CS 240 – Data Structures and Data Management

Module 7: Dictionaries via Hashing

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Based on lecture notes by many previous cs240 instructors

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Outline

7 Dictionaries via Hashing
- Hashing Introduction
- Hashing with Chaining
- Probe Sequences
- Cuckoo hashing
- Hash Function Strategies

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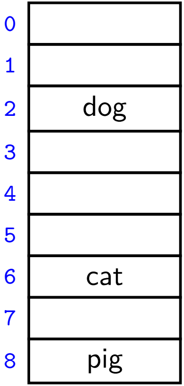
Direct Addressing

Special situation: For a known $M \in \mathbb{N}$, every key k is an integer with $0 \leq k < M$.

We can then implement a dictionary easily: Use an array A of size M that stores (k, v) via $A[k] \leftarrow v$.



0	
1	
2	dog
3	
4	
5	
6	cat
7	
8	pig

- *search*(k): Check whether $A[k]$ is `NULL`
- *insert*(k, v): $A[k] \leftarrow v$
- *delete*(k): $A[k] \leftarrow$ `NULL`

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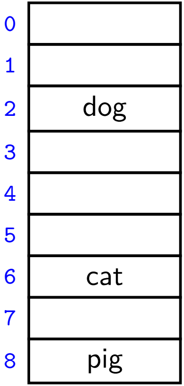

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Total space is $\Theta(M)$.

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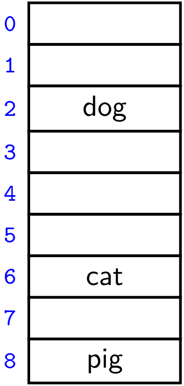

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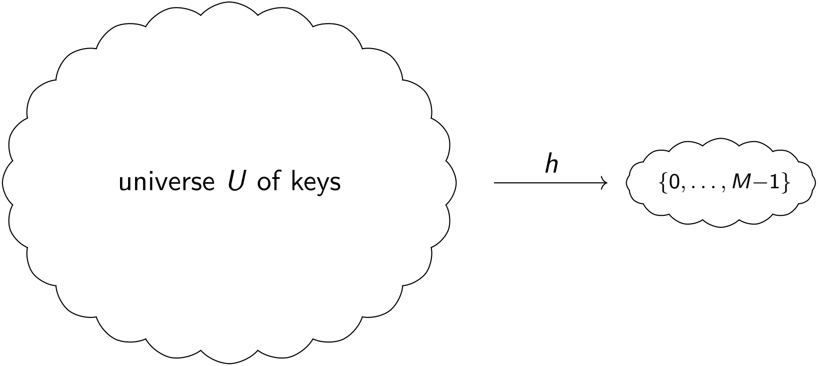
What sorting algorithm does this remind you of?
Bucket Sort

Hashing Idea

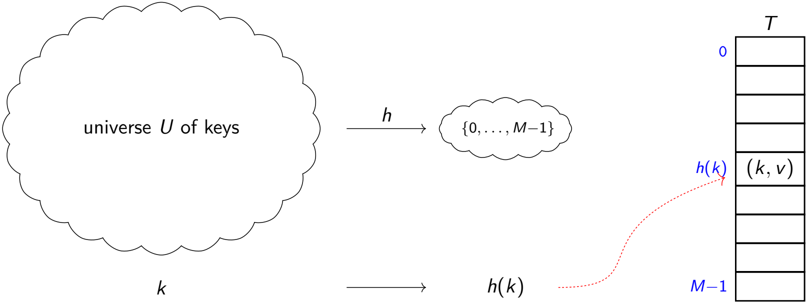
Two disadvantages of direct addressing:

- It cannot be used if the keys are not integers.
- It wastes space if M is unknown or $n \ll M$.

Hashing idea: Map (arbitrary) keys to integers in range $\{0, \ldots, M-1\}$ (for an integer M of our choice), then use direct addressing.



Hashing Details



- **Assumption:** We know that all keys come from some **universe** U. (Typically U = non-negative integers, sometimes $|U|$ finite.)
- We pick a **table-size** M.
- We pick a **hash function** $h : U \to \{0, 1, \ldots, M-1\}$. (Commonly used: $h(k) = k \bmod M$. We will see other choices later.)

- Store dictionary in **hash table**, i.e., an array T of size M.
- An item with key k wants to be stored in **slot** $h(k)$, i.e., at $T[h(k)]$.

Hashing example

$U = \mathbb{N}$, $M = 11$, $\qquad h(k) = k \bmod 11$.

The hash table stores keys 7, 13, 43, 45, 49, 92. (Values are not shown).

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Collisions

- Generally hash function h is not injective, so many keys can map to the same integer.
    - For example, $h(46) = 2 = h(13)$ if $h(k) = k \bmod 11$.
- We get **collisions**: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.

