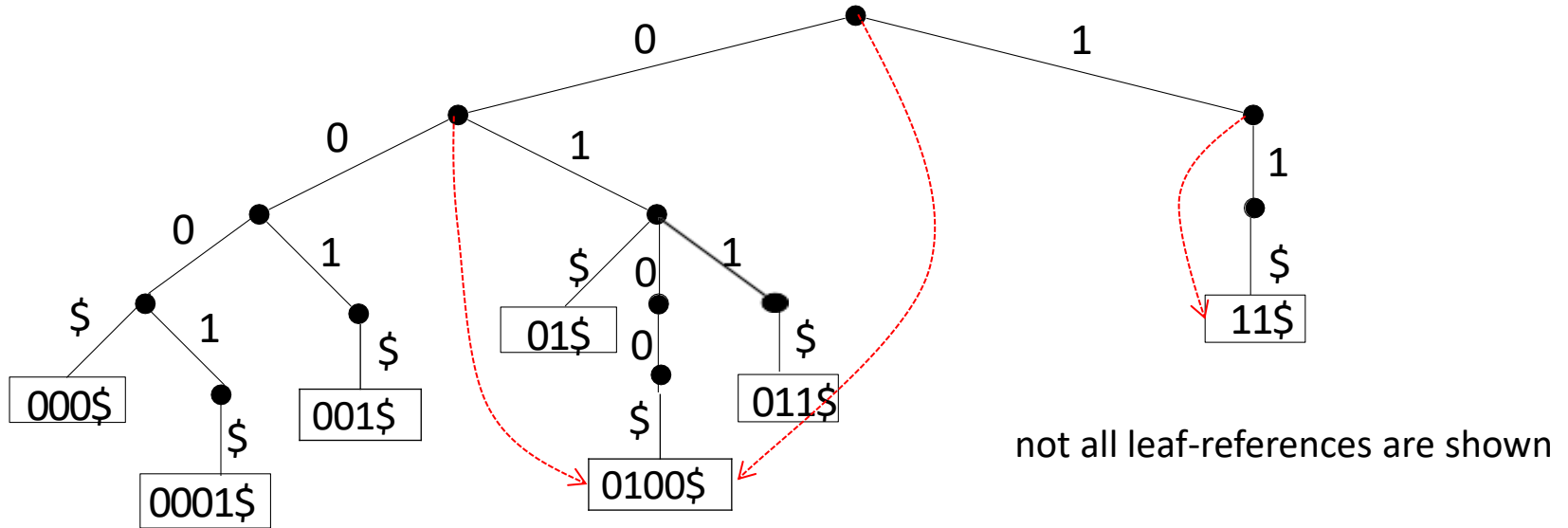
- For later applications of tries, want **prefix-search**( $w$ )
  - find word  $v$  in a trie for which  $w$  is a prefix
  - prefix  $w$  does not end with \$



**prefix-search**(01) can return: 01\$ or 0100\$ or 011\$

# Tries: Leaf-References

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  - find word  $v$  in a trie for which  $w$  is a prefix

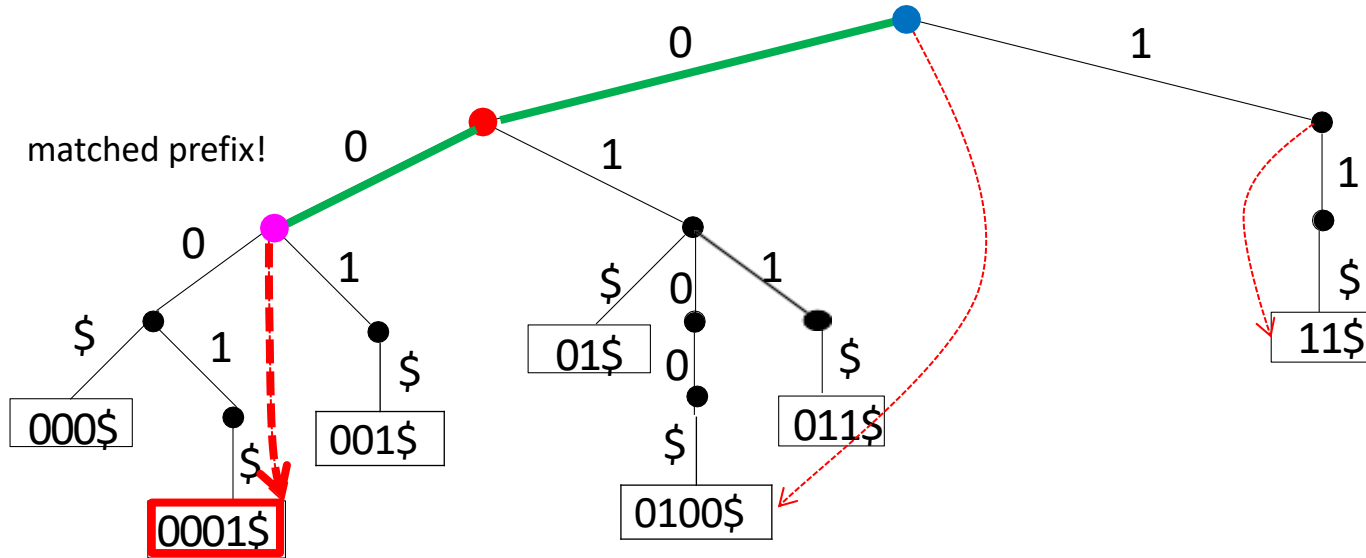


- To find  $v$  quickly, need **leaf-references**
- Convention: reference to leaf with longest word in the subtree
  - ties broken arbitrarily

# Tries: Leaf-References

- Example: `Trie::prefix-search(00)`

$P =$  ● ● ●  
 (magenta, red, blue dots)



- If match, stack size is larger by exactly 1 than size of prefix  $w$ 
  - 1 node for the root
  - 1 node for each character of  $w$

`Trie::prefix-search( $w$ )`

$P \leftarrow \text{get-path-to}(w)$

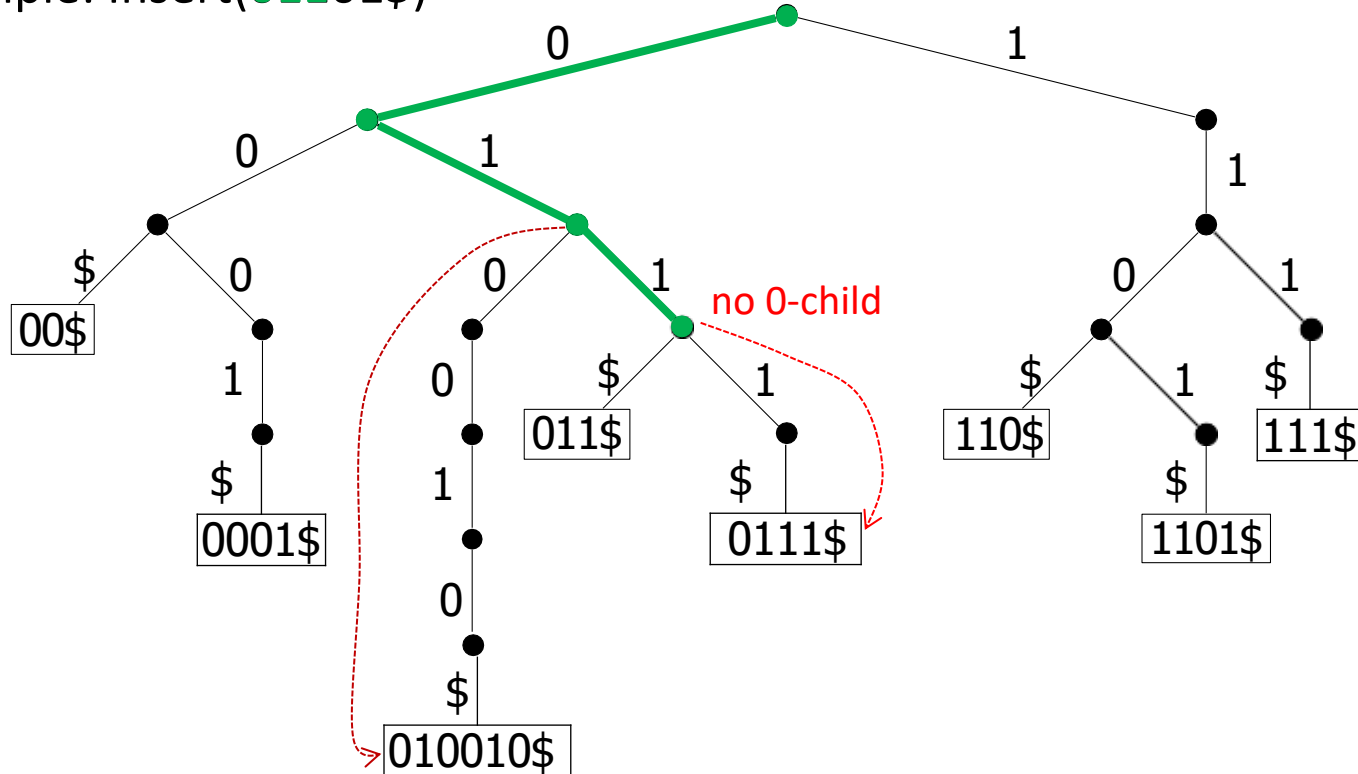
if number of nodes on  $P$  is  $w.size$  or less then  
 return "not string with prefix  $w$  found"

$p \leftarrow P.top()$

return  $p.leaf$

# Tries: Insert

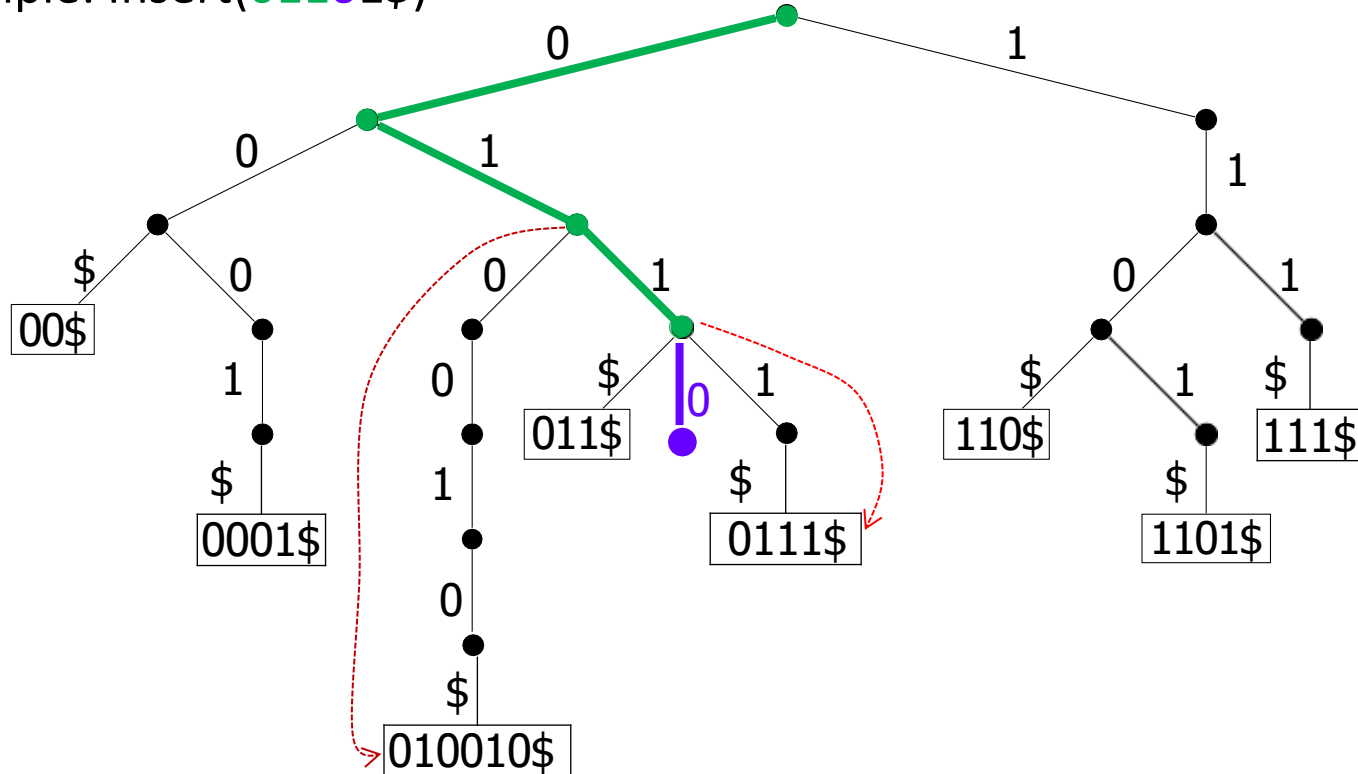
- $P \leftarrow \text{get-path-to}(w)$  gives ancestors that exist already
- Expand trie from  $p \leftarrow P.\text{top}()$  by adding nodes for the extra bits of  $w$
- Update leaf-references for new nodes and also for nodes in  $P$ 
  - $w$  could be longer than the leaves nodes in  $P$  currently point to
- Example: Insert(01101\$)





