CS 240 – Data Structures and Data Management

Module 3: Sorting, Average-case and Randomization

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Based on lecture notes by many previous cs240 instructors

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Outline

3. Sorting, Average-case and Randomization
   - Analyzing average-case run-time
   - Randomized Algorithms
   - SELECTION and *quick-select*
   - SORTING and *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

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For this module:

- Assume that the set $\mathcal{I}_n$ of size-n instances is finite (or can be mapped to a finite set in a natural way)
- Assume that all instances occur equally frequently

Then we can use the following *simplified formula*

$$T^{\text{avg}}(n) = \frac{\sum_{I:\text{size}(I)=n} T(I)}{\#\text{instances of size } n} = \frac{1}{|\mathcal{I}_n|} \sum_{I \in \mathcal{I}_n} T(I)$$

To learn how to analyze this, we will do simpler examples first.

A simple (contrived) example

silly-test(π, n)
π: a permutation of $\{0, \ldots, n-1\}$, stored as an array
1. **if** $\pi[0] = 0$ **then for** $j \leftarrow 1$ to n **do** print 'bad case'
2. **else** print 'good case'

$$T^{\text{avg}}(n) = \frac{1}{n!}\sum_{\pi \in \Pi_n} T(\pi) = \frac{1}{n!}\Big(\sum_{\substack{\pi \in \Pi_n \\ \text{in bad case}}} T(\pi) + \sum_{\substack{\pi \in \Pi_n \\ \text{in good case}}} T(\pi)\Big)$$

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- bad case $\Rightarrow$ run-time $\leq c \cdot n$ (for some constant $c > 0$)
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$$\leq \frac{1}{n!}\Big((n-1)! \cdot cn + n! \cdot c \Big) = \frac{1}{n}cn + c = 2c \in O(1)$$

A second (not-so-contrived) recursive example

all-0-test(w, n)
// test whether all entries of bitstring $w[0..n-1]$ are 0
1. **if** $(n = 0)$ **return** `true`
2. **if** $(w[n-1] = 1)$ **return** `false`
3. *all-0-test*$(w, n-1)$

(In real life, you would write this non-recursive.)

Define $T(w) = \#$ bit-comparisons (i.e., line 2) on input w. This is asymptotically the same as the run-time.

Worst-case run-time: Always go into the recursion until $n = 0$.
$T(n) = 1 + T(n-1) = 1 + 1 + \cdots + T(0) = n \in \Theta(n)$.

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Best-case run-time: Return immediately. $T(n) = 1 \in \Theta(1)$.

Average-case run-time?

Average-case run-time of *all-0-test*

$$T^{\text{avg}}(n) = \frac{1}{|\mathcal{B}_n|} \sum_{w \in \mathcal{B}_n} T(w). \qquad (\mathcal{B}_n = \{\text{bitstrings of length } n\}, |\mathcal{B}_n| = 2^n)$$

Recursive formula for one non-empty bitstring w:

$$T(w) \;=\; \begin{cases} 1 & \text{if } w[n-1] = 1 \\ 1 + T(\underbrace{w[0..n-2]}_{\text{length } n-1}) & \text{otherwise} \end{cases}$$

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Natural guess for the recursive formula for $T^{\text{avg}}(n)$:

$$T^{\text{avg}}(n) = \underbrace{\tfrac{1}{2}}_{\substack{\text{half have} \\ w[n-1]=1}} \cdot 1 + \underbrace{\tfrac{1}{2}}_{\substack{\text{half have} \\ w[n-1]=0}} (1 + T^{???}(n-1))$$

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- This holds with $\leq$ (but is useless) if '???' is 'worst'.
- This is *not obvious* if '???' is 'avg'.

Average-case run-time of *all-0-test*

$$T^{\text{avg}}(n) = \frac{1}{|\mathcal{B}_n|} \sum_{w \in \mathcal{B}_n} T(w)$$

$$=$$

$$= 1 + \frac{1}{2} T^{\text{avg}}(n-1)$$

Easy induction proof: $T^{\text{avg}}(n) \leq 2 \in O(1)$.

Average-case analysis and recursions

Why can't we always write 'avg' for '???' in $T^{\text{avg}}(n) = 1 + \frac{1}{2}T^{???}(n-1)$?

Consider the following (contrived) example:

silly-all-0-test(w, n)
w: array of size at least n that stores bits
1. **if** $(n = 0)$ **then return** `true`
2. **if** $(w[n-1] = 1)$ **then return** `false`
3. **if** $(n > 1)$ **then** $w[n-2] \leftarrow 0$ `// this is the only change`
4. *silly-all-0-test*(w, $n-1$)

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silly-all-0-test(w, n)
w: array of size at least n that stores bits
    if (n = 0) then return true
    if (w[n−1] = 1) then return false
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    silly-all-0-test(w, n−1)
```

- Only one more line of code in each recursion, so same formula applies.
- But observe that now $T(w) = \begin{cases} 1 & \text{if } w[n-1] = 1 \\ n & \text{if } w[n-1] = 0 \end{cases}$.

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- But observe that now $T(w) = \begin{cases} 1 & \text{if } w[n-1] = 1 \\ n & \text{if } w[n-1] = 0 \end{cases}$.
- So $T^{\text{avg}}(n) = 1 + \frac{n}{2} \in \Theta(n)$. The "obvious" recursion did not hold.

Average-case analysis is highly non-trivial for recursive algorithms.

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Randomized algorithms

- A **randomized algorithm** is one which relies on some random numbers in addition to the input.

(Computers cannot generate randomness. We assume that there exists a *pseudo-random number generator (PRNG)*, a deterministic program that uses an initial value or *seed* to generate a sequence of seemingly random numbers. The quality of randomized algorithms depends on the quality of the PRNG!)

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randomized-silly(A, n)
A: array of size at least n
1.  sum ← 0
2.  if (random(3)>0) then
3.      for i ← 0 to n − 1 do
4.          sum ← sum + A[i]
5.  return sum
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- We assume the existence of a function *random*(n) that returns an integer uniformly from $\{0, 1, 2, \ldots, n-1\}$. So $Pr(random(3)=0) = \frac{1}{3}$.

Expected run-time

The run-time of the algorithm now depends on the random numbers, as well as the input.

Define $T_{\mathcal{A}}(I, R)$ to be the run-time of a randomized algorithm $\mathcal{A}$ for an instance I and the sequence R of outcomes of random trials.

The **expected run-time $T^{\exp}(I)$ for instance I** is the expected value:

$$T^{\exp}(I) = \mathbf{E}[T(I, R)] = \sum_{R} T(I, R) \cdot \Pr(R)$$

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Now take the *maximum* over all instances of size n to define the **expected run-time** (or formally: *worst-instance expected-luck run-time*) **of $\mathcal{A}$**.

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We can still have good luck or bad luck, so occasionally we also discuss the very worst that could happen, i.e., $\max_I \max_R T(I,R)$.

Expected run-time example

randomized-silly(A, n)
A: array of size at least n

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1.   sum ← 0
2.   if (random(3)>0) then
3.         for i ← 0 to n − 1 do
4.               sum ← sum + A[i]
5.   return sum
```