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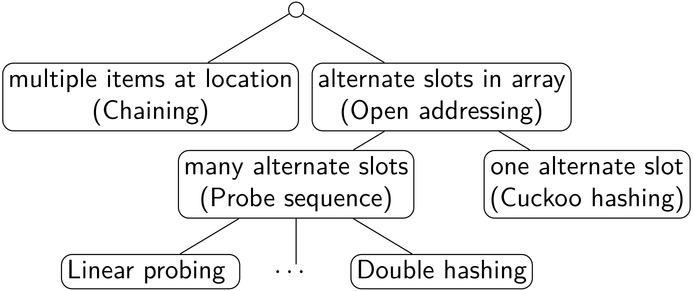
- We get **collisions**: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.
- There are many strategies to resolve collisions:



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7. Dictionaries via Hashing
   - Hashing Introduction
   - **Hashing with Chaining**
   - Probe Sequences
   - Cuckoo hashing
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Hashing with Chaining

Simplest collision-resolution strategy: Each slot stores a **bucket** containing 0 or more KVPs.

- A bucket could be implemented by any dictionary realization (even another hash table!).
- The simplest approach is to use unsorted lists with MTF for buckets. This is called collision resolution by **chaining**.

Hashing with Chaining

Simplest collision-resolution strategy: Each slot stores a **bucket** containing 0 or more KVPs.

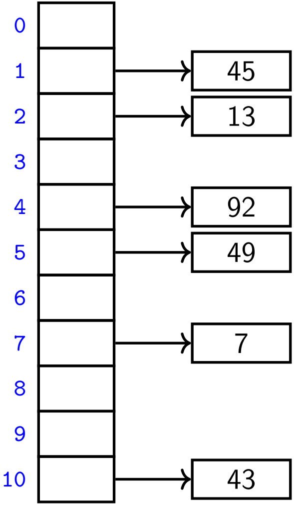
- A bucket could be implemented by any dictionary realization (even another hash table!).
- The simplest approach is to use unsorted lists with MTF for buckets. This is called collision resolution by **chaining**.

- *insert*(k, v): Add (k, v) to the front of the list at $T[h(k)]$.
- *search*(k): Look for key k in the list at $T[h(k)]$.
  Apply MTF-heuristic!
- *delete*(k): Perform a search, then delete from the linked list.

insert takes time $O(1)$.
search and *delete* have run-time $O(1 + \text{length of list at } T(h(k)))$.

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

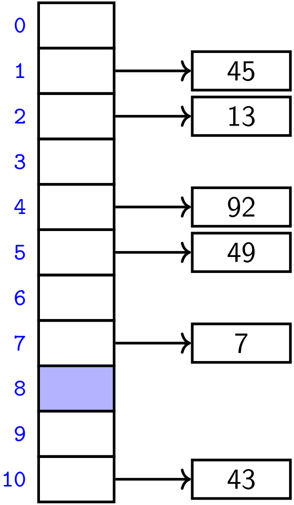


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(41)

$h(41) = 8$



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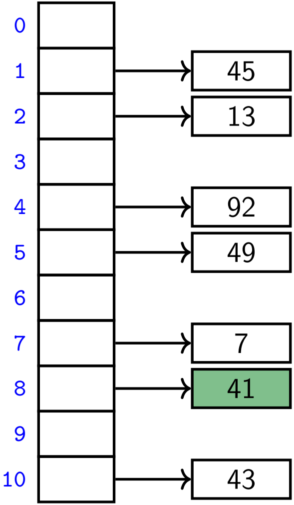
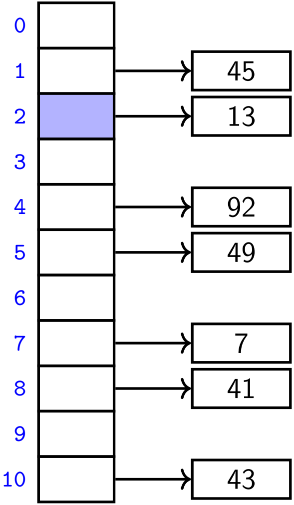


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(46)

$h(46) = 2$



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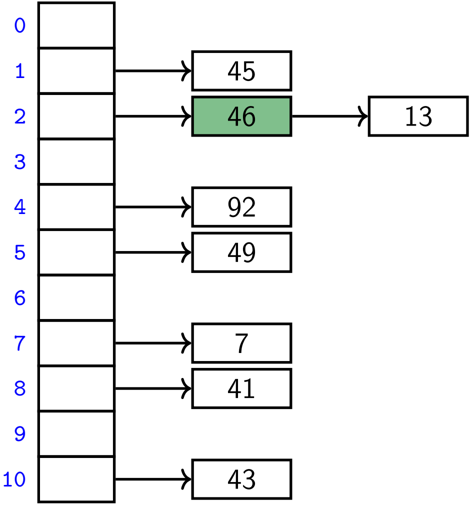


insert(46)

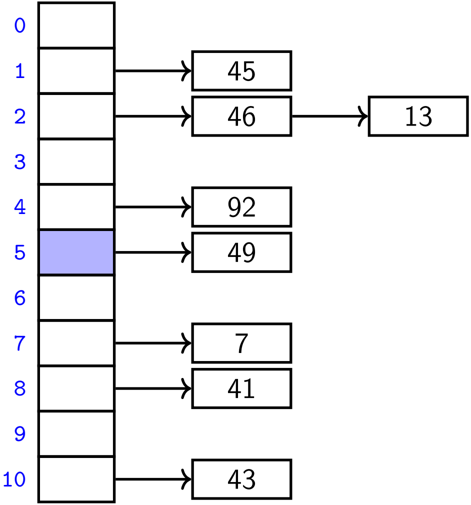
$h(46) = 2$

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(16)