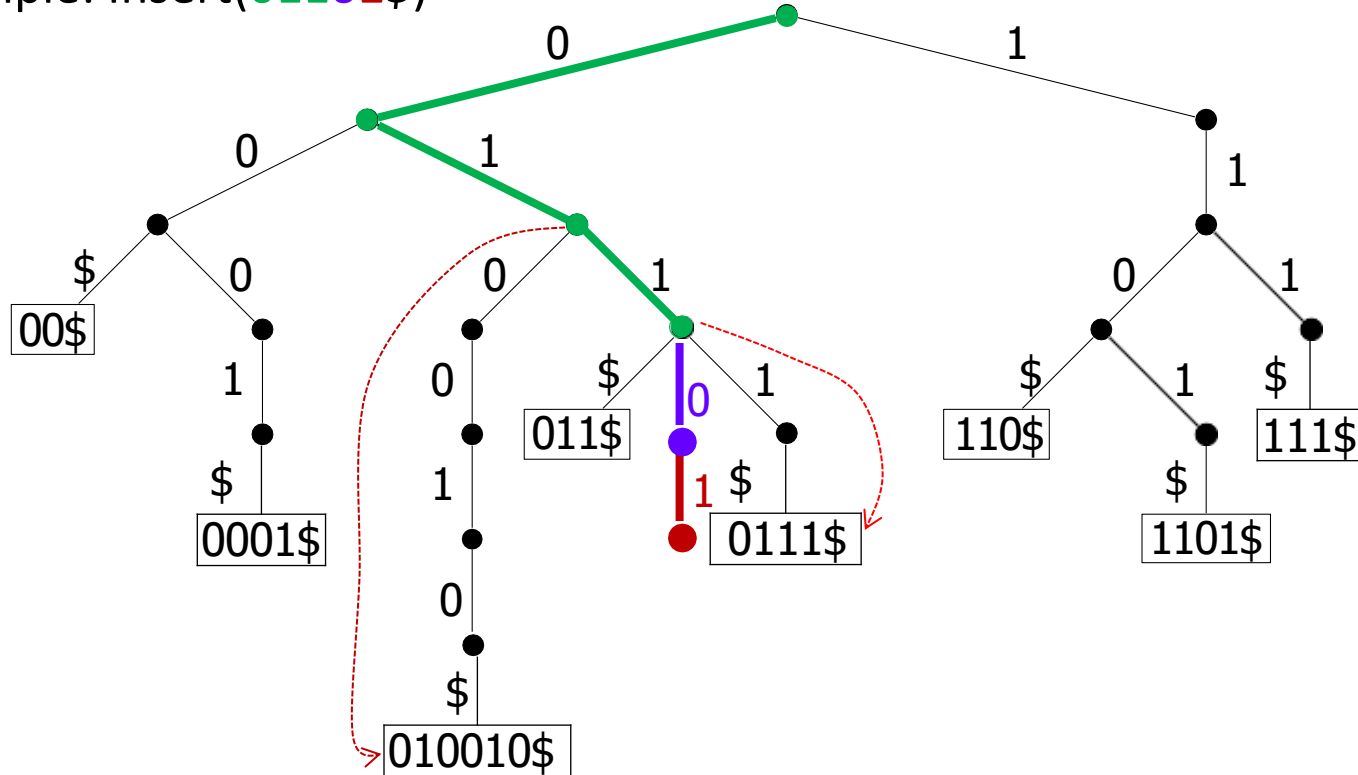
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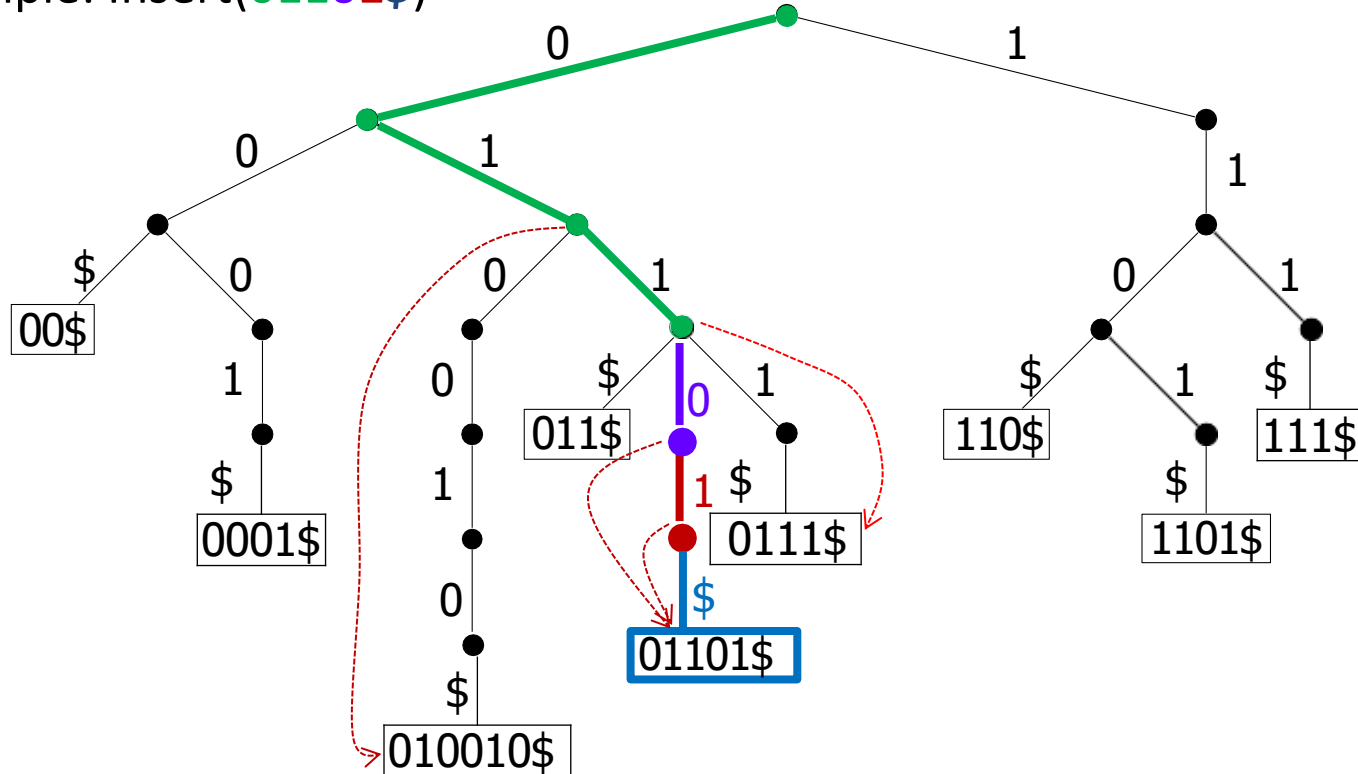
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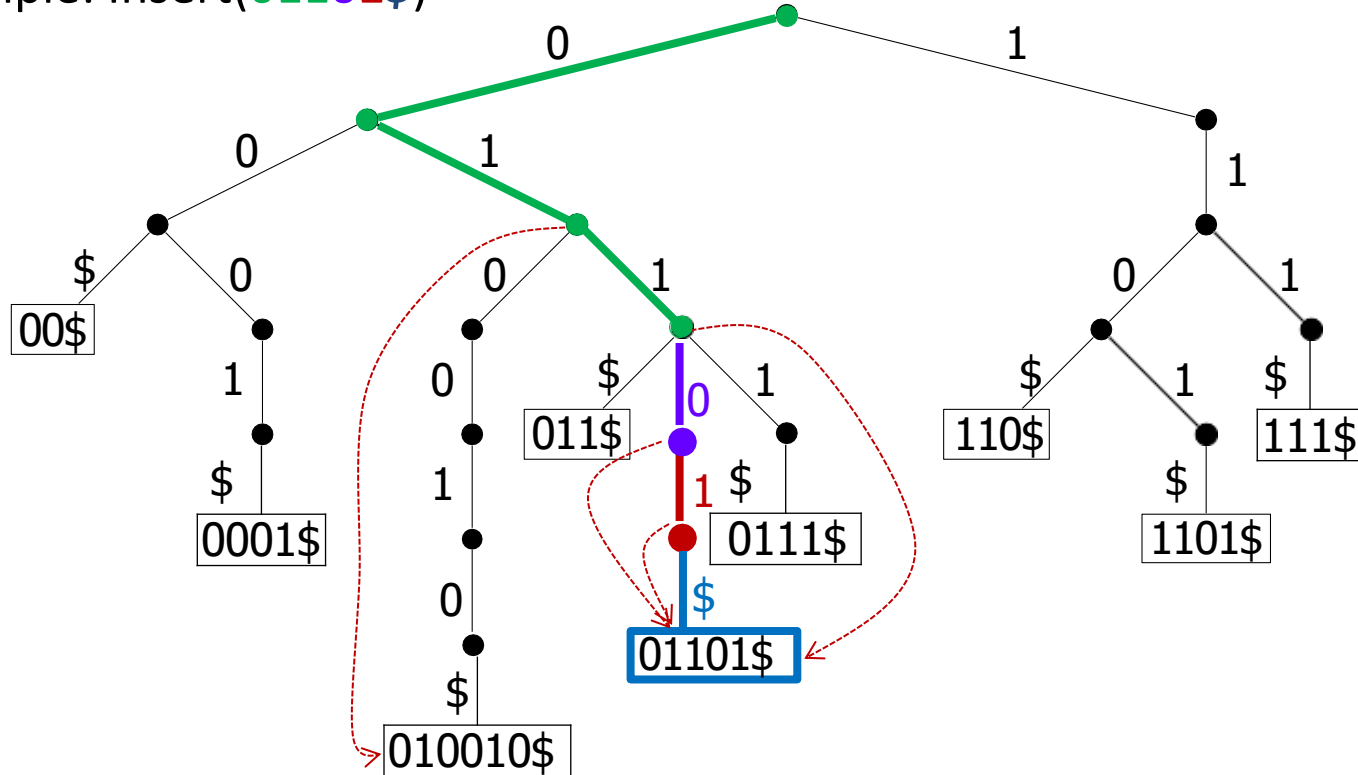
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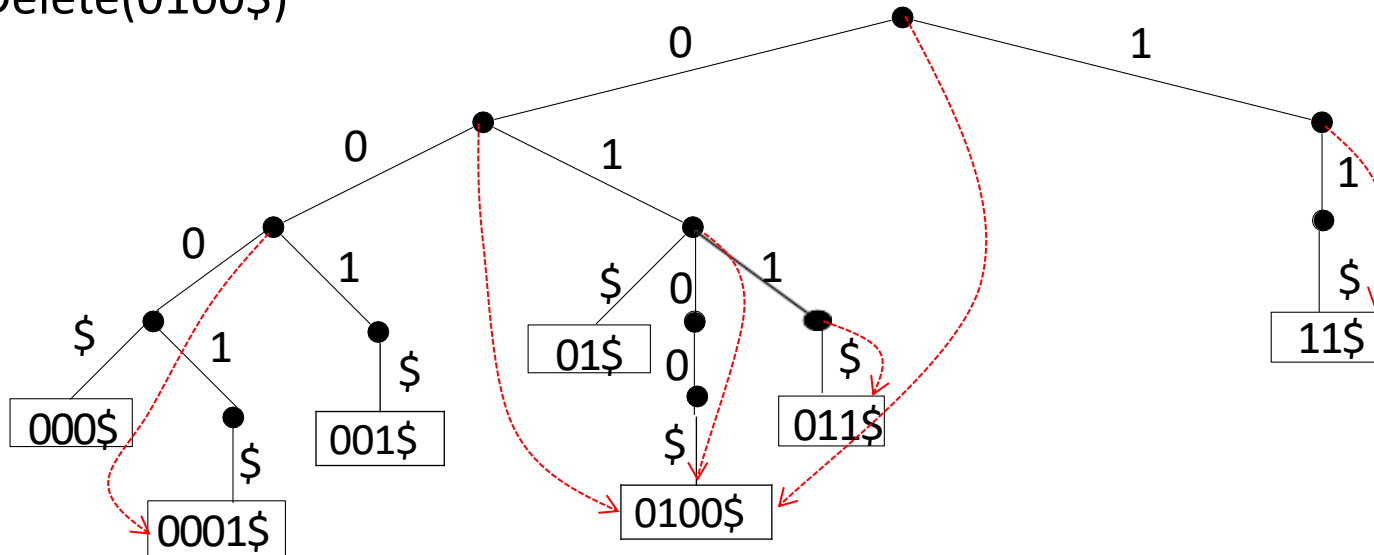
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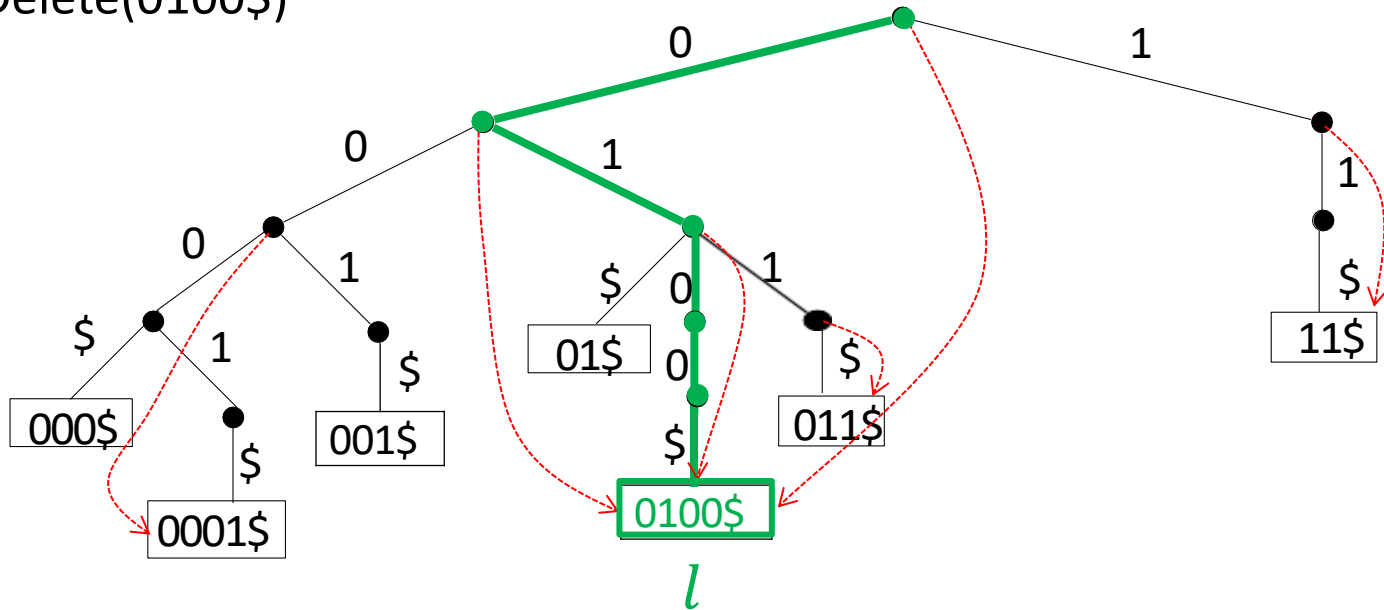
# Tries: Delete

- $P \leftarrow \text{get-path-to}(w)$  gives all ancestors
- Let  $l$  be the leaf where  $w$  is stored
- Delete  $l$  and nodes on  $P$  until ancestor has two or more children
- Update leaf-references on the rest of  $P$ 
  - if  $z \in P$  referred to  $l$ , find new  $z$ . leaf from current children of  $z$
- Delete(0100\$)