$$T^{\exp}(A) = \sum_R Pr(R) \cdot T(A, R)$$

- Here only one random outcome: $Pr(0) = Pr(1) = Pr(2) = \frac{1}{3}$
- If outcome is 0: $O(1)$ time, say c time units (for some constant c)
- If outcome is 1 or 2: $O(n)$ time, say cn time units
- $T^{\exp}(A) = \frac{1}{3}c + \frac{1}{3}cn + \frac{1}{3}cn \in \Theta(n)$
- All instances have the same expected run-time, so $T^{\exp}(n) \in \Theta(n)$

Why Randomized Algorithms?

- Doing randomization is often a good idea if an algorithm has bad worst-case time but seems to perform much better on most instances.

- **Goal:** Shift the dependency of run-time from what we can't control (the input) to what we *can* control (the random numbers).

  No more bad instances, just unlucky numbers.

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  No more bad instances, just unlucky numbers.
- Not all randomizations achieve this automatically; it must be proved.
- Doing randomization can also (with restrictions) be used to bound the avg-case run-time.

Randomizations of algorithms

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w: array of size at least n that stores bits
    if n = 0 return true
    if (random(2)=0) then
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```

This is *all-0-test*, except that we flip last bit based on a coin toss.

In each recursion, we use the outcome $x \in \{0, 1\}$ of one coin toss.
We return without recursing if $x = w[n-1]$ (this has probability $\frac{1}{2}$).

Expected run-time of *randomized-all-0-test*

Let $T(w, R)$ be the # of bit-comparisons used on input w if the random outcomes are R.

- The random outcomes R consist of two parts $R = \langle x, R' \rangle$:
  - x: outcome of first coin toss
  - R': random outcomes (if any) for the recursions

  We have $\Pr(R) = \Pr(x) \cdot \Pr(R')$ (random choices are independent).

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- Recursive formula for one instance:

$$T(w, R) = T(w, \langle x, R' \rangle) = \begin{cases} 1 & \text{if } x = w[n-1] \\ 1 + T(w[0..n-2], R') & \text{otherwise} \end{cases}$$

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- Natural guess for the recursive formula for $T^{\exp}(n)$:

$$T^{\exp}(n) = \underbrace{\tfrac{1}{2}}_{\Pr(x=w[n-1])} \cdot 1 + \underbrace{\tfrac{1}{2}}_{\Pr(x \neq w[n-1])} (1 + T^{\exp}(n-1)) = 1 + \tfrac{1}{2} T^{\exp}(n-1)$$

Expected run-time of *randomized-all-0-test*

In contrast to average-case analysis, the natural guess usually is correct for the expected run-time.

Proof for *randomized-all-0-test*:

$$T^{\exp}(w) \; = \; \sum_{R} \Pr(R)\, T(w, R) =$$

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Therefore $T^{\exp}(n) = \max_{w \in \mathcal{B}_n} T^{\exp}(w) \leq 1 + \frac{1}{2} T^{\exp}(n-1)$

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- We had $T^{\exp}_{\textit{rand-all-0-test}}(n) \le 1 + \frac{1}{2} T^{\exp}_{\textit{rand-all-0-test}}(n-1)$
- We earlier had $T^{\text{avg}}_{\textit{all-0-test}}(n) \le 1 + \frac{1}{2} T^{\text{avg}}_{\textit{all-0-test}}(n-1)$
- Same recursion $\Rightarrow$ same upper bound $\Rightarrow$ $T^{\exp}_{\textit{rand-all-0-test}}(n) \in O(1)$.

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Recall: *randomized-all-0-test* was very similar to *all-0-test*

(The only different was a random bitflip.)

Is it a coincidence that the two recursive formulas are the same?

Or does the expected time of a randomized version always have something to do with the average-case time?

Expected run-time of *randomized-all-0-test*

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Or does the expected time of a randomized version always have something to do with the average-case time?

- Not in general! (It depends how we randomize.)
- Yes if the randomization is a *shuffle* (choose instance randomly).

Avg-case run-time via expected run-time

Consider the following randomization of a deterministic algorithm $\mathcal{A}$.

shuffle-$\mathcal{A}(n)$
1. Among all instances $\mathcal{I}_n$ of size n for $\mathcal{A}$, choose I randomly
2. $\mathcal{A}(I)$

(*shuffle-*$\mathcal{A}$ usually does not solve what $\mathcal{A}$ solves)

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- If we do not count the time for line 1:

$$T^{\text{avg}}_{\mathcal{A}}(n) = \frac{1}{|\mathcal{I}_n|} \sum_{I \in \mathcal{I}_n} T(I) = \sum_{I \in \mathcal{I}_n} Pr(I \text{ chosen}) \cdot T(I) = T^{\text{exp}}_{\textit{shuffle-}\mathcal{A}}(n)$$

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- So the average-case run-time of $\mathcal{A}$ is the same as this **run-time of $\mathcal{A}$ on randomly chosen input**.
- This gives us a different way to compute $T^{\text{avg}}_{\mathcal{A}}(n)$.