$h(16) = 5$



Chaining example

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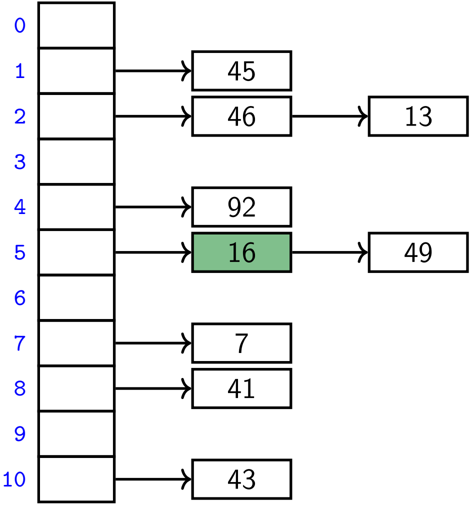
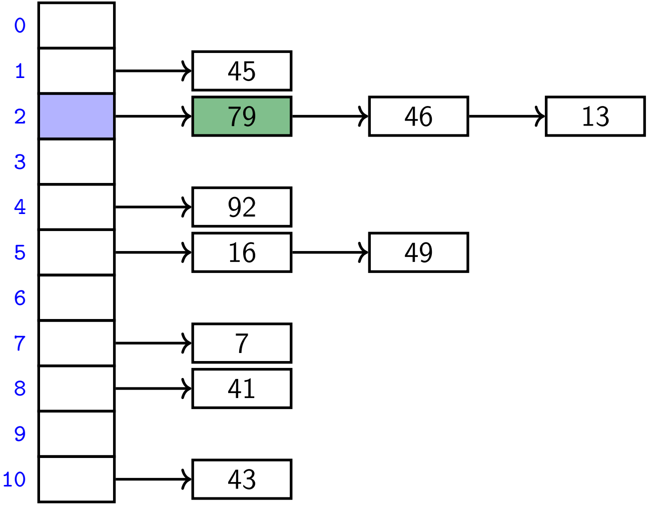


insert(16)

$h(16) = 5$

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$



insert(79)

$h(79) = 2$

Complexity of chaining

Run-times: *insert* takes time $\Theta(1)$.
search and *delete* have run-time $\Theta\Big(1 + \text{size of bucket } T[h(k)]\Big)$.

- The *average* bucket-size is $\frac{n}{M} =: \alpha$.
  (α is also called the **load factor**.)

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- The *average* bucket-size is $\frac{n}{M} =: \alpha$.
  (α is also called the **load factor**.)

- However, this does not imply that the *average-case* cost of *search* and *delete* is $\Theta(1+\alpha)$.
  - Consider the case where all keys hash to the same slot
  - The average bucket-size is still α
  - But the operations take $\Theta(n)$ time on average

- To get meaningful average-case bounds, we need some assumptions on the hash-functions and the keys!

Complexity of chaining

- To analyze what happens ‘on average’, switch to *randomized* hashing.
- How can we randomize?

Complexity of chaining

- To analyze what happens 'on average', switch to *randomized* hashing.
- How can we randomize?
  Assume that the *hash-function* is chosen randomly.
  - We will later see examples how to do this.
- To be able to analyze, we assume the following:

Uniform Hashing Assumption*: Any possible hash-function is equally likely to be chosen as hash-function.*

(This is not at all realistic, but the assumption makes analysis possible.)

Complexity of chaining

UHA implies that the distribution of keys is unimportant.

- **Claim 1:** Hash-values are uniform.
  Formally: $P(h(k) = i) = \frac{1}{M}$ for any key k and slot i.

- **Claim 2:** Hash-values of any two keys are independent of each other.

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Back to complexity of chaining:

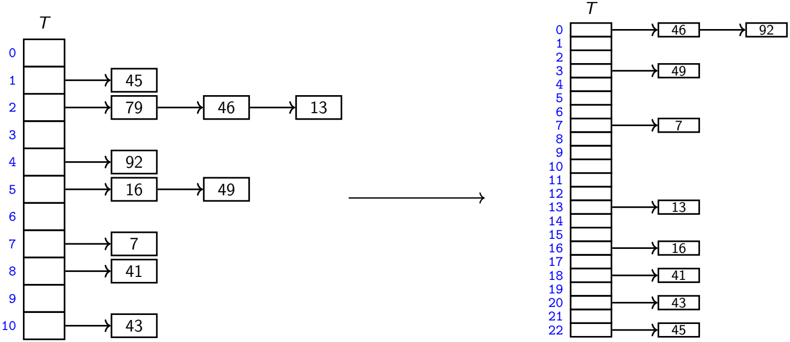
- Each bucket has expected length $\frac{n}{M} \le \alpha$
  - n other keys are in this slot with probability $\frac{1}{M}$
- Each key in dictionary is expected to collide with $\frac{n-1}{M}$ other keys
  - $n-1$ other keys are in same slot with probability $\frac{1}{M}$
- Expected cost of *search* and *delete* is hence $\Theta(1+\alpha)$

Load factor and re-hashing

- For hashing with chaining (and also other collision resolution strategies), the run-time bound depends on α

  (Recall: *load factor* $\alpha = n/M$.)