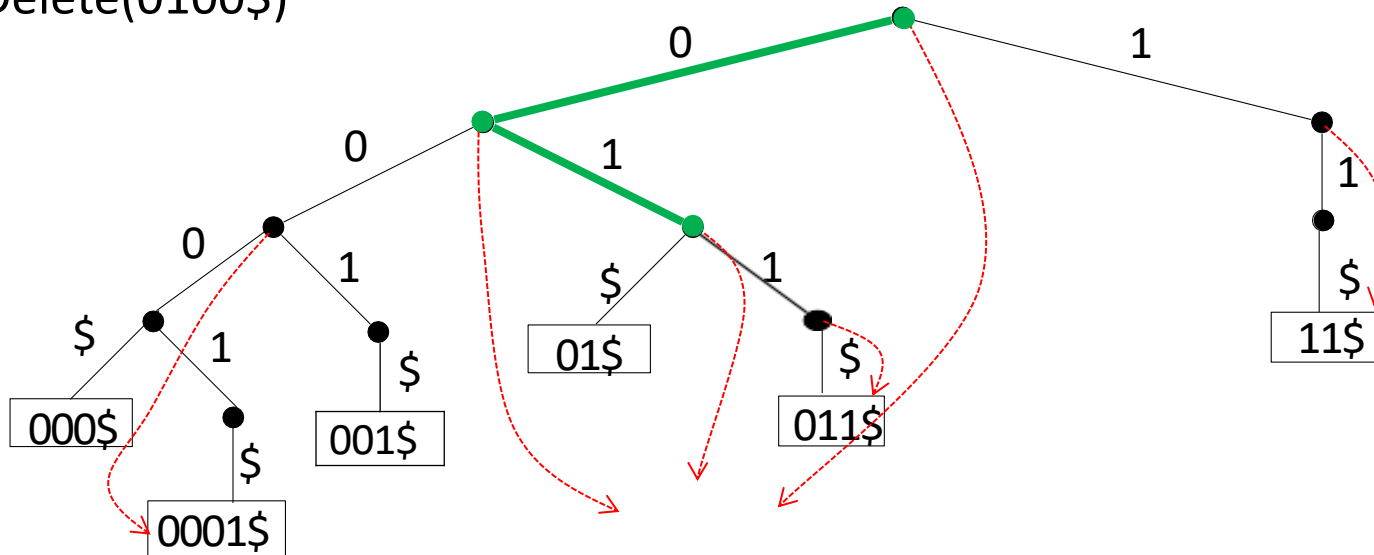
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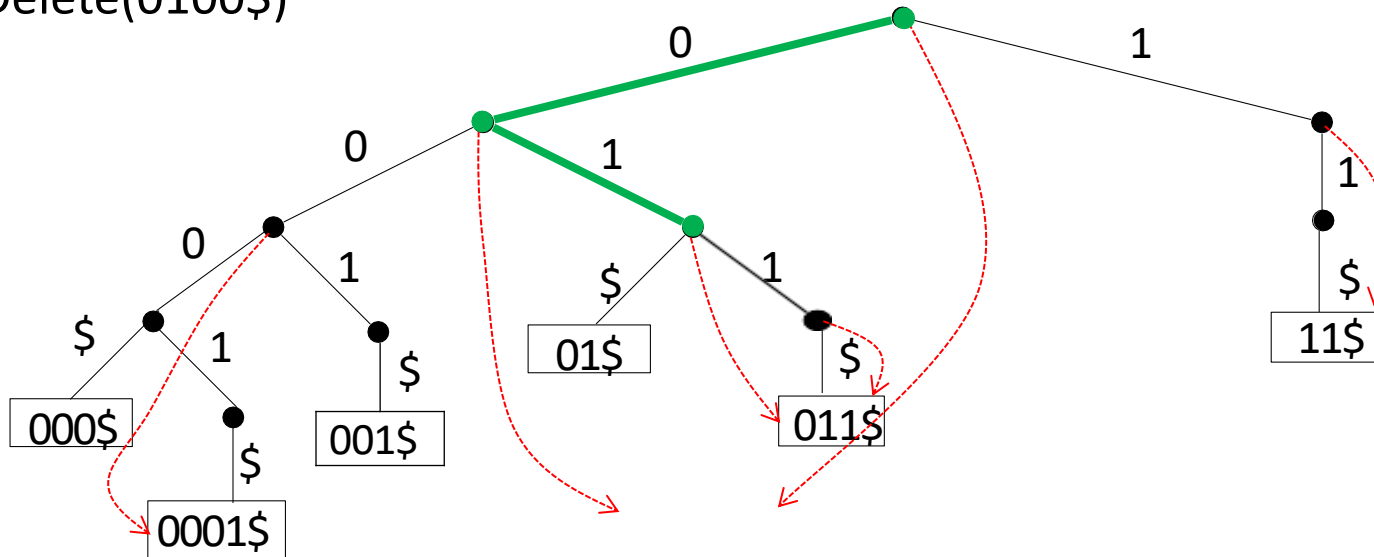
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# Tries: Delete

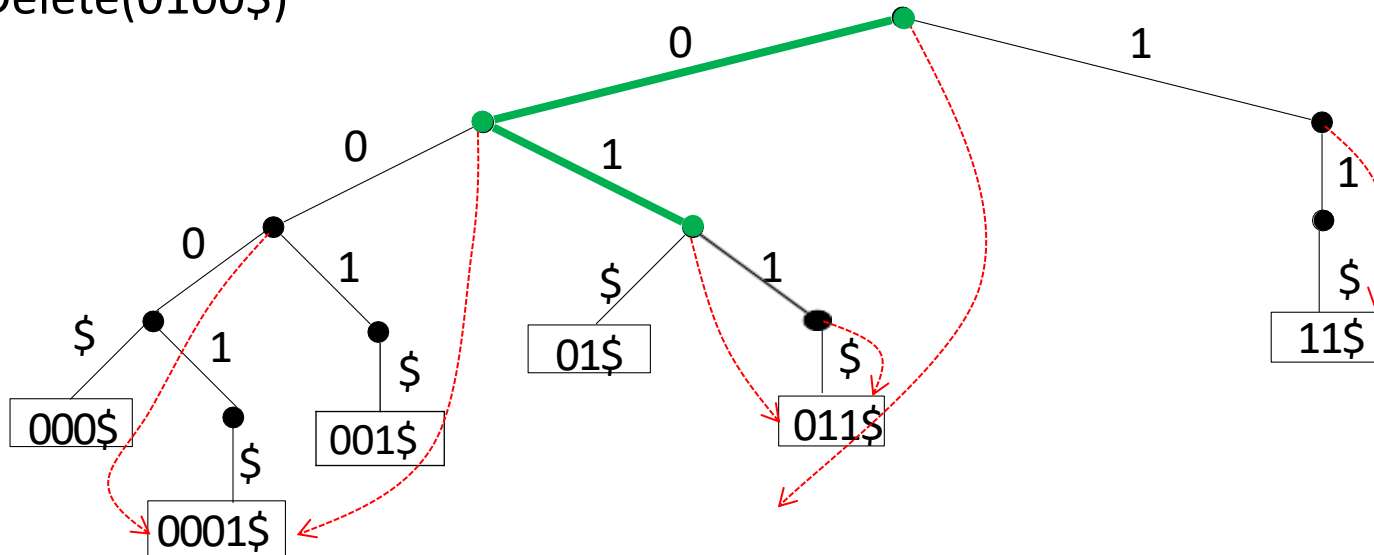
- $P \leftarrow \text{get-path-to}(w)$  gives all ancestors
- Let  $l$  be the leaf where  $w$  is stored
- Delete  $l$  and nodes on  $P$  until ancestor has two or more children
- Update leaf-references on the rest of  $P$ 
  - if  $z \in P$  referred to  $l$ , find new  $z$ . leaf from current children of  $z$
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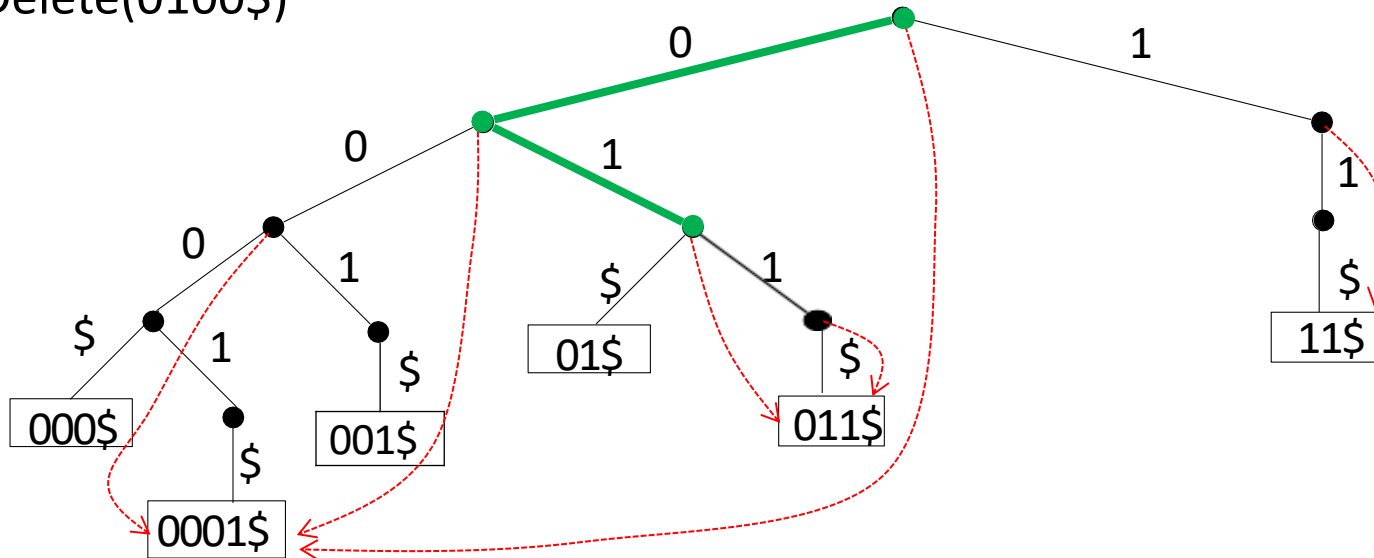
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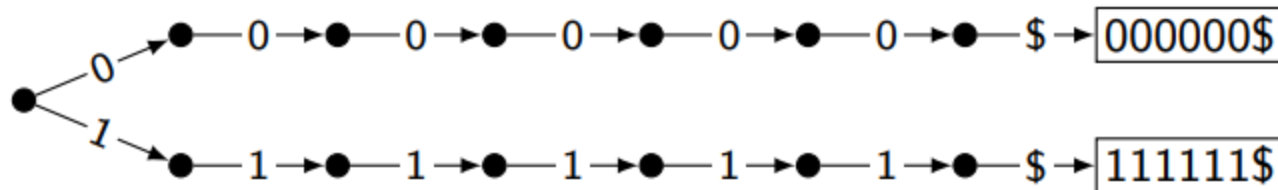
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# Standard Trie Summary

- `search(w)`, `prefix-search(w)`, `insert(w)`, `delete(w)` all take  $\Theta(|w|)$  time
  - time is independent of  $n$ , the number of words stored in the trie
  - time is small for short words
- Trie for a given set of words is unique
  - except for order of children and ties among leaf-references
- Disadvantages
  - can be wasteful with respect to space
    - the problem is 'chains'



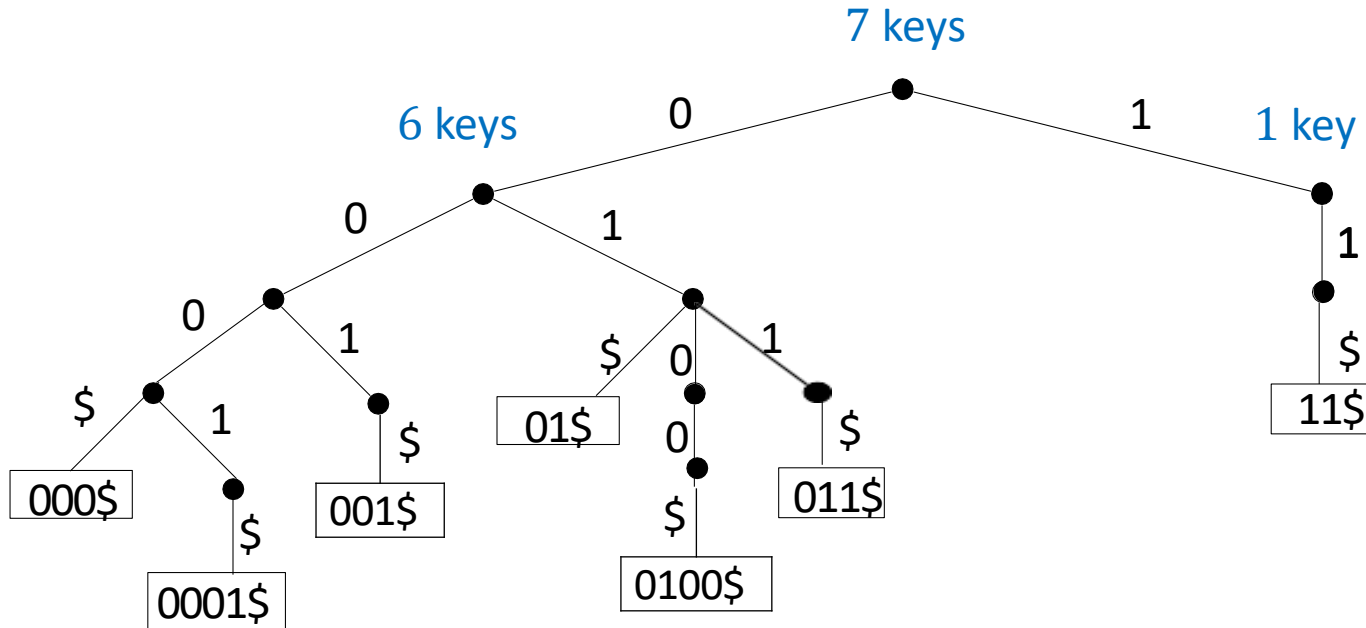
- Worst case space is  $\Theta(n \cdot \text{maximum word length})$
- How to save space?