Avg-case run-time via expected run-time

Example: *all-0-test* (rephrased with for-loops):

```
shuffle-all-0-test(n)
for (i ← n−1; i ≥ 0; i--) do
        w[i] ← random(2)
for (i ← n−1; i ≥ 0; i--) do
        if (w[i] = 1) return false
return true
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randomized-all-0-test(w, n)
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- These algorithms are not quite the same.
    - Randomization outside respectively inside the for-loop.
- But this does not matter for the expected number of bit-comparisons.
    - Either way, at time of comparison the bit is 1 with probability $\frac{1}{2}$.

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- But this does not matter for the expected number of bit-comparisons.
  - Either way, at time of comparison the bit is 1 with probability $\frac{1}{2}$.
- So $T^{\text{avg}}_{\textit{all-0-test}}(n) = T^{\text{exp}}_{\textit{shuffle-all-0-test}}(n) = T^{\text{exp}}_{\textit{rand-all-0-test}}(n) \in O(1)$

  can be deduced without analyzing $T^{\text{avg}}_{\textit{all-0-test}}(n)$ directly.

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average-case run-time	expected run-time
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There is a relationship *only* if the randomization effectively achieves "choose the input instance randomly".

Outline

3. Sorting, Average-case and Randomization
   - Analyzing average-case run-time
   - Randomized Algorithms
   - SELECTION and *quick-select*
   - SORTING and *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

The SELECTION Problem

We saw SELECTION: Given an array A of n numbers, and $0 \le k < n$, find the element that would be at position k of the sorted array.

(We also call this the element of **rank** k.)

0	1	2	3	4	5	6	7	8	9
30	60	10	0	50	80	90	10	40	70

select(3) should return 30.

SELECTION can be done with heaps in time $\Theta(n + k \log n)$.

Special case: MEDIANFINDING = SELECTION with $k = \lfloor \frac{n}{2} \rfloor$. With previous approaches, this takes time $\Theta(n \log n)$, no better than sorting.

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Question: Can we do selection in linear time?
Yes! We will develop algorithm *quick-select* below.

The encountered sub-routines will also be useful otherwise.

Crucial Subroutines

quick-select and the related *quick-sort* rely on two subroutines:

- *choose-pivot*(A): Return an index p in A. We will use the **pivot-value** $v \leftarrow A[p]$ to rearrange the array.

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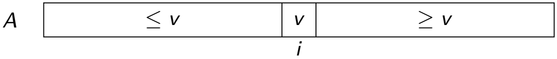

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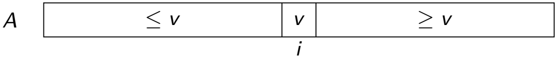
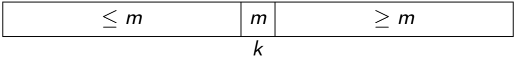


  Easy to implement so that it uses at most n key-comparisons.

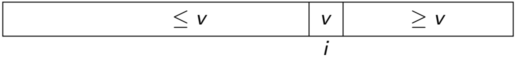
- $p =$ index of pivot-value before *partition* (we choose it)
  $i =$ index of pivot-value after *partition* (no control)

quick-select Algorithm

Goal: Find element m of rank k by rearranging A:



Recall: *partition* method achieves



Where is m if $i = k$? If $i < k$? If $i > k$?

quick-select(A, k)
A: array of size n, k: integer s.t. $0 \le k < n$

$p \leftarrow$ *choose-pivot*(A)
$i \leftarrow$ *partition*(A, p)
if $i = k$ **then return** $A[i]$
else if $i > k$ **then return** *quick-select*$(A[0 \ldots i{-}1], k)$
else if $i < k$ **then return** *quick-select*$(A[i{+}1 \ldots n{-}1], k - (i{+}1))$

Analysis of *quick-select*

Let $T(A,k)$ be the number of key-comparisons for *quick-select*(A,k).
Write A' for rearranged A after *partition*, and i for the pivot-rank.

$$T(A,k) = \begin{cases} n & \text{if } i = k \\ n + T(A'[0..i-1], k) & \text{if } i > k \quad \text{(sub-array has size } i\text{)} \\ n + T(A'[i+1..n-1], k-i-1) & \text{if } i < k \quad (\ldots \text{size } n-i-1) \end{cases}$$

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- Sub-array always gets smaller, so $\leq n$ recursions $\Rightarrow O(n^2)$ time.
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  $T^{\text{worst}}(n,0) \geq n + (n{-}1) + (n{-}2) + \cdots + 1 \in \Omega(n^2)$

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Average case run-time? Doing this directly would be *very* complicated.
Instead we will do it via a randomized version.

Randomizing quick-select: Shuffling

Goal: Create a randomized version of *quick-select*.

- This will give a proof of the avg-case run-time of *quick-select*.
- This will be a better algorithm in practice.

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- This will be a better algorithm in practice.

First idea: Shuffle the input, then do *quick-select*.

shuffle-quick-select(A, k)
1. **for** $(j \leftarrow 1$ to $n{-}1)$ **do** *swap*$(\,A[j], A[$*random*$(j{+}1)]\,)$ `// shuffle`
2. *quick-select*(A, k)

- Shuffling (permuting) the input-array is (by assumption) equivalent to randomly choosing an input instance.
- So we know $T^{\text{avg}}_{\textit{quick-select}}(n) = T^{\text{exp}}_{\textit{shuffle-quick-select}}(n)$

(Recall: $T(\cdot)$ counts key-comparisons, so shuffling is free.)

Randomizing quick-select: Random Pivot

Second idea: Do the shuffling inside the recursion.
(Equivalently: Randomly choose which value is used for the pivot.)

```
randomized-quick-select(A, k)
    swap A[n−1] with A[random(n)]
    i ← partition(A, n−1)
    if i = k then return A[i]
    else if i > k then
        return randomized-quick-select(A[0 . . . i−1], k)
    else if i < k then
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    - **return** *randomized-quick-select*$(A[i{+}1 \ldots n{-}1], k - (i{+}1))$

- $T^{\text{exp}}_{\textit{rand.-quick-select}}(n) = T^{\text{exp}}_{\textit{shuffle-quick-select}}(n)$.

(This is not completely obvious, but believable. No proof.)

Expected run-time of *randomized-quick-select*

Let $T(A, k, R) = \#$ key-comparisons of *randomized-quick-select* on input $\langle A, k \rangle$ if the random outcomes are R.