- We keep the load factor small by **rehashing** when needed:



  - Keep track of n and M throughout operations
  - If α gets too large, create new (roughly twice as big) hash-table, new hash-function(s) and re-insert all items in the new table.

Hashing with Chaining summary

- For Hashing with Chaining: Rehash so that $\alpha \in \Theta(1)$ throughout
- Rehashing costs $\Theta(M + n)$ time (plus the time to find a new hash function).
- Rehashing happens rarely enough that we can ignore this term when amortizing over all operations.
- We should also re-hash when α gets too small, so that $M \in \Theta(n)$ throughout, and the space is always $\Theta(n)$.

Summary: The amortized expected cost for hashing with chaining is $O(1)$ and the space is $\Theta(n)$

(assuming uniform hashing and $\alpha \in \Theta(1)$ throughout)

Theoretically perfect, but too slow in practice.

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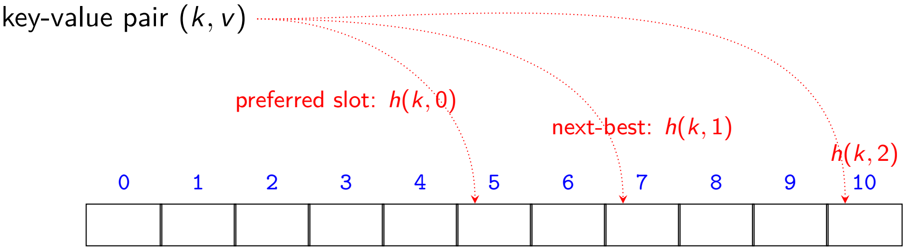
Open addressing

Main idea: Avoid the links needed for chaining by permitting only one item per slot, but allowing a key k to be in multiple slots.

search and *insert* follow a **probe sequence** of possible locations for key k: $\langle h(k,0), h(k,1), h(k,2), \ldots h(k, M-1)\rangle$ until an empty spot is found.



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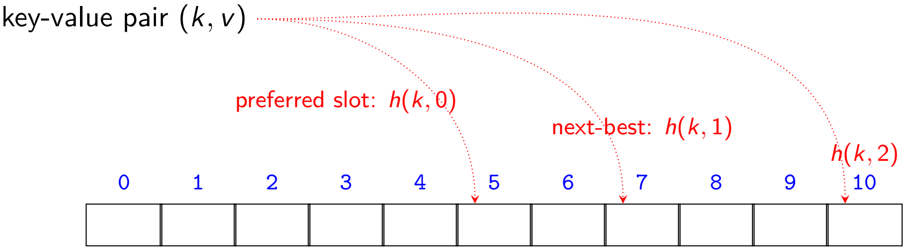


Simplest method for open addressing: *linear probing*
$h(k, j) = (h(k) + j) \bmod M$, for some hash function h.

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

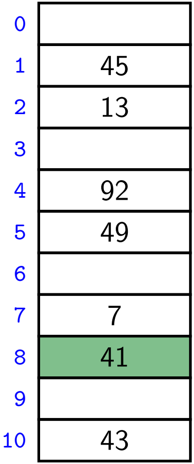
0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k)+j) \bmod 11.$

insert(41)

$h(41,0) = 8$



0	
1	45
2	13
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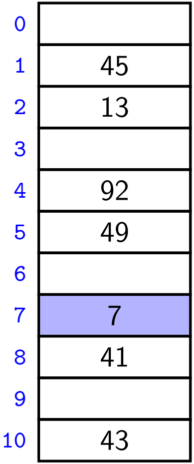


0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

insert(84)

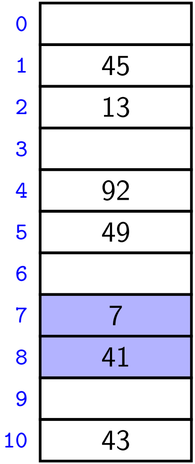
$h(84,0) = 7$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k)+j) \bmod 11.$

insert(84)

$h(84,1) = 8$



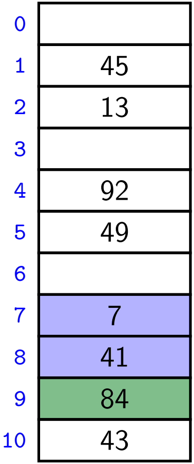
0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k)+j) \bmod 11.$

insert(84)

$h(84, 2) = 9$



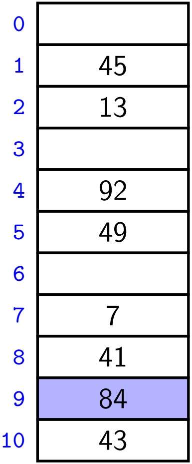
0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k)+j) \bmod 11.$

insert(20)