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  - Compressed Trie
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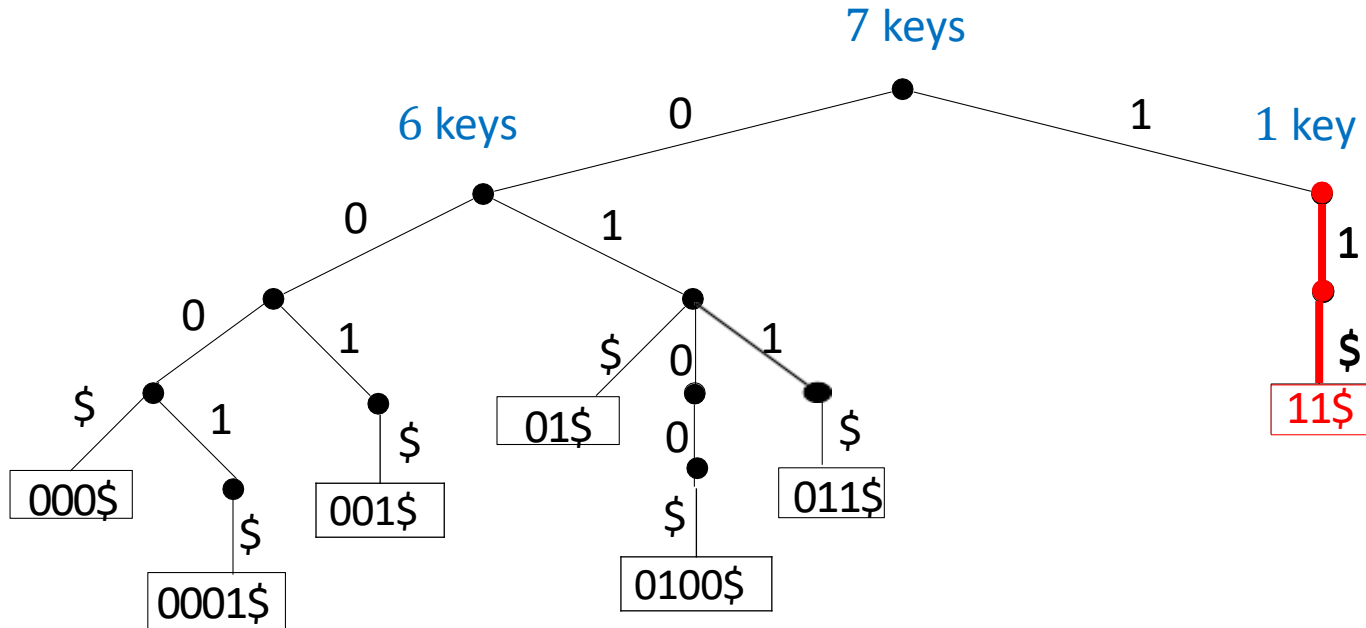
# Pruned Trie

- Sub-trie with one key has only one node
- Convert standard trie into pruned trie



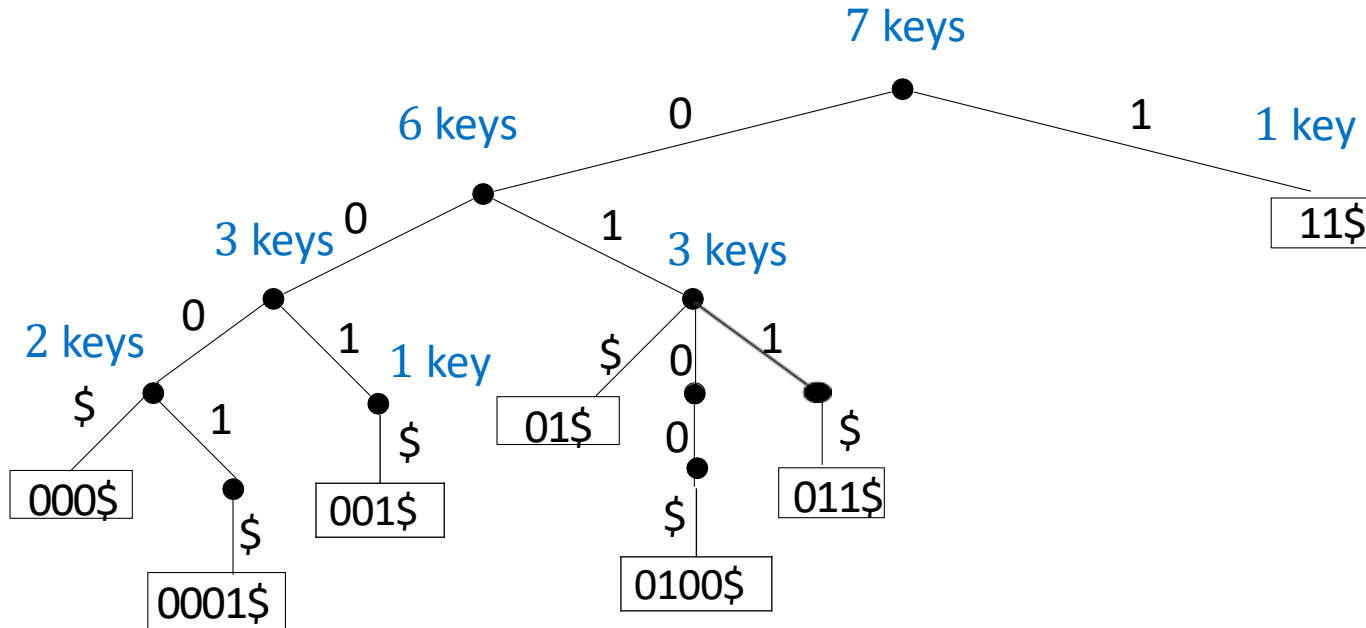
# Pruned Trie

- Sub-trie with one key has only one node
- Convert standard trie into pruned trie



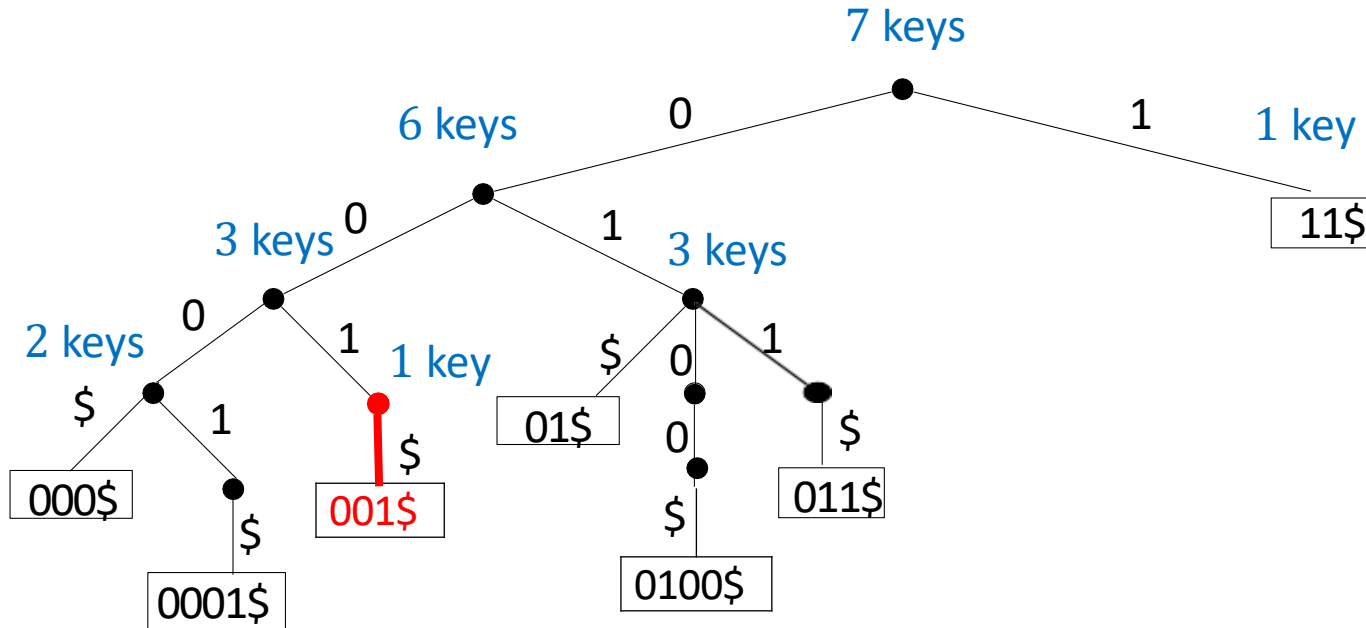
# Pruned Trie

- Sub-trie with one key has only one node
- Convert standard trie into pruned trie



# Pruned Trie

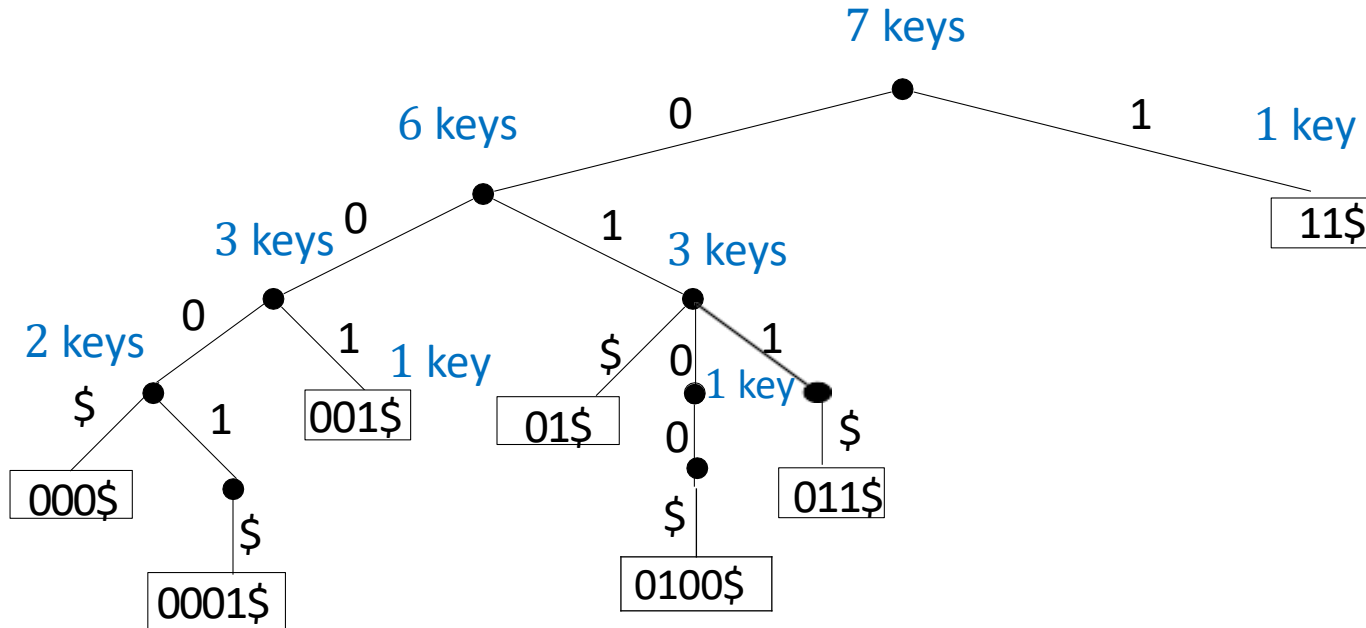
- Sub-trie with one key has only one node
- Convert standard trie into pruned trie





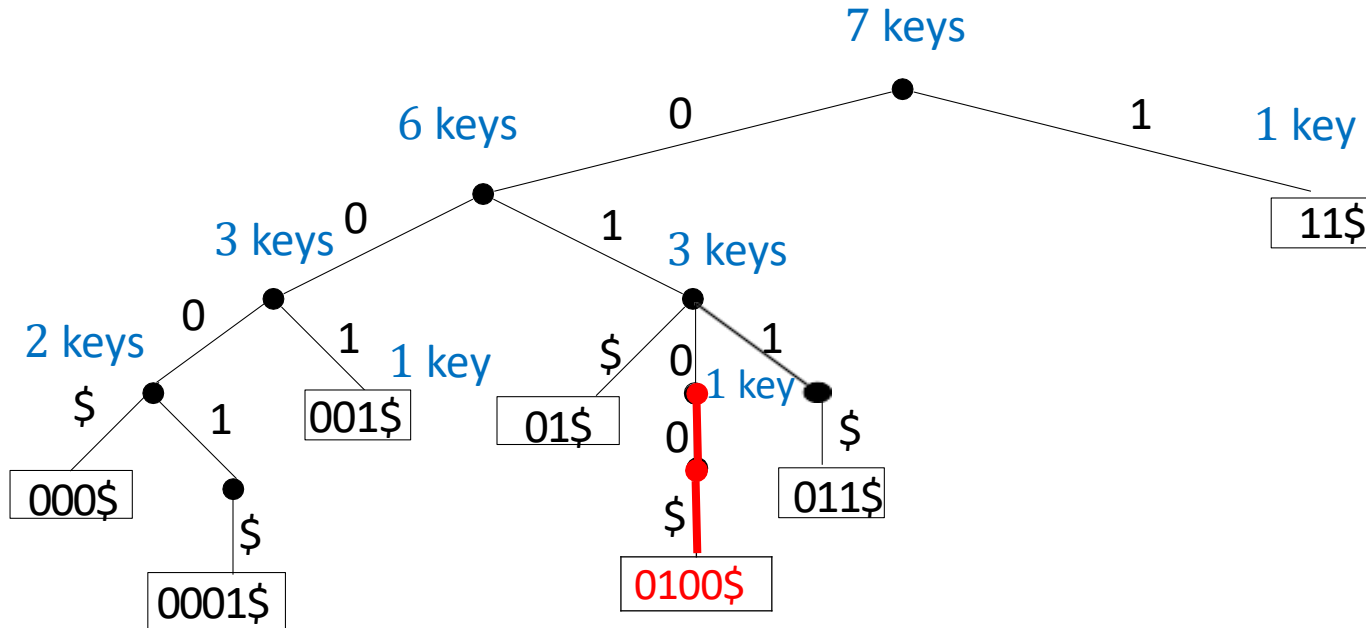
# Pruned Trie

- Sub-trie with one key has only one node
- Convert standard trie into pruned trie



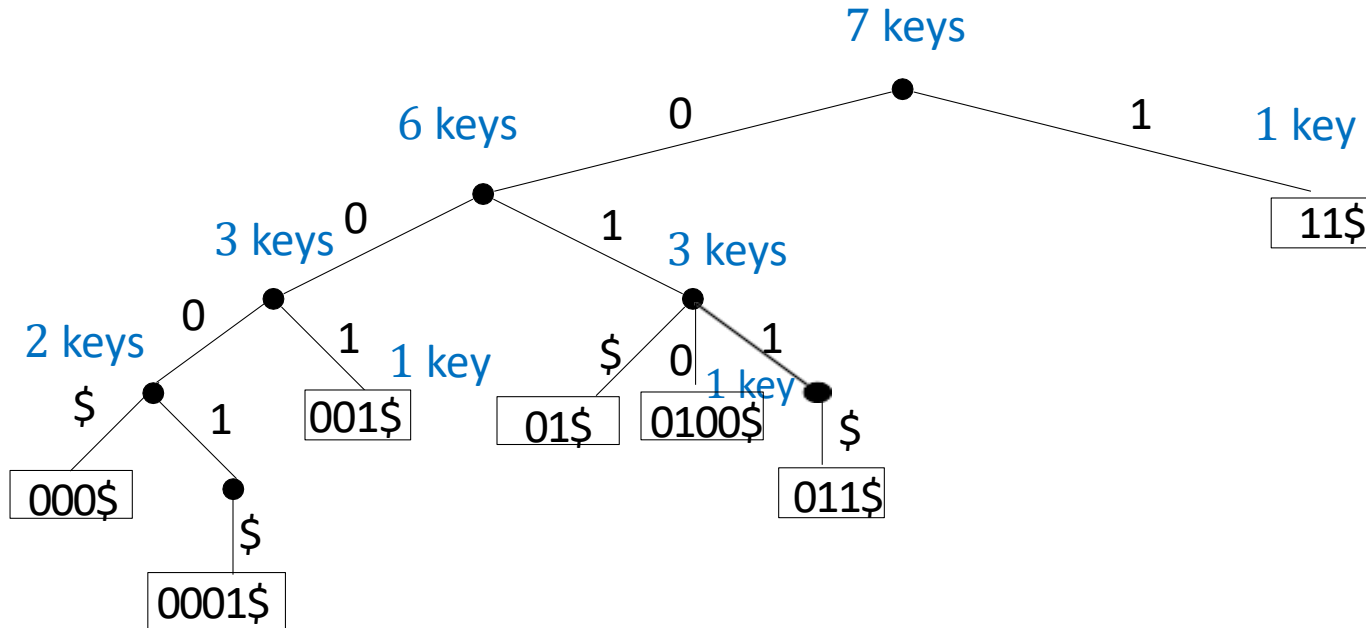
# Pruned Trie

- Sub-trie with one key has only one node
- Convert standard trie into pruned trie