- Write random outcomes R as $R = \langle i, R' \rangle$ (where 'i' stands for 'the first random number was such that the pivot-rank is i')
- Observe: $\Pr(\text{pivot-rank is } i) = \frac{1}{n}$
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- Recursive formula for one instance (and fixed $R = \langle i, R' \rangle$):

$$T(A, k, \langle i, R' \rangle) = n + \begin{cases} T(\text{ size-}i \text{ array}, k, R') & \text{if } i > k \\ T(\text{ size-}(n-i-1) \text{ array}, k-i-1, R') & \text{if } i < k \\ 0 & \text{otherwise} \end{cases}$$

Analysis of *randomized-quick-select*

Since the expected run-time uses the *worst-case instance*, the recursive formula can now be shown easily:

$$T^{\exp}(A,k) \left(\begin{aligned} &= \sum_R P(R)\cdot T(\langle A,k\rangle,R) = \sum_{i=0}^{n-1}\sum_{R'} P(i)\cdot P(R')\cdot T(\langle A,k\rangle,\langle i,R'\rangle) \\ &= \frac{1}{n}\sum_{i=0}^{k-1}\sum_{R'} P(R')\Big(n + T(\langle A'[i{+}1..n{-}1],k{-}i{-}1\rangle,R')\Big) \\ &\qquad + \underbrace{\frac{1}{n}\cdot n}_{i=k} \quad + \frac{1}{n}\sum_{i=k+1}^{n-1}\sum_{R'} P(R')\Big(n + T(\langle A'[0..i{-}1,k\rangle,R')\Big) \\ &= n + \frac{1}{n}\sum_{i=0}^{k-1}\sum_{R'} P(R')\,T(\langle A'[i{+}1..n{-}1],k{-}i{-}1\rangle,R') \\ &\qquad\qquad + \frac{1}{n}\sum_{i=k+1}^{n-1}\sum_{R'} P(R')\,T(\langle A'[0..i{-}1,k\rangle,R') \\ &= n + \frac{1}{n}\sum_{i=0}^{k-1}\underbrace{T^{\exp}(\langle A'[i{+}1..n{-}1],k{-}i{-}1\rangle)}_{\le T^{\exp}(n-i-1)} + \frac{1}{n}\sum_{i=k+1}^{n-1}\underbrace{T^{\exp}(\langle A'[0..i{-}1],k\rangle)}_{\le T^{\exp}(i)} \end{aligned} \right) \text{tedious but straightforward}$$

$$\le n + \frac{1}{n}\sum_{i=0}^{n-1}\max\{T^{\exp}(i),\, T^{\exp}(n{-}i{-}1)\}$$ *independent* of A, k

Analysis of *randomized-quick-select*

In summary, the expected run-time of *randomized-quick-select* satisfies:

$$T^{\exp}(n) \le n + \frac{1}{n}\sum_{i=0}^{n-1} \max\{\,T^{\exp}(i),\, T^{\exp}(n-i-1)\}$$

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Claim: This recursion resolves to $O(n)$.

Summary of Selection

- *randomized-quick-select* has expected run-time $\Theta(n)$.
  - The run-time bound is tight since *partition* takes $\Omega(n)$ time
  - If we're unlucky in the random numbers then the run-time is still $\Omega(n^2)$
- So the expected run-time of *shuffle-quick-select* is $\Theta(n)$.
- So the run-time of *quick-select* on randomly chosen input is $\Theta(n)$.
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- *randomized-quick-select* is generally the fastest solution to SELECTION.

- There exists a variation that solves SELECTION with worst-case run-time $\Theta(n)$, but it uses double recursion and is slower in practice. ($\rightarrow$ *cs*341, maybe)

Outline

3. Sorting, Average-case and Randomization
   - Analyzing average-case run-time
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   - Selection and *quick-select*
   - **Sorting and *quick-sort***
   - Lower Bound for Comparison-Based Sorting
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quick-sort

Hoare developed *partition* and *quick-select* in 1960.
He also used them to *sort* based on partitioning:

quick-sort(A)
A: array of size n
1. **if** $n \leq 1$ **then return**
2. $p \leftarrow$ *choose-pivot*(A)
3. $i \leftarrow$ *partition*(A, p)
4. *quick-sort*$(A[0, 1, \ldots, i{-}1])$
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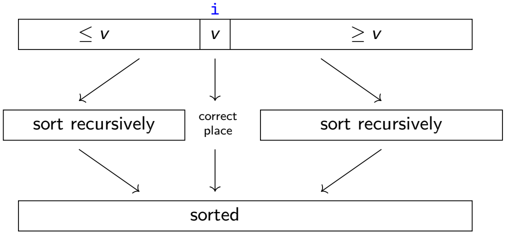

quick-sort analysis

Set $T(A) :=$ # of key-comparison for *quick-sort* in array A.

Worst-case run-time: $\Theta(n^2)$

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Average-case run-time? We again prove this via randomization.

Randomizing quick-sort

randomized-quick-sort(A)
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This implies

$$T^{\exp}(n) = \underbrace{\ldots \;=\; \ldots \;\leq\; \ldots}_{\text{long but straightforward}} = n + \frac{1}{n}\sum_{i=0}^{n-1}\Big(T^{\exp}(i) + T^{\exp}(n{-}i{-}1)\Big)$$

Expected run-time of *randomized-quick-sort*

$$T^{\exp}(n) \le n + \frac{1}{n}\sum_{i=0}^{n-1}\Big(T^{\exp}(i)+T^{\exp}(n-i-1)\Big) = n+\frac{2}{n}\sum_{i=1}^{n-1} T^{\exp}(i)$$

(since $T(0) = 0$)

Claim: $T^{\exp}(n) \in O(n \log n)$.

Proof:

Expected run-time of *randomized-quick-sort*

$$T^{\exp}(n) \le n + \frac{1}{n}\sum_{i=0}^{n-1}\Big(T^{\exp}(i) + T^{\exp}(n-i-1)\Big) = n + \frac{2}{n}\sum_{i=1}^{n-1} T^{\exp}(i)$$

(since $T(0) = 0$)

Claim: $T^{\exp}(n) \in O(n \log n)$.