$h(20,0) = 9$



0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

Linear probing example

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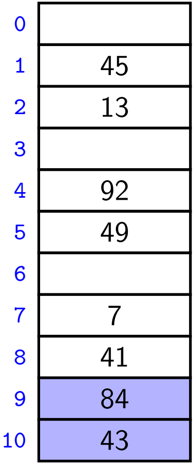
insert(20)

$h(20,1) = 10$



0	
1	45
2	13
3	
4	92
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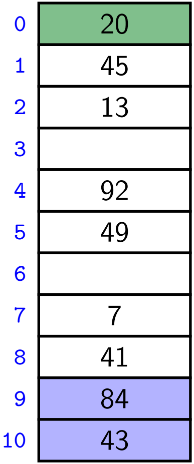
6	
7	7
8	41
9	84
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k)+j) \bmod 11.$

insert(20)

$h(20,2) = 0$



0	20
1	45
2	13
3	
4	92
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Probe sequence operations

delete becomes problematic:

- Cannot leave an empty spot behind; the next search might otherwise not go far enough.

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  - Mark spot as *deleted* (rather than `NULL`)
  - Search continues past deleted spots.
  - Insertion reuses deleted spots.

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- Better idea: **lazy deletion**:
    - Mark spot as *deleted* (rather than `NULL`)
    - Search continues past deleted spots.
    - Insertion reuses deleted spots.

  Keep track of how many items are 'deleted' and re-hash (to keep space at $\Theta(n)$) if there are too many.

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

delete(43)
$h(43, 0) = 10$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

search(63)
$h(63, 0) = 8$

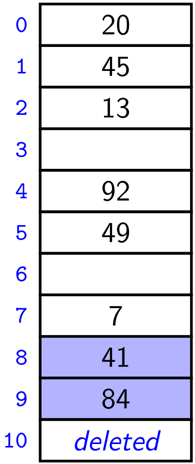
0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

search(63)

$h(63,1) = 9$



0	20
1	45
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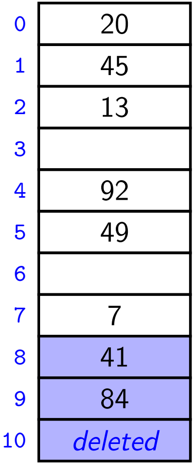
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11$, $h(k) = k \bmod 11$, $h(k,j) = (h(k)+j) \bmod 11$.

search(63)
$h(63,2) = 10$



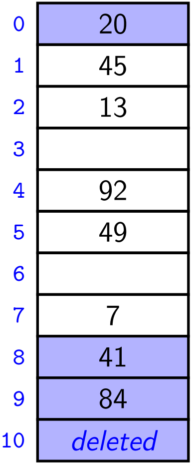
0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

search(63)

$h(63,3) = 0$

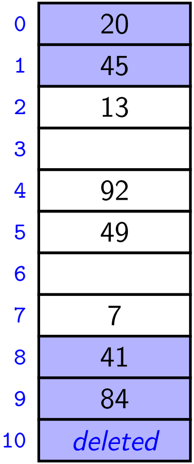


0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

search(63)
$h(63,4) = 1$

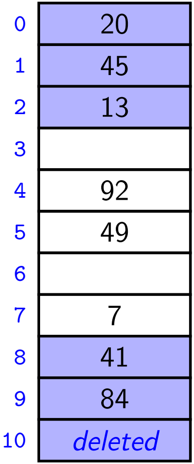


0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k)+j) \bmod 11.$

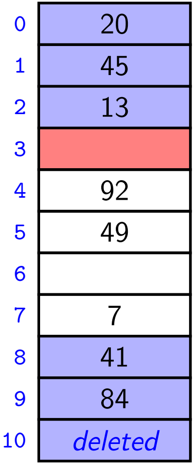
search(63)
$h(63,5) = 2$



0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$



0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	*deleted*

search(63)
$h(63, 6) = 3$
not found

Probe sequence operations

```
probe-sequence::insert($T, (k, v)$)
for ($j = 0; j < M; j$++)
    if $T[h(k, j)]$ is NULL or “deleted”
        $T[h(k, j)] = (k, v)$
        return “success”
return “failure to insert”      // need to re-hash
```

```
probe-sequence-search($T, k$)
for ($j = 0; j < M; j$++)
    if $T[h(k, j)]$ is NULL return “item not found”
    if $T[h(k, j)]$ has key $k$ return $T[h(k, j)]$
    // key is incorrect or "deleted"
    // try next probe, i.e., continue for-loop
return “item not found”
```

Independent hash functions

- Some hashing methods require *two* hash functions h_0, h_1.
- These hash functions should be *independent* in the sense that the random variables $P(h_0(k) = i)$ and $P(h_1(k) = j)$ are independent.
- Using two modular hash-functions often leads to dependencies.

Independent hash functions

- Some hashing methods require *two* hash functions h_0, h_1.
- These hash functions should be *independent* in the sense that the random variables $P(h_0(k) = i)$ and $P(h_1(k) = j)$ are independent.
- Using two modular hash-functions often leads to dependencies.
- Better idea: Use *multiplication method* for second hash function:
    - Fix some floating-point number A with $0 < A < 1$

$$h(k) = \underbrace{\left\lfloor M \cdot (\underbrace{\underbrace{A \cdot k}_{\text{multiply}} - \underbrace{\lfloor A \cdot k \rfloor}_{\text{integral part}}}_{\text{fractional part, in } [0,1)}) \right\rfloor}_{\text{integer in } [0,M)}.$$

    - Our examples use $\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618033988749....$ as A.