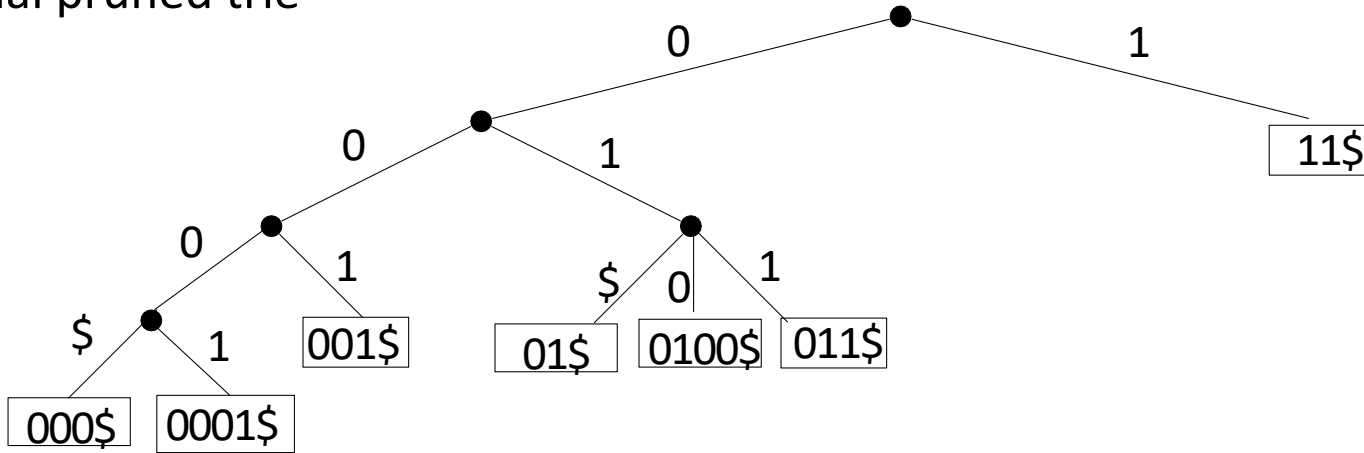
# Pruned Trie

- Sub-trie with one key has only one node
- Convert standard trie into pruned trie



# Pruned Trie

- Sub-trie with one key has only one node
- Final pruned trie



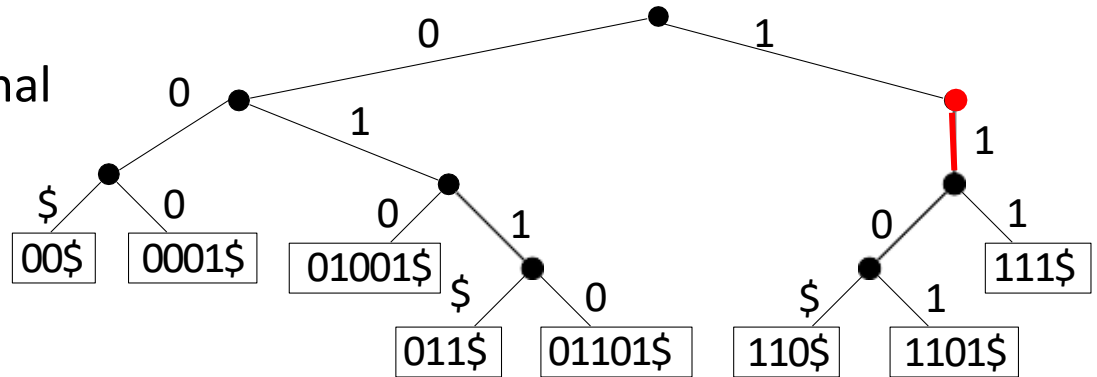
- node has a child only if it has two or more descendants
- saves space if there are only few bitstrings that are long
- can even store really long bitstrings more efficiently (real numbers)
- more efficient version of tries, but operations get a bit more complicated
  - in particular, have to change *prefix-search*( $w$ ) to search at the leaf if full prefix is not found prior to visiting the leaf

# Outline

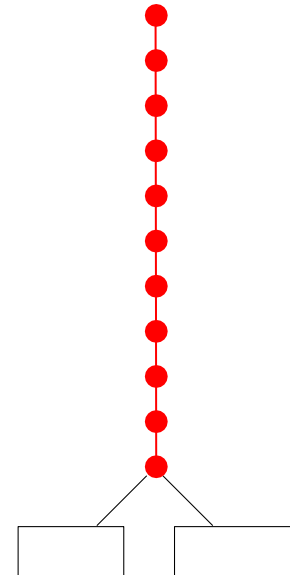
- Lower bound for search
- Interpolation Search
- **Tries**
  - Standard Trie
  - Pruned Trie
  - **Compressed Trie**
  - Multiway Trie

# Pruned Trie: Internal Nodes with One Child

- Pruned trie can have internal nodes with one child

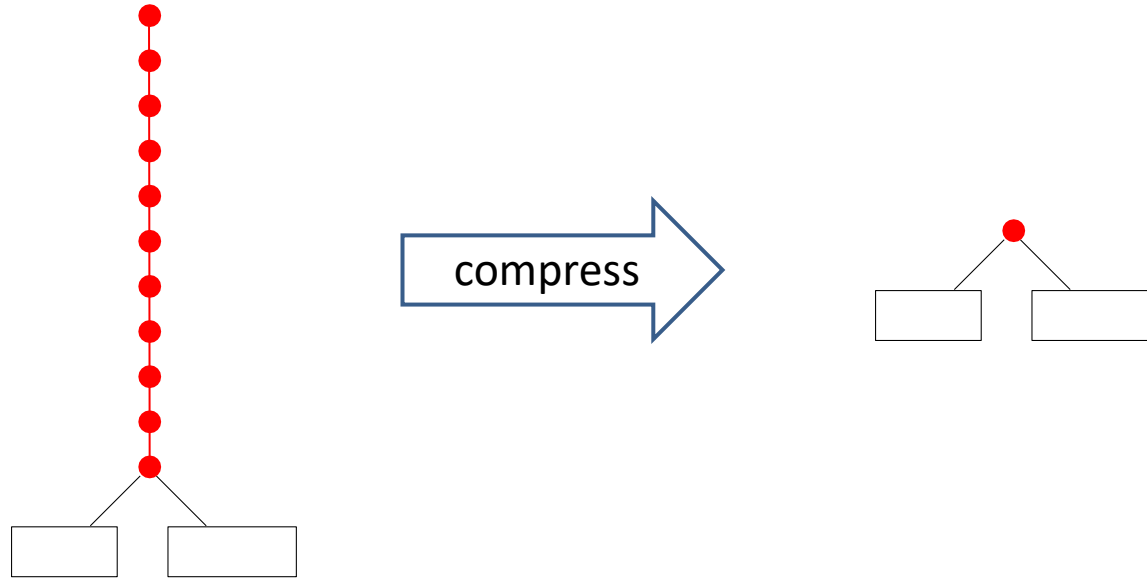


- Extreme example



- Such 'chains' in a trie waste space and reduce efficiency

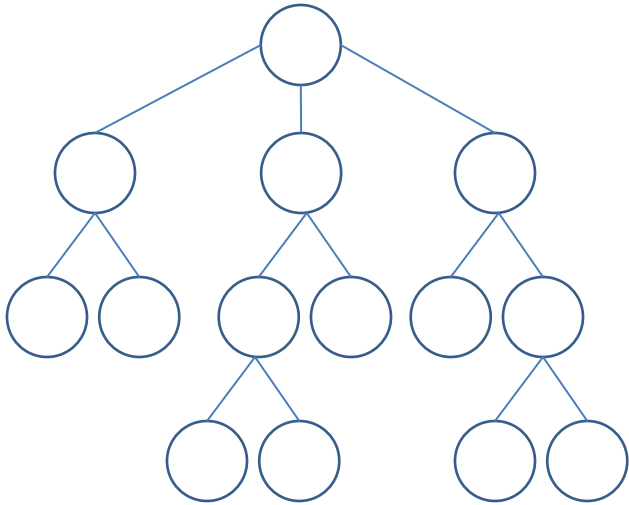
# Compressing Singly Linked Chains



- Singly linked 'chains' in a trie waste space and reduce efficiency
- If compress chains into one node, each internal node will have at least 2 children
- Let  $n$  be the number of leaf nodes (i.e. the number of stored keys)
- Will show that if each internal node has 2 or more children, then there are at most  $n - 1$  internal nodes
- Therefore at most  $2n - 1$  total nodes
  - $n$  external + at most  $n - 1$  internal
  - space is  $O(n)$ , not much wasted space

# Tree with no 'chains' Theorem

- Let  $T$  be a tree with  $m$  leaves. If every non-leaf (internal) node has at least 2 children, then the tree has at most  $m - 1$  internal nodes

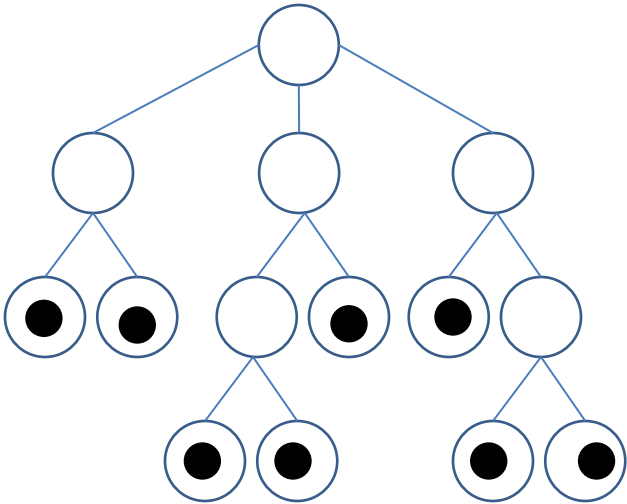


- Visual proof
  - put a stone on each leaf



# Tree with no 'chains' Theorem

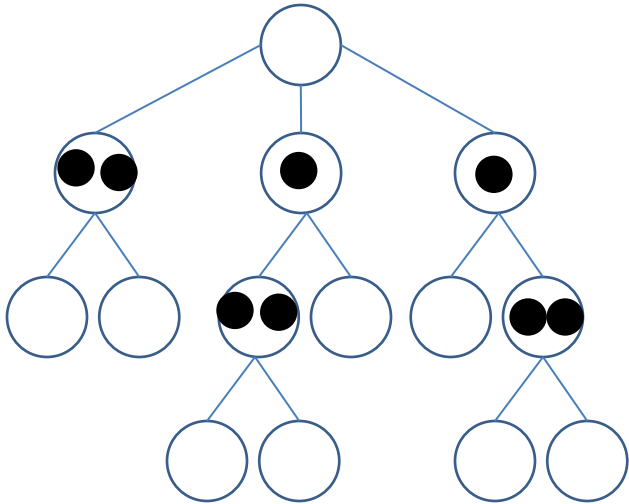
- Let  $T$  be a tree with  $m$  leaves. If every non-leaf (internal) node has at least 2 children, then the tree has at most  $m - 1$  internal nodes



- Visual proof
  - put a stone on each leaf
  - there are  $m$  stones
  - all leaves pass a stone to the parent

# Tree with no 'chains' Theorem

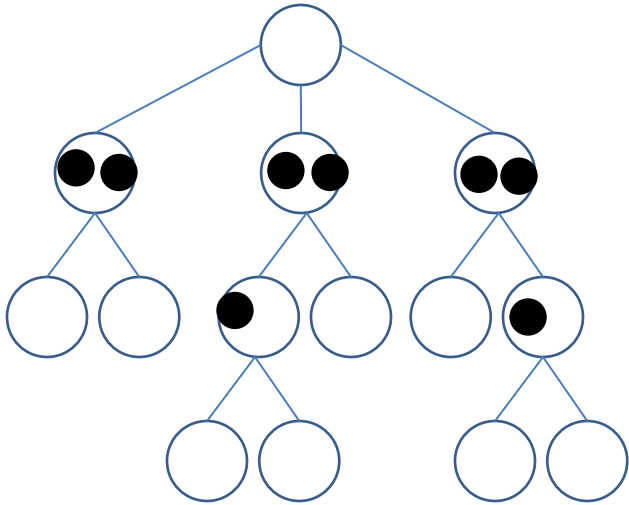
- Let  $T$  be a tree with  $m$  leaves. If every non-leaf (internal) node has at least 2 children, then the tree has at most  $m - 1$  internal nodes



- Visual proof
  - put a stone on each leaf
  - there are  $m$  stones
  - all leaves pass a stone to the parent
  - all internal nodes at level  $h - 1$  have at least 2 stones, they leave one stone and pass one stone to parent

# Tree with no 'chains' Theorem

- Let  $T$  be a tree with  $m$  leafs. If every non-leaf (internal) node has at least 2 children, then the tree has at most  $m - 1$  internal nodes

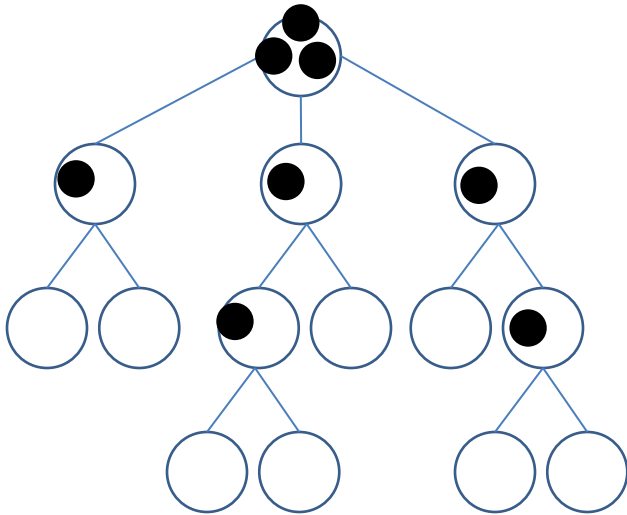


- Visual proof

- put a stone on each leaf
- there are  $m$  stones
- all leaves pass a stone to the parent
- all internal nodes at level  $h - 1$  have at least 2 stones, they leave one stone and pass one stone to parent
- all internal nodes at level  $h - 2$  have at least 2 stones, they leave one stone and pass one stone to the parent