Proof:

Summary of *quick-sort*

- *randomized-quick-sort* has expected run-time $\Theta(n \log n)$.
  - The run-time bound is tight since the best-case run-time is $\Omega(n \log n)$
  - If we're unlucky in the random numbers then the run-time is still $\Omega(n^2)$
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  - So $\Theta(n)$ auxiliary space (can be improved to $\Theta(\log n)$)
- There are numerous tricks to improve *randomized-quick-select*
- With these, this is in practice the fastest solution to SORTING (but *not* in theory).

quick-sort with tricks

```
randomized-quick-sort-improved(A, n)
Initialize a stack S of index-pairs with { (0, n−1) }
while S is not empty
    (ℓ, r) ← S.pop()                          // avoid recursions
    while (r−ℓ+1 > 10) do                     // stop recursions early
        p ← ℓ + random(ℓ−r+1)
        i ← Hoare-partition(A, ℓ, r, p)       // use better routine
        if (i−ℓ > r−i) do                     // reduce aux. space
            S.push( (ℓ, i−1) )
            ℓ ← i+1                           // remove tail-recursion
        else
            S.push( (i+1, r) )
            r ← i−1
insertion-sort(A)
```

Outline

3. Sorting, Average-case and Randomization
   - Analyzing average-case run-time
   - Randomized Algorithms
   - SELECTION and *quick-select*
   - SORTING and *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

Lower bounds for sorting

We have seen many sorting algorithms:

Sort	Running time	Analysis
selection-sort	$\Theta(n^2)$	worst-case
insertion-sort	$\Theta(n^2)$	worst-case
	$\Theta(n)$	best-case
merge-sort	$\Theta(n \log n)$	worst-case
heap-sort	$\Theta(n \log n)$	worst-case
quick-sort	$\Theta(n \log n)$	average-case
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Question: Can one do better than $\Theta(n \log n)$ running time?
Answer: Yes and no! *It depends on what we allow*.

- No: Comparison-based sorting lower bound is $\Omega(n \log n)$.
- Yes: Non-comparison-based sorting can achieve $O(n)$ (under restrictions!). ($\rightarrow$ later)

Lower bound for sorting in the comparison model

All algorithms so far are **comparison-based**: Data is accessed only by

- comparing two elements (a *key-comparison*)
- moving elements around (e.g. copying, swapping)

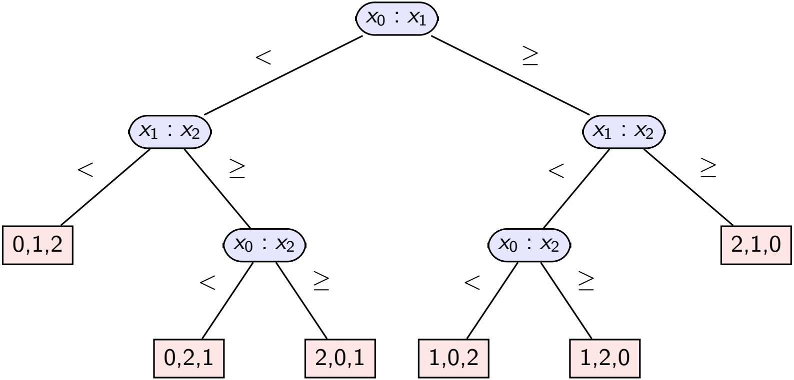
Theorem. Any *comparison-based* sorting algorithm requires in the worst case $\Omega(n \log n)$ comparisons to sort n distinct items.

Proof.

Decision trees

Any comparison-based algorithms can be expressed as **decision tree**.

To sort $\{x_0, x_1, x_2\}$:



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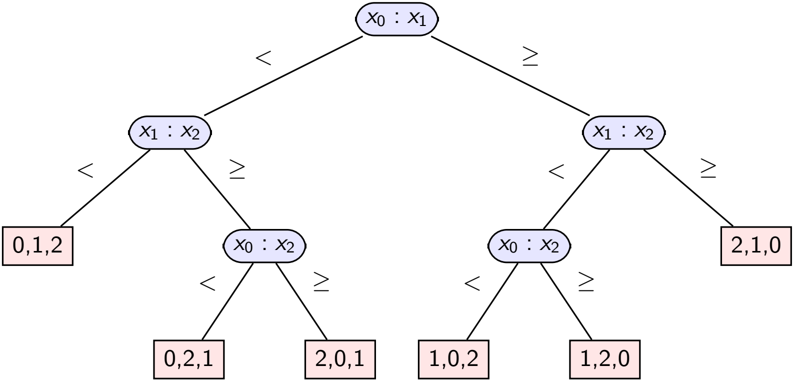
To sort $\{x_0, x_1, x_2\}$:

Example: $\{x_0{=}4, x_1{=}2, x_2{=}7\}$



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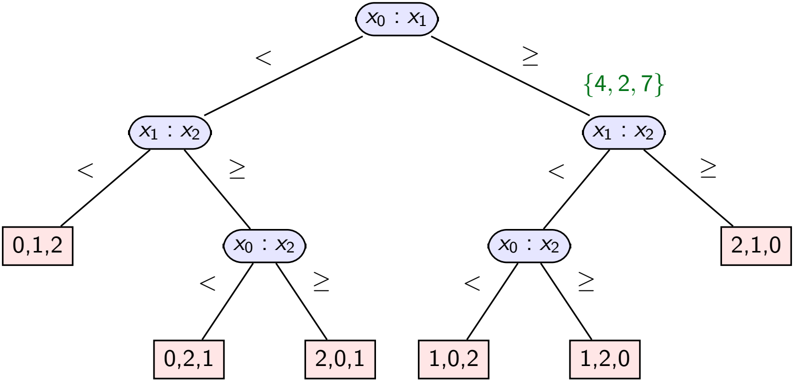

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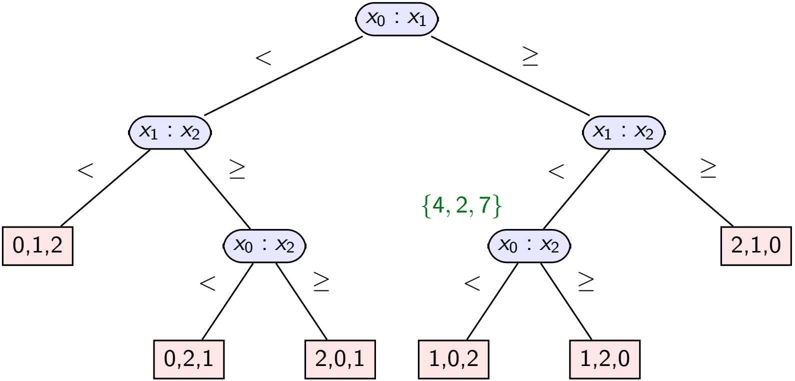