Double Hashing

- Assume we have two hash independent functions h_0, h_1.
- Assume further that $h_1(k) \neq 0$ and that $h_1(k)$ is relative prime with the table-size M for all keys k.
  - Choose M prime.
  - Modify standard hash-functions to ensure $h_1(k) \neq 0$
    E.g. modified multiplication method: $h(k) = 1 + \lfloor (M-1)(kA - \lfloor kA \rfloor) \rfloor$
- **Double hashing**: open addressing with probe sequence

$$h(k, j) = \big(h_0(k) + j \cdot h_1(k)\big) \bmod M$$

- *search*, *insert*, *delete* work just like for linear probing, but with this different probe sequence.

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

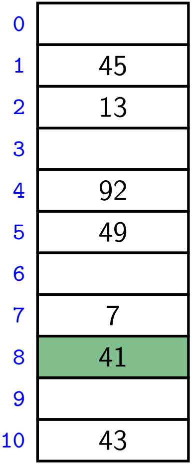
Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(41)

$h_0(41) = 8$

$h(41, 0) = 8$



0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

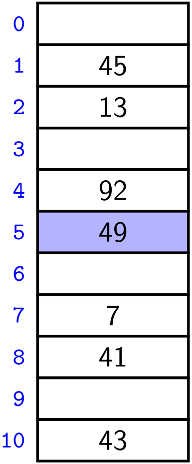
insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

$h_1(194) = 9$

$h(194, 1) = 5$



0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(194)

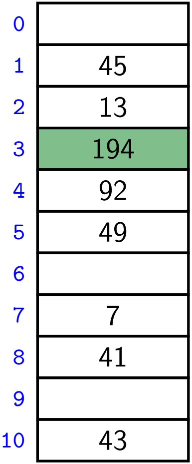
$h_0(194) = 7$

$h(194, 0) = 7$

$h_1(194) = 9$

$h(194, 1) = 5$

$h(194, 2) = 3$



0	
1	45
2	13
3	194
4	92
5	49
6	
7	7
8	41
9	
10	43

Outline

7. Dictionaries via Hashing
    - Hashing Introduction
    - Hashing with Chaining
    - Probe Sequences
    - Cuckoo hashing
    - Hash Function Strategies

Cuckoo hashing

We use two independent hash functions h_0, h_1 and two tables T_0, T_1.

Main idea: An item with key k can *only* be at $T_0[h_0(k)]$ or $T_1[h_1(k)]$.

search and *delete* then *always* take constant time.

	T_0
0	44
1	
2	
3	
4	59
5	
6	
7	51
8	
9	
10	

	T_1
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	92
10	

Cuckoo Hashing Insertion

insert *always* initially puts the new item into $T_0[h_0(k)]$

- Evict item that may have been there already.
- If so, evicted item inserted at alternate position
- This may lead to a loop of evictions.
  - **Can show:** If insertion is possible, then there are at most $2n$ evictions.
  - So abort after too many attempts.

```
cuckoo::insert(k, v)
(k_insert, v_insert) ← new key-value pair with (k, v)
i ← 0
do at most 2n times:
    (k_evict, v_evict) ← T_i[h_i(k_insert)]            // save old KVP
    T_i[h_i(k_insert)] ← (k_insert, v_insert)          // put in new KVP
    if (k_evict, v_evict) is NULL return "success"
    else                                               // repeat in other table
        (k_insert, v_insert) ← (k_evict, v_evict); i ← 1 − i
return "failure to insert"                             // need to re-hash
```

Cuckoo hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

T_0

0	44
1	
2	
3	
4	59
5	
6	
7	
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	92
10	

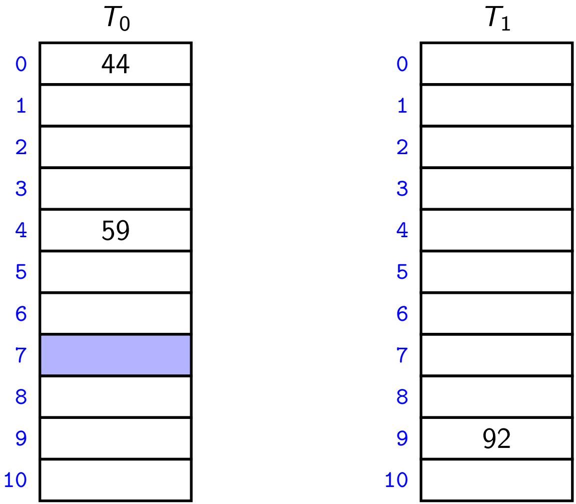
Cuckoo hashing example

$M = 11,$ $h_0(k) = k \bmod 11,$ $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(51)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$



	T_0
0	44
1	
2	
3	
4	59
5	
6	
7	
8	
9	
10	

	T_1
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	92
10	

Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(51)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$