# Tree with no 'chains' Theorem

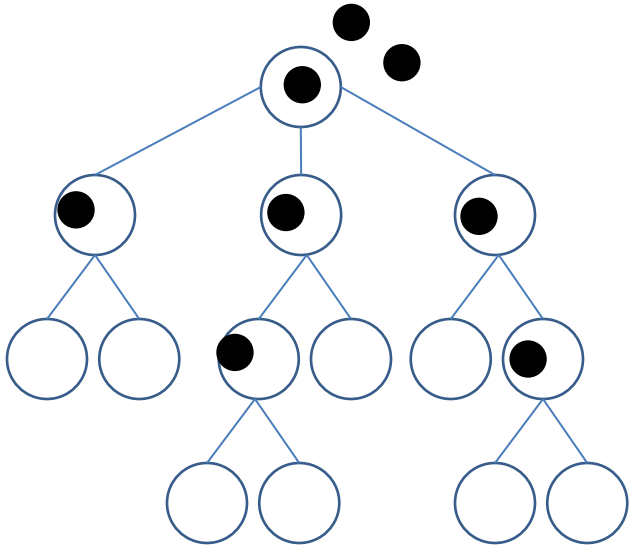
- Let  $T$  be a tree with  $m$  leafs. If every non-leaf (internal) node has at least 2 children, then the tree has at most  $m - 1$  internal nodes



- Visual proof
  - continue until reach the root
  - now each internal node has 1 stone and root has 2 or more stones

# Tree with no 'chains' Theorem

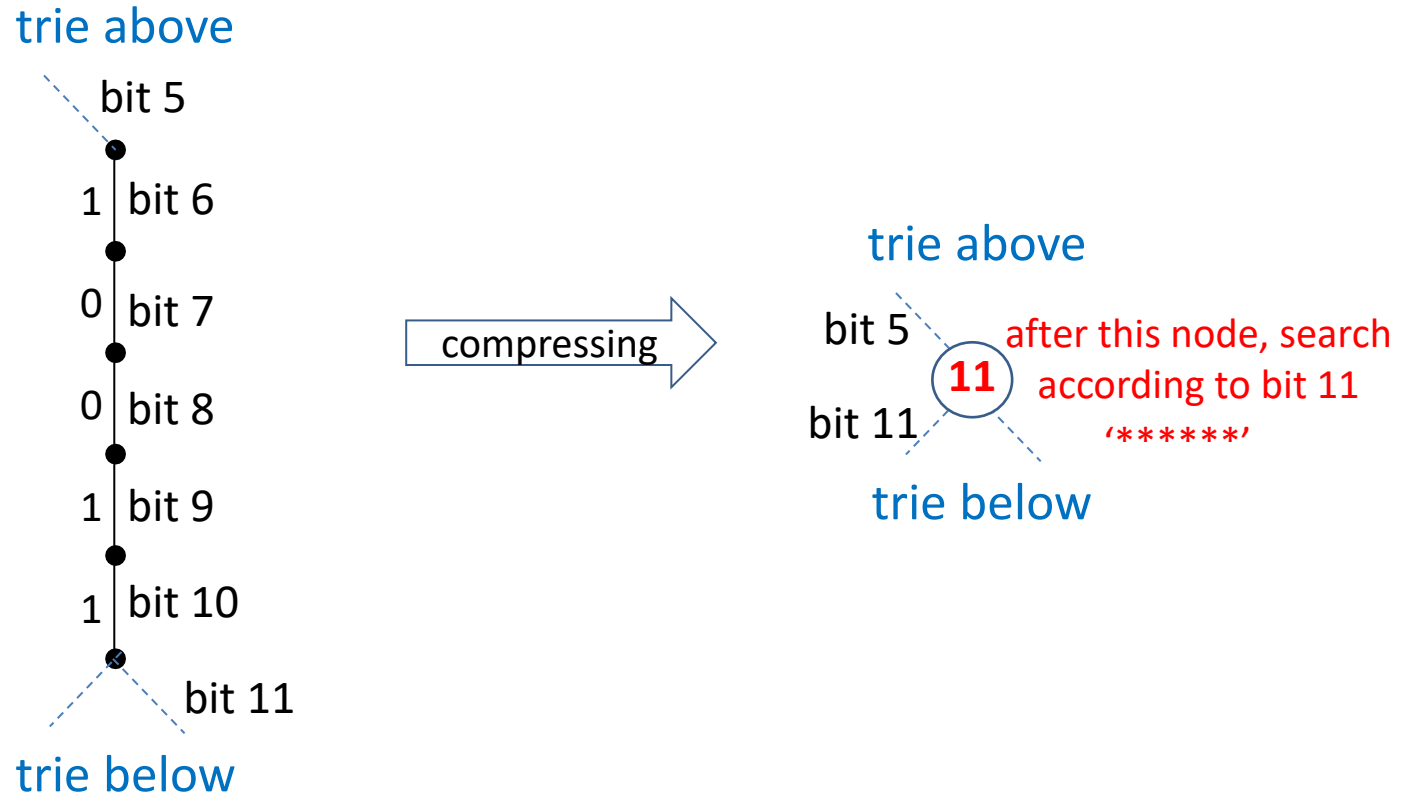
- Let  $T$  be a tree with  $m$  leafs. If every non-leaf (internal) node has at least 2 children, then the tree has at most  $m - 1$  internal nodes



- Visual proof

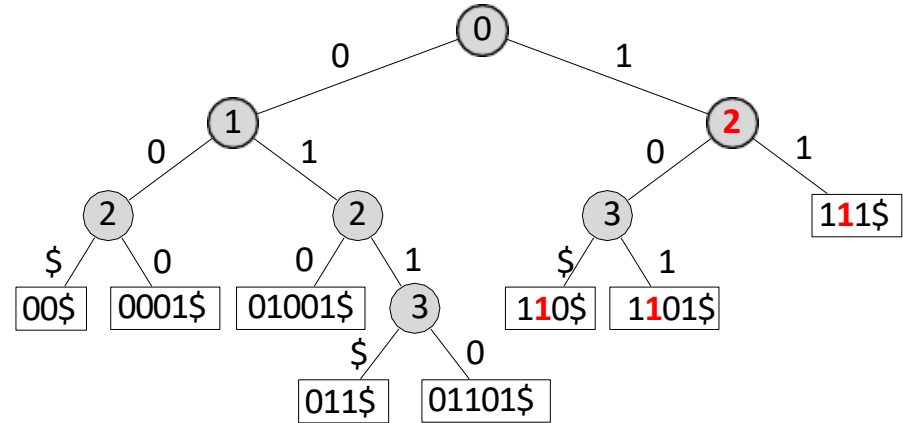
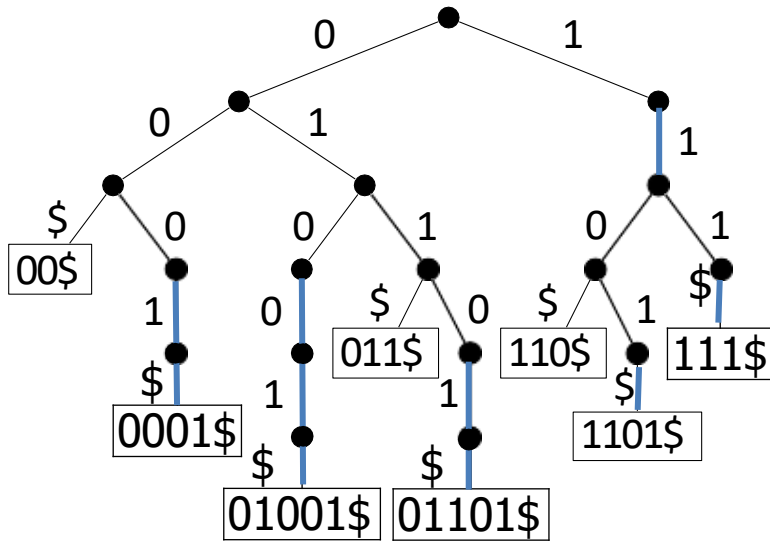
- continue until reach the root
- now each internal node has 1 stone and root has 2 or more stones
- root leaves 1 stone and throws the rest outside the tree
- now each internal node has 1 stone, and there is one or more stones outside the tree
- since number of stones is  $m$ , the number of internal nodes is strictly less than  $m$

# Compressing Chains



- But now we lost part of the binary string '10011'
- Check if the leaf we reach stores the search key

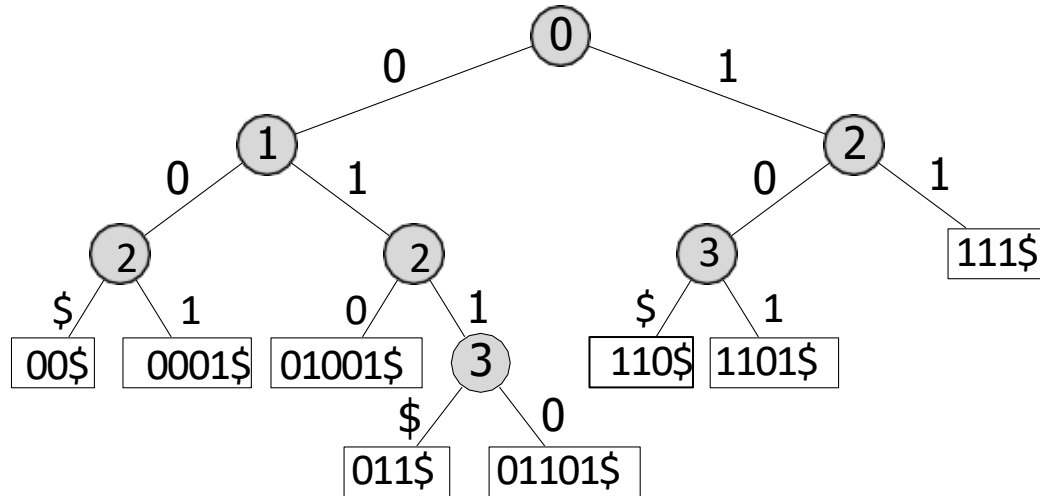
# Compressed Tries (Patricia Tries)



- Morrison (1968): *Patricia-Tries*
- Practical Algorithm to Retrieve Information Coded in Alphanumeric
- **Idea:** compress paths of nodes with only one child
- Each node stores an *index*: next bit to be tested during a search
- Compressed trie with  $n$  keys has at most  $n - 1$  internal (non-leaf) nodes

# Compressed Tries: Search Example

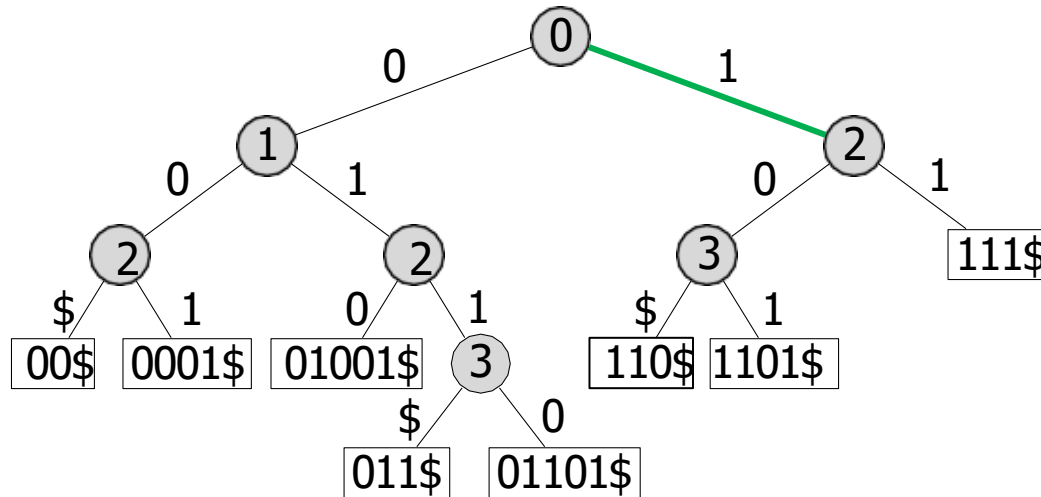
Example: Search(10\$)





# Compressed Tries: Search Example

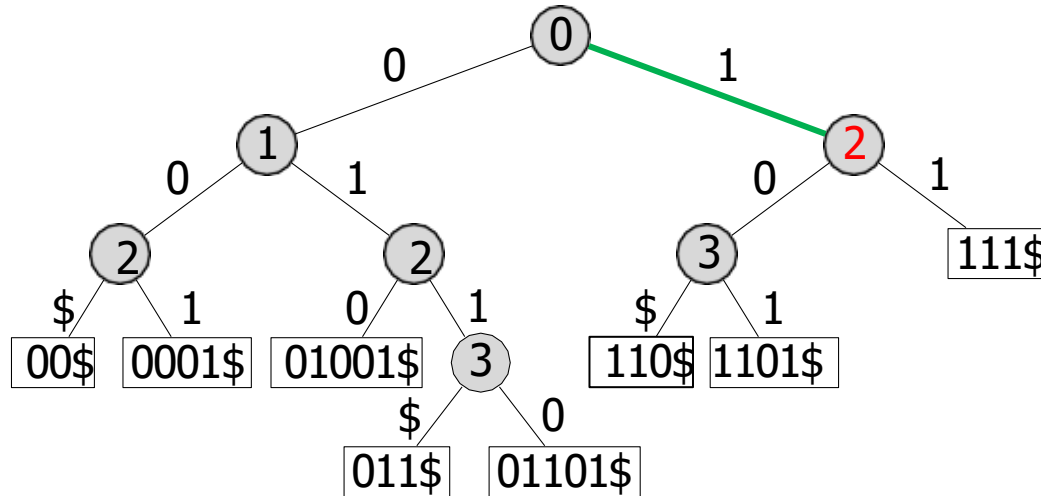
Example: Search(10\$)



# Compressed Tries: Search Example

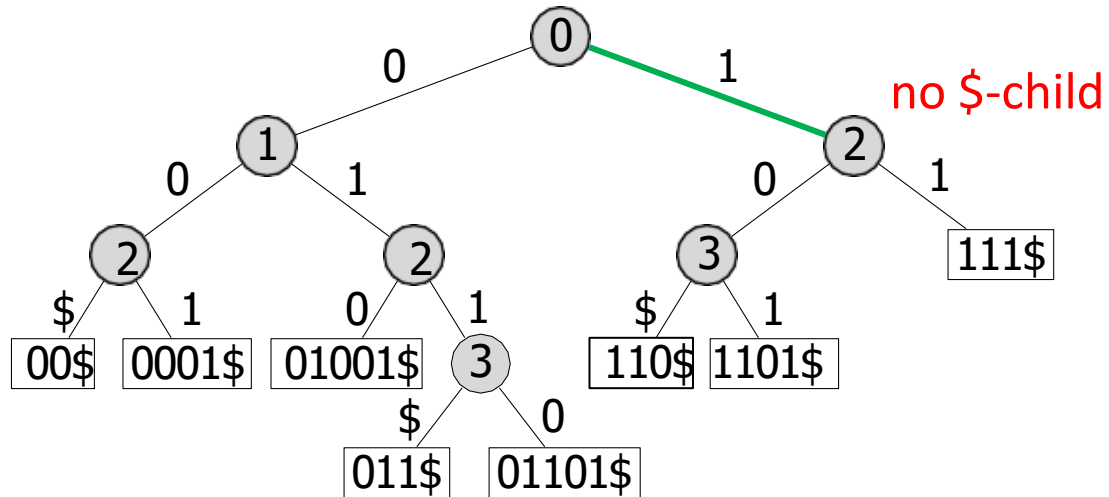
Example: Search(10\$)