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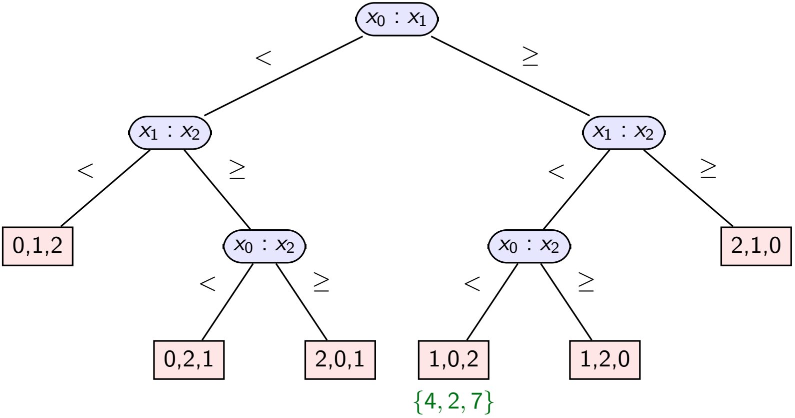

Output: $\{4, 2, 7\}$ has sorting permutation $\langle 1, 0, 2 \rangle$

(i.e., $x_1{=}2 \leq x_0{=}4 \leq x_2{=}7$)

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Non-Comparison-Based Sorting

- Assume keys are numbers in base R (R: **radix**)
  - So all digits are in $\{0, \ldots, R-1\}$
  - $R = 2, 10, 128, 256$ are the most common, but R need not be constant

Example ($R = 4$):

123	230	21	320	210	232	101

- Assume all keys have the same number w of digits.
  - Can achieve after padding with leading 0s.
  - In typical computers, $w = 32$ or $w = 64$, but w need not be constant

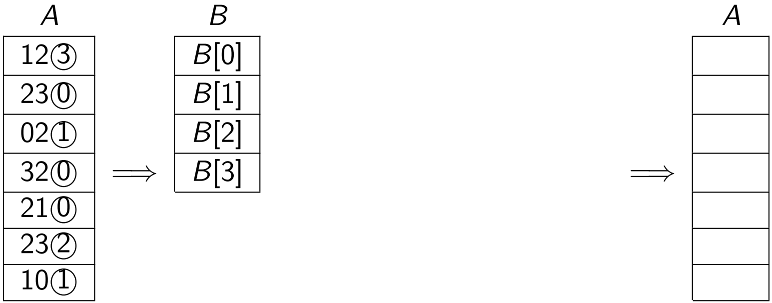
Example ($R = 4$):

123	230	021	320	210	232	101

- Can sort based on individual digits.
  - How to sort 1-digit numbers?
  - How to sort multi-digit numbers based on this?

(Single-digit) *bucket-sort*

Sort array A by last digit:



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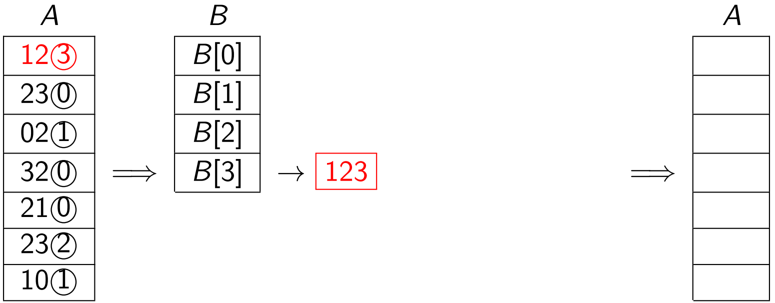

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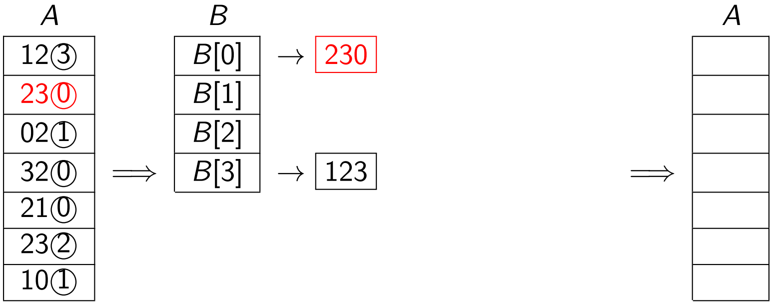

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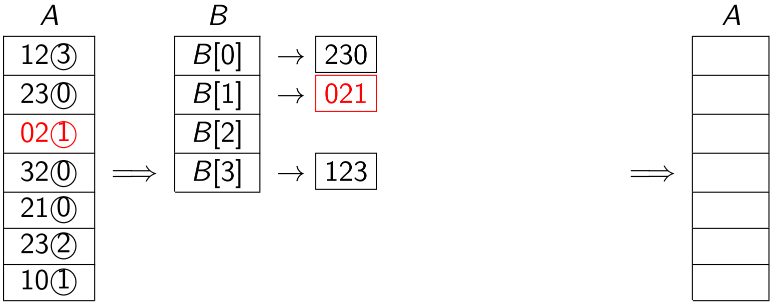

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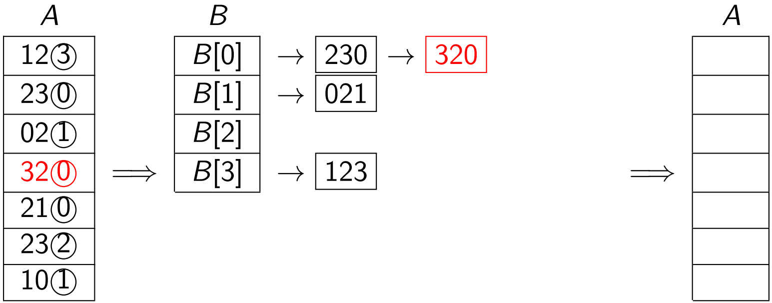

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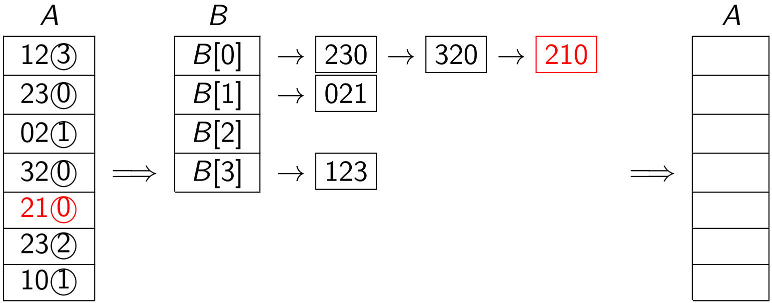