	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		
6		
7	51	
8		
9		92
10		

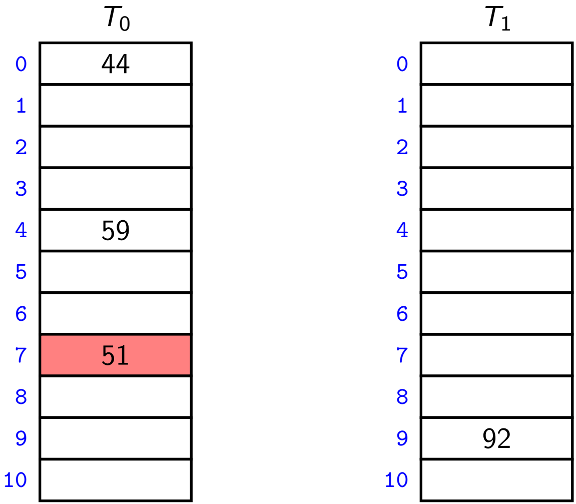
Cuckoo hashing example

$M = 11,$ $h_0(k) = k \bmod 11,$ $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 0$
$k = 95$

$h_0(k) = 7$
$h_1(k) = 7$



	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		
6		
7	51	
8		
9		92
10		

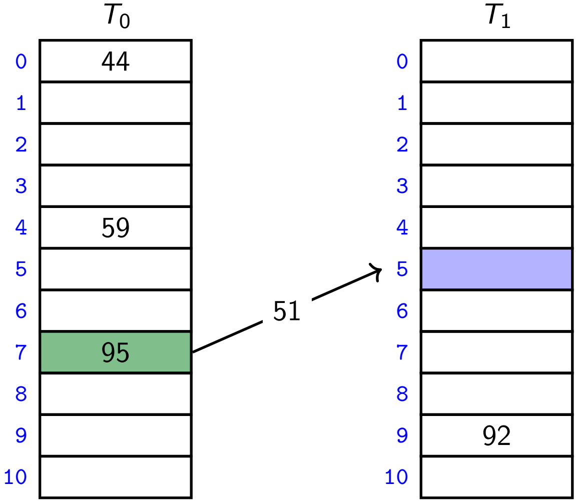
Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 1$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$



Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 1$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$

	T_0
0	44
1	
2	
3	
4	59
5	
6	
7	95
8	
9	
10	

	T_1
0	
1	
2	
3	
4	
5	51
6	
7	
8	
9	92
10	

Cuckoo hashing example

$M = 11,$ $h_0(k) = k \bmod 11,$ $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 0$
$k = 26$

$h_0(k) = 4$
$h_1(k) = 0$

T_0

0	44
1	
2	
3	
4	59
5	
6	
7	95
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	51
6	
7	
8	
9	92
10	

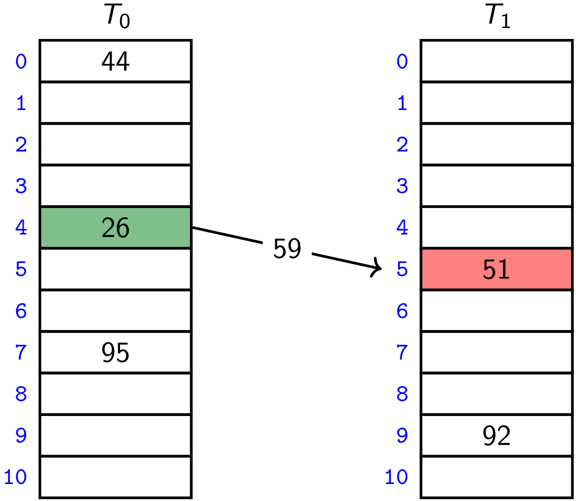
Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 1$
$k = 59$

$h_0(k) = 4$
$h_1(k) = 5$



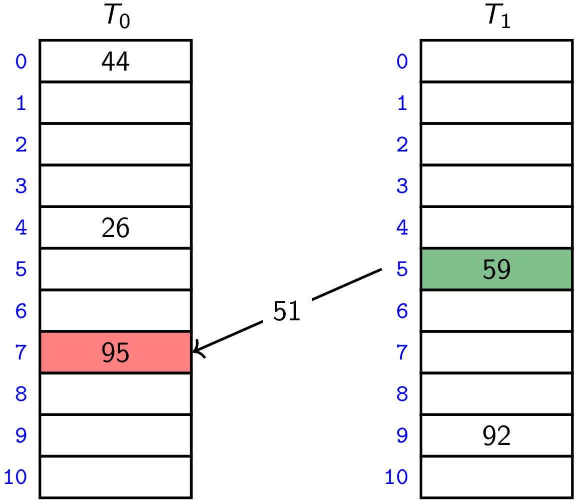
Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$