↑  
skip



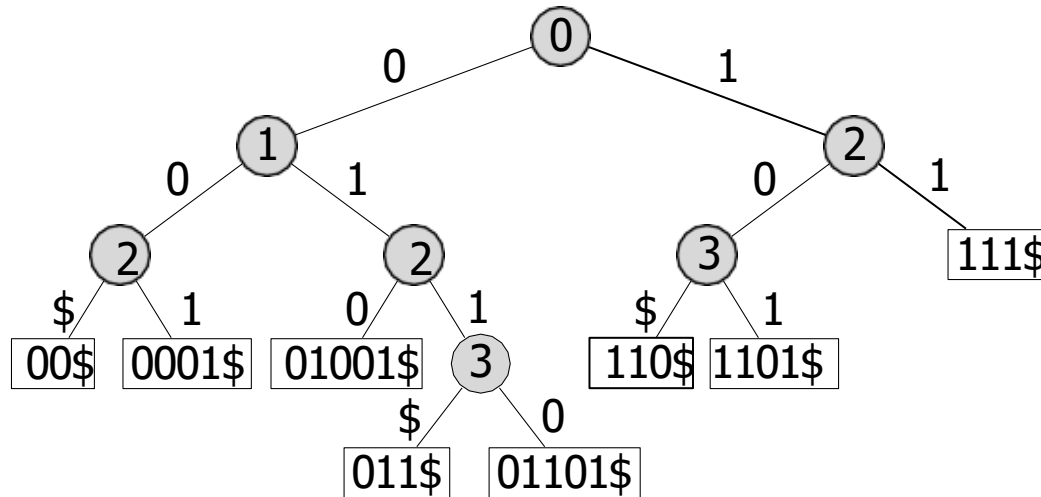
# Compressed Tries: Search Example

Example: Search(10\$) **unsuccessful**



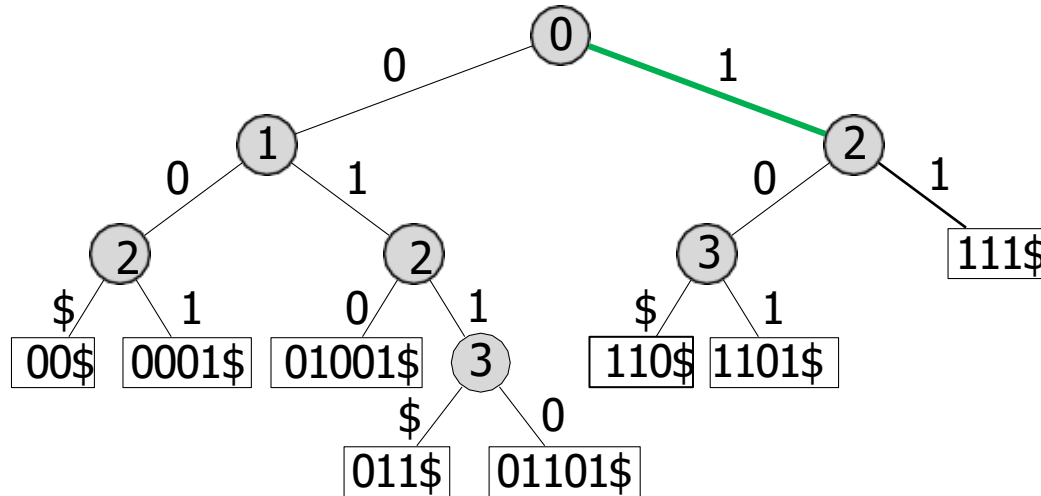
# Compressed Tries: Search Example

Example: Search(101\$)



# Compressed Tries: Search Example

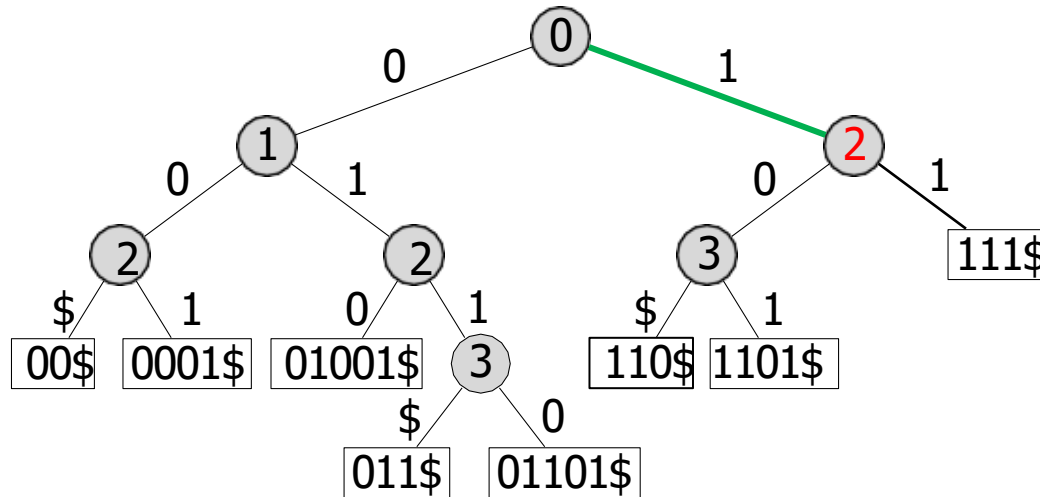
Example: Search(101\$)



# Compressed Tries: Search Example

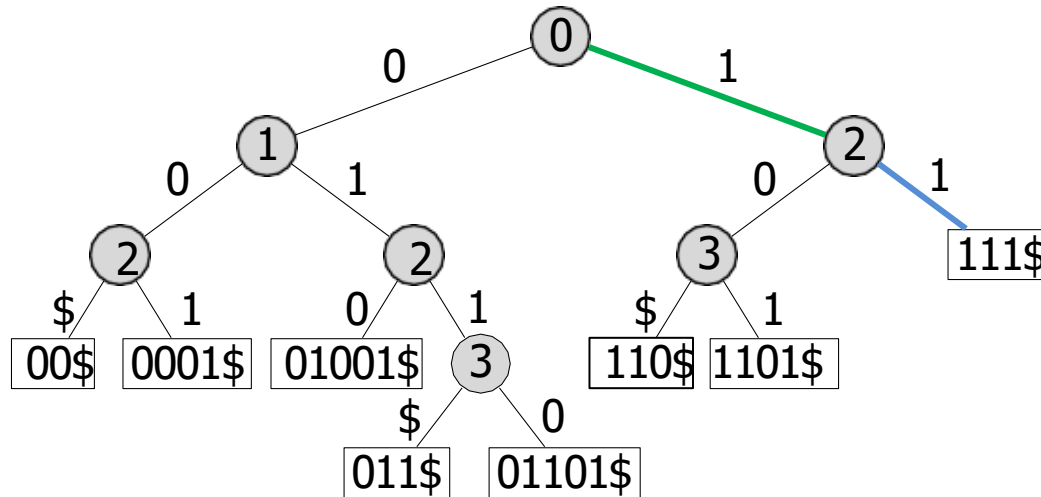
Example: Search(101\$)

↑  
skip



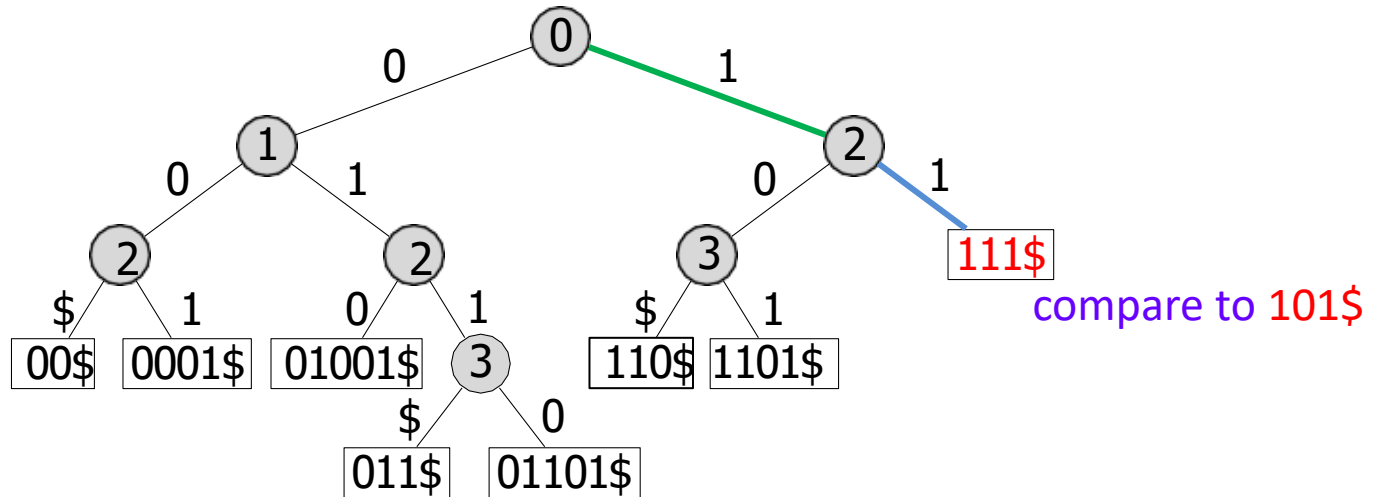
# Compressed Tries: Search Example

Example: Search(101\$)



# Compressed Tries: Search Example

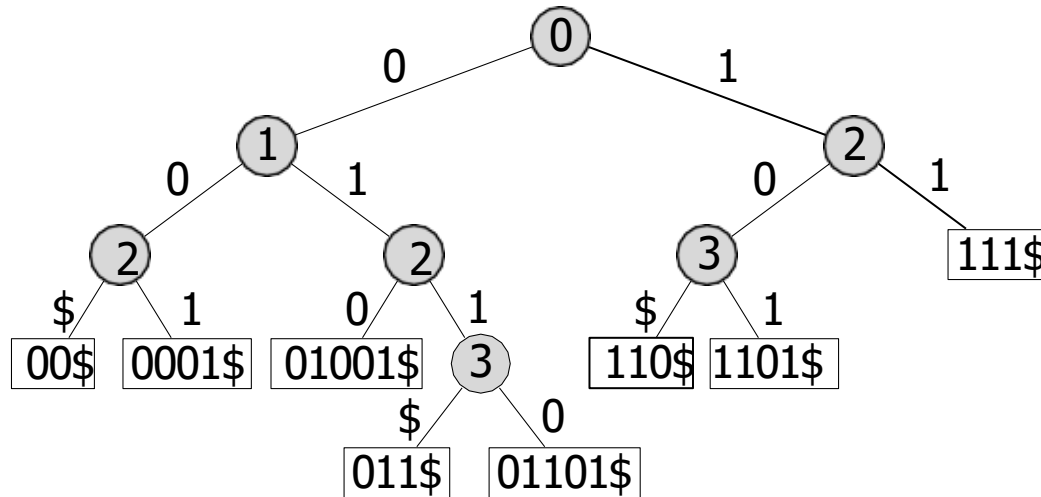
Example: Search(101\$) **Unsuccessful**





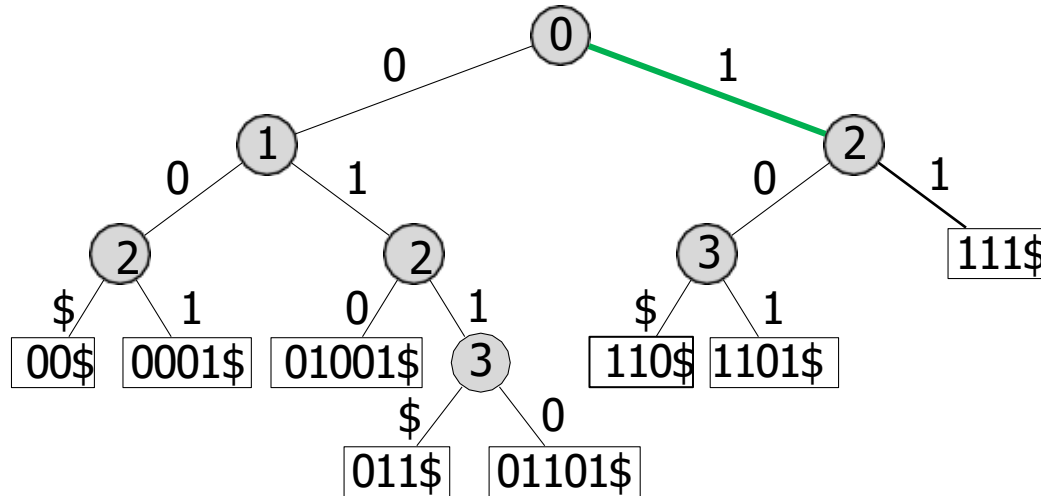
# Compressed Tries: Search Example

Example: Search(111\$)



# Compressed Tries: Search Example

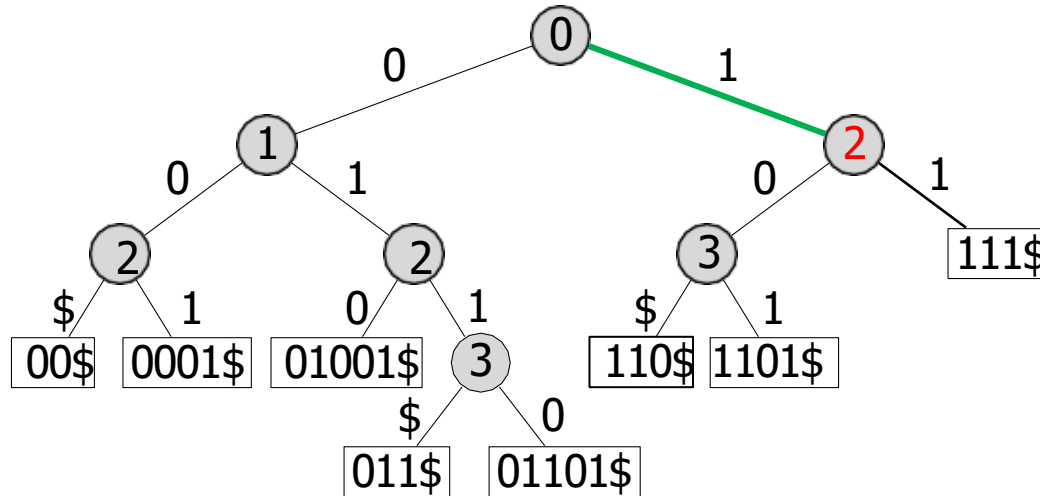
Example: Search(111\$)