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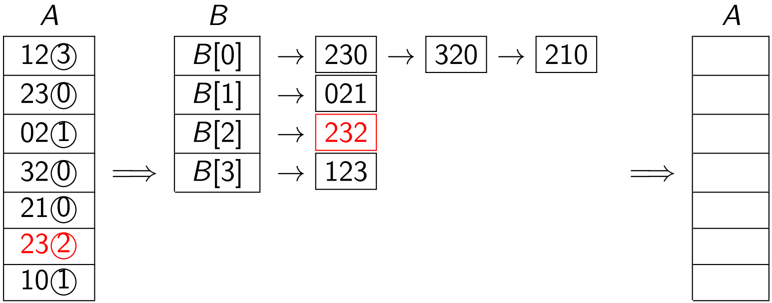

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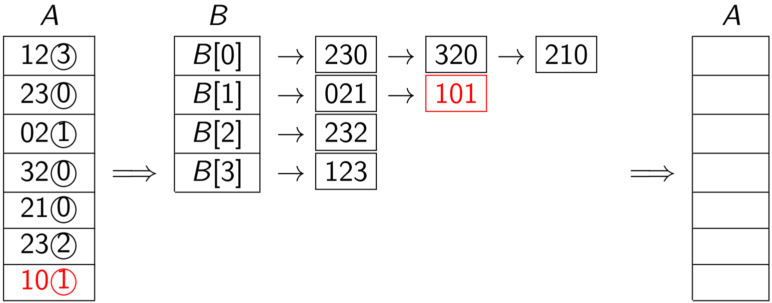

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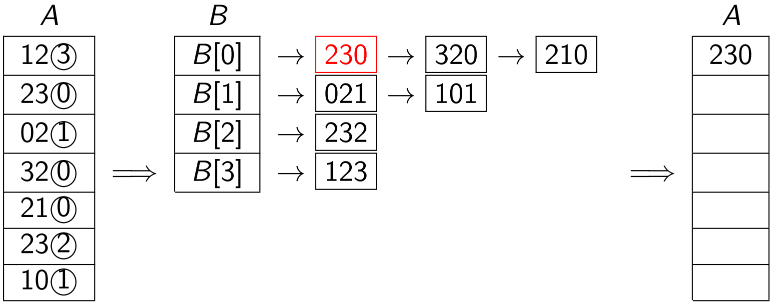

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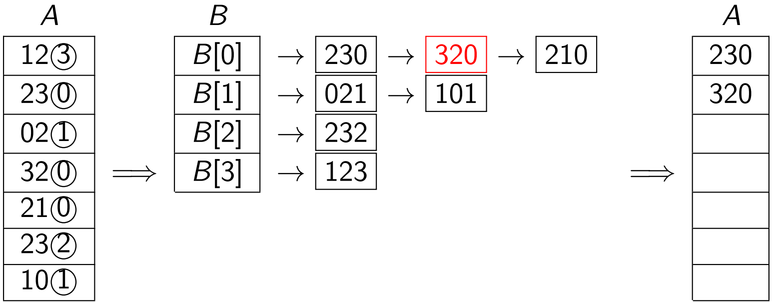

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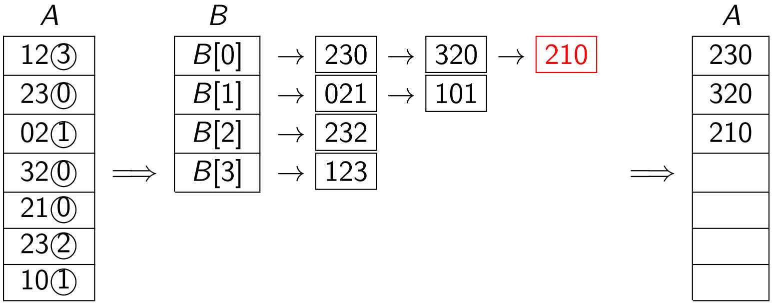

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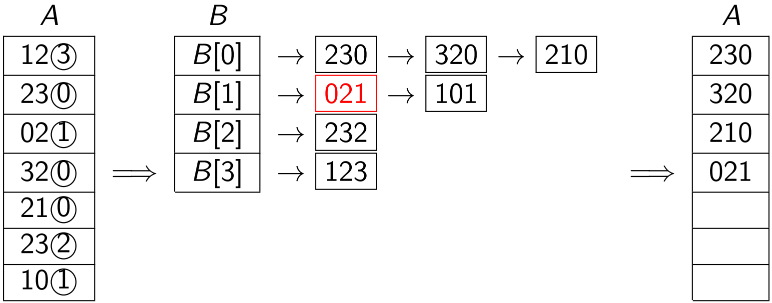

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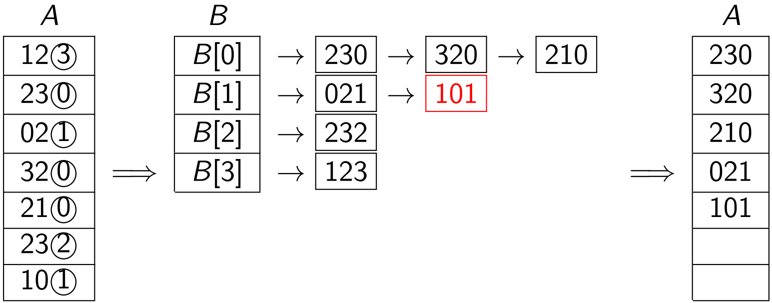

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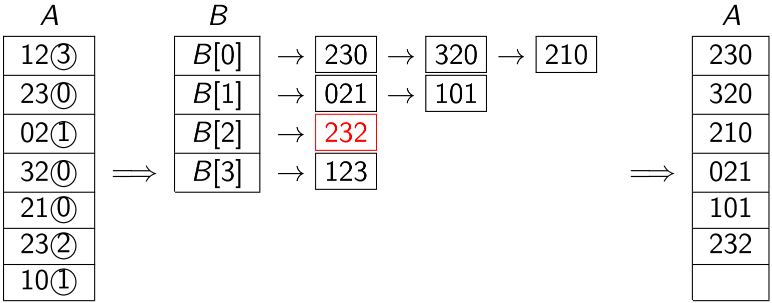

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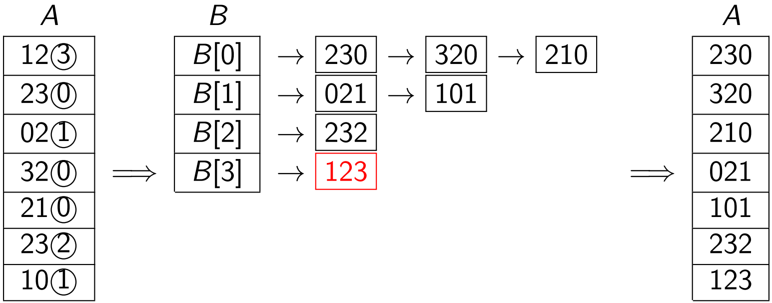

(Single-digit) *bucket-sort*

bucket-sort$(A, n,$ *sort-key*$(\cdot))$

A: array of size n

sort-key$(\cdot)$: maps items of A to $\{0, \ldots, R-1\}$

```
Initialize an array B[0...R − 1] of empty queues (buckets)
for i ← 0 to n−1 do
    Append A[i] at end of B[sort-key(A[i])]
i ← 0
for j ← 0 to R − 1 do
    while B[j] is non-empty do
        move front element of B[j] to A[i++]
```

- In our example *sort-key*$(A[i])$ returns the last digit of $A[i]$

(Single-digit) *bucket-sort*

```
bucket-sort(A, n, sort-key(·))
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- In our example *sort-key*($A[i]$) returns the last digit of $A[i]$
- *bucket-sort* is **stable**: equal items stay in original order.
- Run-time $\Theta(n + R)$, auxiliary space $\Theta(n + R)$
- It is possible to replace the lists by arrays $\rightsquigarrow$ *count-sort* (no details).

Most-significant-digit(MSD)-radix-sort

Sort array of w-digit radix-R numbers recursively:
sort by 1st digit, then each group by 2nd digit, etc.

①23
②32
⓪21
③20
②10
②30
①01

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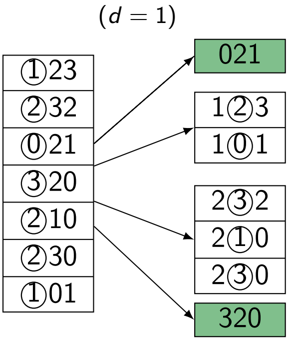

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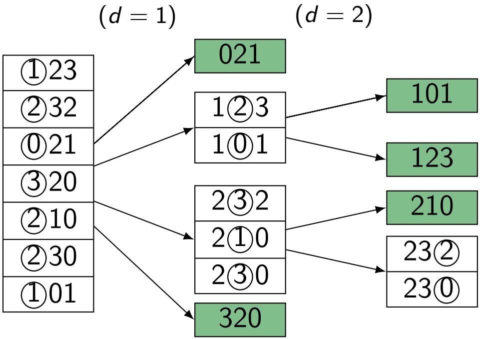

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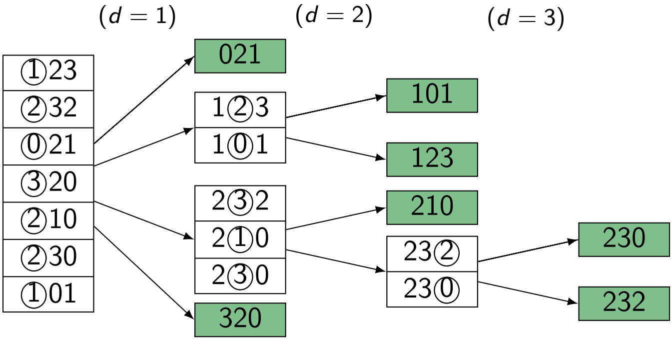

MSD-radix-sort

MSD-radix-sort$(A, n, \quad d \leftarrow 1)$
A: array of size n, contains w-digit radix-R numbers

if $(d \leq w$ and $(n > 1))$
    bucket-sort(A, n,'return dth digit of $A[i]$')
    $\ell \leftarrow 0$ `// find sub-arrays and recurse`
    for $j \leftarrow 0$ to $R - 1$
        Let $r \geq \ell - 1$ be maximal s.t. $A[\ell..r]$ have dth digit j
        MSD-radix-sort$(A[\ell..r], r-\ell+1, d+1)$
        $\ell \leftarrow r + 1$

Analysis:

- $\Theta(w)$ levels of recursion in worst-case.
- $\Theta(n)$ subproblems on most levels in worst-case.
- $\Theta(R + \text{(size of sub-array)})$ time for each *bucket-sort* call.

$\Rightarrow$ Run-time $\Theta(wnR)$ — slow. Many recursions and allocated arrays.

Least-significant-digit(LSD)-radix-sort

LSD-radix-sort(A, n)
A: array of size n, contains m-digit radix-R numbers
```
for d ← least significant to most significant digit do
        bucket-sort(A, n, 'return dth digit of A[i]')
```

	$(d = 3)$		$(d = 2)$		$(d = 1)$	
12③	$\Longrightarrow$	2③0	$\Longrightarrow$	①01	$\Longrightarrow$	021
23⓪		3②0		②10		101
02①		2①0		③20		123
32⓪		0②1		⓪21		210
21⓪		1⓪1		①23		230
23②		2③2		②30		232
10①		1②3		②32		320

- Loop-invariant: A is sorted w.r.t. digits $d, \ldots, w$ of each entry.
- **Time cost**: $\Theta(w(n + R))$ **Auxiliary space**: $\Theta(n + R)$

Summary

- SORTING is an important and *very* well-studied problem
- Can be done in $\Theta(n \log n)$ time; faster is not possible for general input
- *heap-sort* is the only $\Theta(n \log n)$-time algorithm we have seen with $O(1)$ auxiliary space.
- *merge-sort* is also $\Theta(n \log n)$, selection & insertion sorts are $\Theta(n^2)$.
- *quick-sort* is worst-case $\Theta(n^2)$, but often the fastest in practice
- *bucket-sort* and *radix-sort* achieve $o(n \log n)$ if the input is special

- Randomized algorithms can eliminate "bad cases"
- Best-case, worst-case, average-case can all differ.
- Often it is easier to analyze the run-time on randomly chosen input rather than the average-case run-time.