# Compressed Tries: Search Example

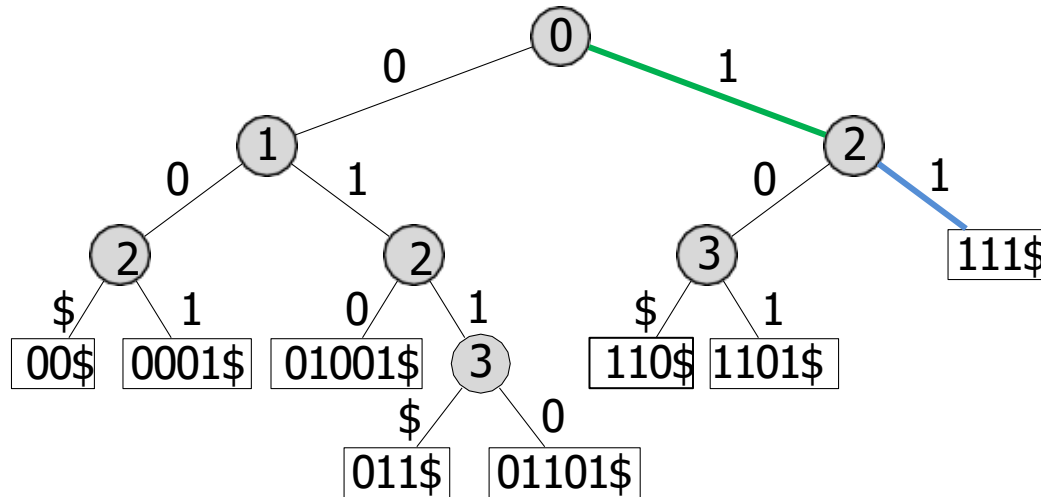
Example: Search(111\$)

↑  
skip



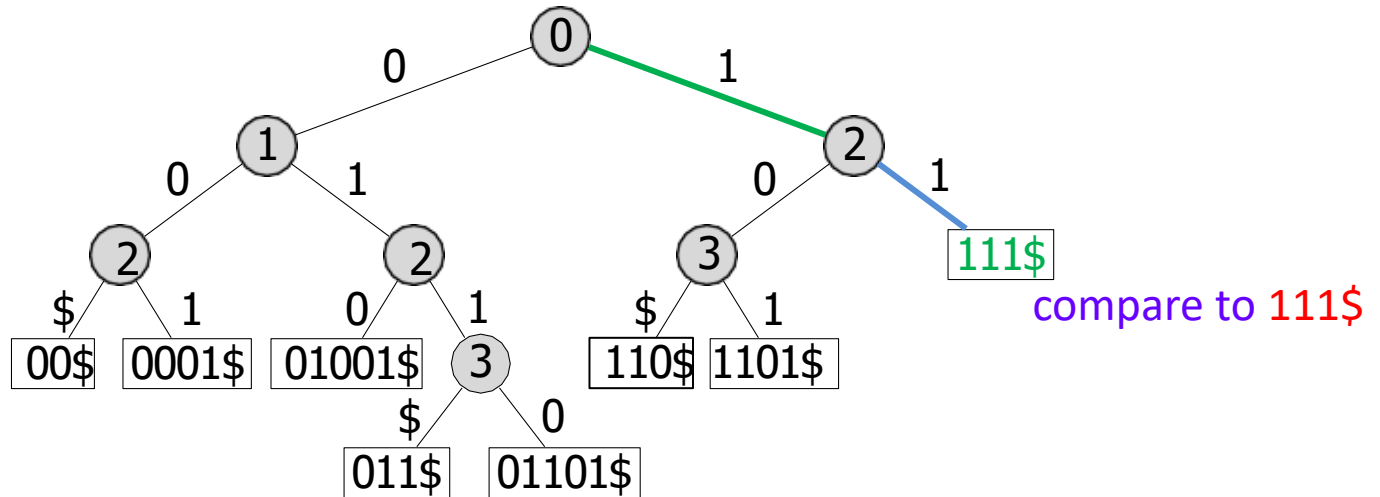
# Compressed Tries: Search Example

Example: Search(111\$)



# Compressed Tries: Search Example

Example: Search(111\$) successful



# Compressed Tries: Search

*CompressedTrie::get-path-to*( $w$ )

$P \leftarrow$  empty stack;  $z \leftarrow$  root;  $P.push(z)$

**while**  $z$  is not a leaf and ( $d \leftarrow z.index \leq w.size$ ) **do**

**if**  $z$  has a child-link labelled with  $w[d]$

$z \leftarrow$  child at this link;  $P.push(z)$

**else break**

**return**  $P$

*CompressedTrie::search*( $w$ )

$P \leftarrow$  *get-path-to*( $w$ );  $z \leftarrow P.top()$

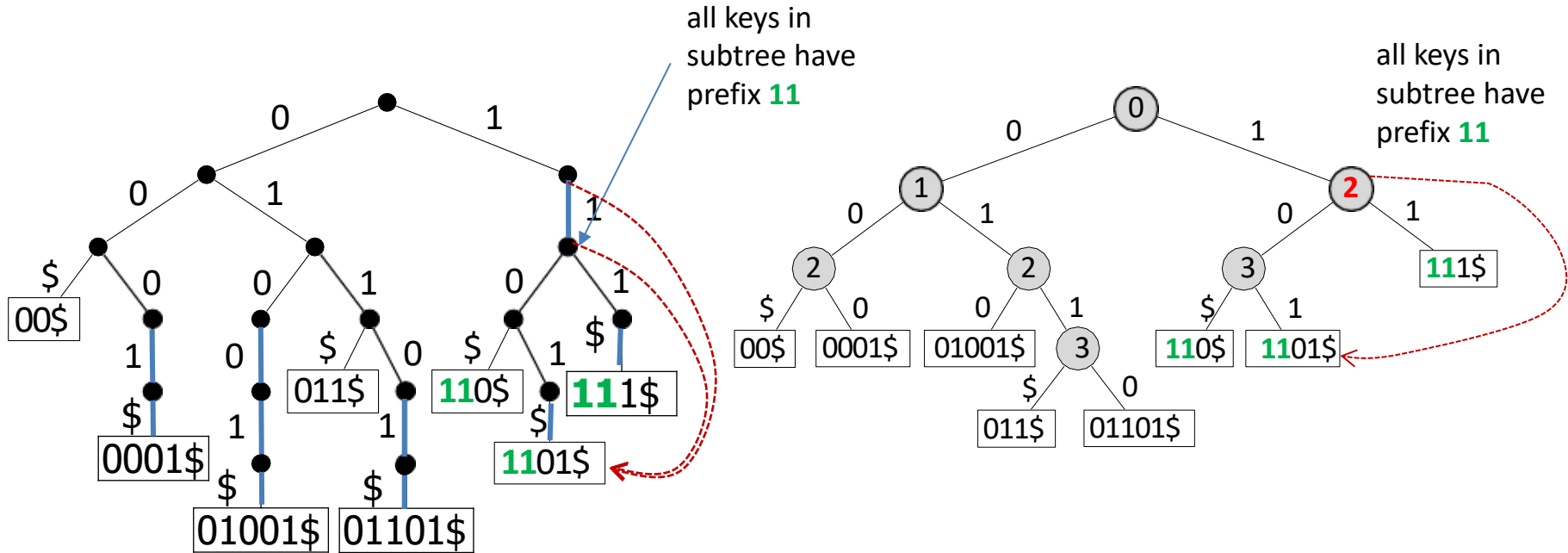
**if**  $z$  is not a leaf or word stored at  $z$  is not  $w$  **then**

**return** “not found”

**return** key-value pair at  $z$

- As in standard tries, follow links that correspond to current bits in  $w$
- Main difference
  - stored indices say which bits to compare
  - also must compare  $w$  to the word found at the leaf

# Compressed Trie: Links to Leaves



- Links to leaves established as in uncompressed tries
- Need to modify *prefix-search*( $w$ ) to check if the leaf has prefix matching  $w$
- All keys in a subtree of the node where search for  $w$  ends have the same prefix of length  $|w|$ 
  - prefix  $w$  either matches all keys in the subtree, or none of subtrees keys

# Compressed Tries: Summary

- *search*( $w$ ) and *prefix-search*( $w$ ) are easy
- *insert*( $w$ ) and *delete*( $w$ ) are conceptually simple
  - search for path  $P$  to word  $w$  (say we reach node  $z$ )
  - uncompress this path (using characters of  $z$ . *leaf*)
  - insert/delete  $w$  as in uncompressed trie
  - compress path from root to where changed happened
- All operations take  $O(|w|)$  time for word  $w$
- Use  $O(n)$  space
- More complicated than standard tries, but space savings are worth it if words are unevenly distributed

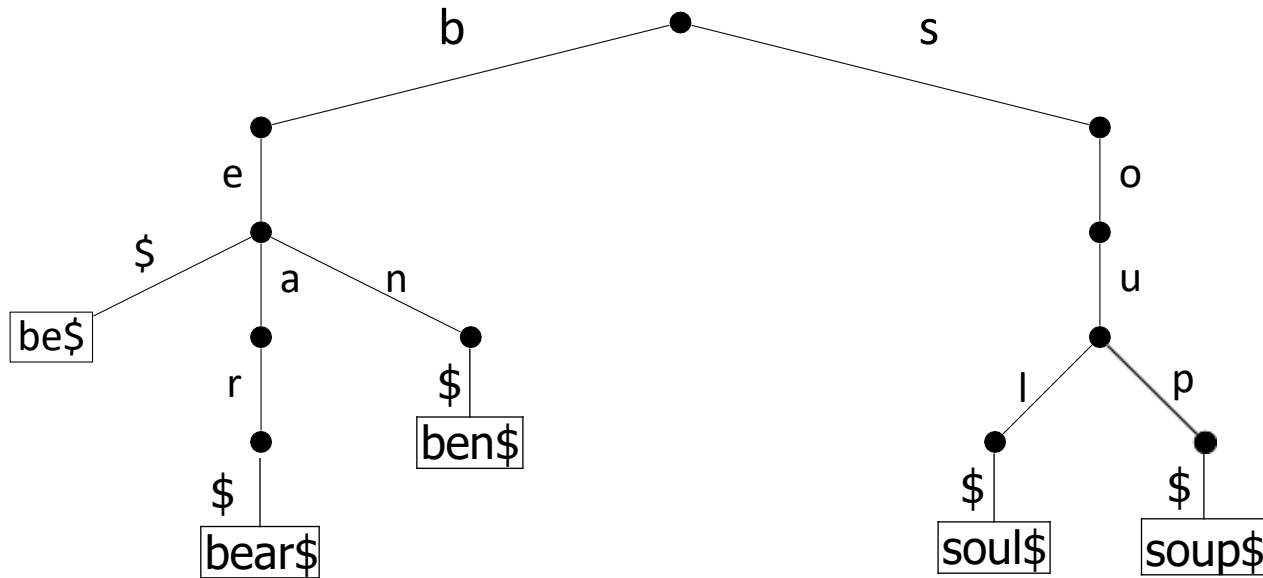


# Outline

- Lower bound for search
- Interpolation Search
- **Tries**
  - Standard Trie
  - Pruned Tries
  - Compressed Trie
  - **Multiway Trie**

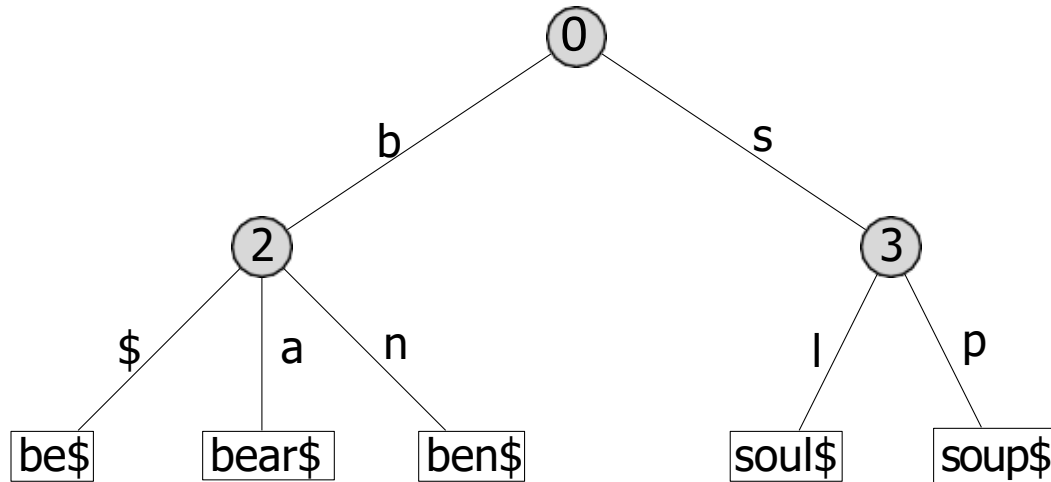
# Multiway Tries: Larger Alphabet

- Represents **Strings** over any **fixed alphabet**  $\Sigma$
- Any node has at most  $|\Sigma| + 1$  children
  - one child for the end-of-word character  $\$$
- Example: A trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



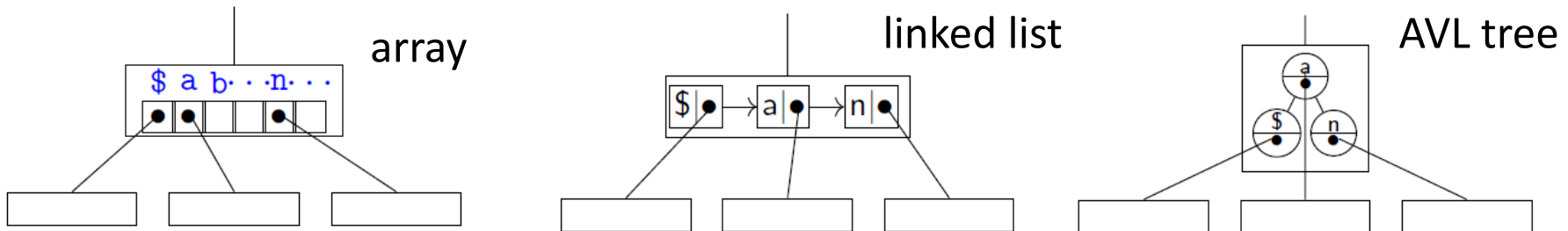
# Compressed Multiway Tries

- Compressed multi-way tries
- Example: A compressed trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



# Multiway Tries: Summary

- Operations  $\text{search}(w)$ ,  $\text{prefix-search}(w)$ ,  $\text{insert}(w)$  and  $\text{delete}(w)$  are as for bitstring tries
- Run-time  $O(|w| \cdot (\text{time to find the appropriate child}))$
- Each node now has up to  $|\Sigma| + 1$  children
- How should children be stored?



- Time/Space tradeoff: arrays are fast, lists are space efficient
  - run-time  $O(|w|)$  with arrays storing children
- AVL tree is best in theory, but not worth it in practice unless  $|\Sigma|$  is huge
- In practice, use hashing (next module)