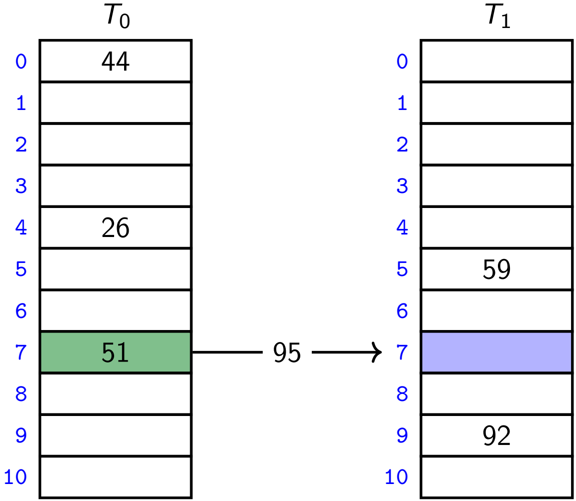
Cuckoo hashing example

$M = 11,$ $\quad h_0(k) = k \bmod 11,$ $\quad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 1$
$k = 95$

$h_0(k) = 4$
$h_1(k) = 7$



Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 1$
$k = 95$

$h_0(k) = 4$
$h_1(k) = 7$

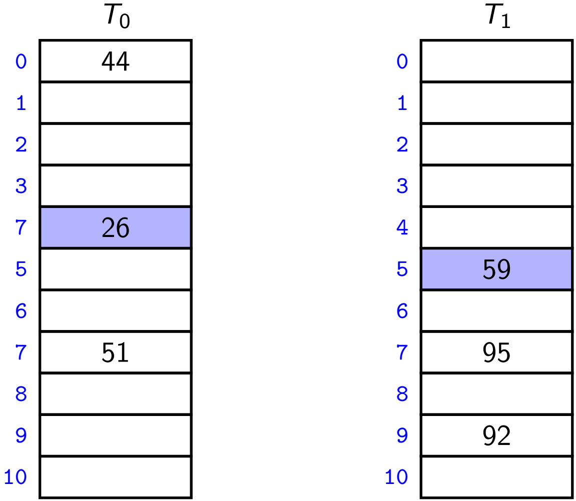
	T_0	T_1
0	44	
1		
2		
3		
4	26	
5		59
6		
7	51	95
8		
9		92
10		

Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

search(59)

$h_0(59) = 4$
$h_1(59) = 5$



	T_0
0	44
1	
2	
3	
7	26
5	
6	
7	51
8	
9	
10	

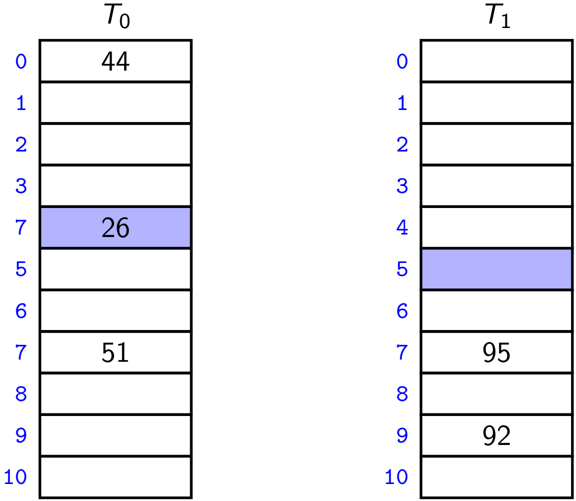
	T_1
0	
1	
2	
3	
4	
5	59
6	
7	95
8	
9	92
10	

Cuckoo hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

delete(59)

$h_0(59) = 4$
$h_1(59) = 5$



	T_0
0	44
1	
2	
3	
7	26
5	
6	
7	51
8	
9	
10	

	T_1
0	
1	
2	
3	
4	
5	
6	
7	95
8	
9	92
10	

Cuckoo hashing discussions

- **Can show**: expected number of evictions during *insert* is $O(1)$.
    - So in practice, stop evictions much earlier than $2n$ rounds.
- This crucially requires load factor $\alpha < \frac{1}{2}$.
    - Here $\alpha = n/(\text{size of } T_0 + \text{size of } T_1)$
- So cuckoo hashing is wasteful on space.
- In fact, space is $\omega(n)$ if *insert* forces lots of re-hashing.
- **Can show**: expected space is $O(n)$.

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- In fact, space is $\omega(n)$ if *insert* forces lots of re-hashing.
- **Can show**: expected space is $O(n)$.

There are many possible variations:

- The two hash-tables could be combined into one.
- Be more flexible when inserting: Always consider both possible positions.
- Use $k > 2$ allowed locations (i.e., k hash-functions).

Complexity of open addressing strategies

For any open addressing scheme, we *must* have $\alpha \leq 1$ (why?).
For the analysis, we require $0 < \alpha < 1$ (not arbitrarily close).
Cuckoo hashing requires $0 < \alpha < 1/2$ (not arbitrarily close).

Under these restrictions (and the universal hashing assumption):

- All strategies have $O(1)$ expected time for *search*, *insert*, *delete*.
- Cuckoo Hashing has $O(1)$ worst-case time for *search*, *delete*.
- Probe sequences use $O(n)$ worst-case space,
  Cuckoo Hashing uses $O(n)$ expected space.

But for any hash-function the worst-case run-time is $\Theta(n)$ for *insert*.

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- Probe sequences use $O(n)$ worst-case space,
  Cuckoo Hashing uses $O(n)$ expected space.

But for any hash-function the worst-case run-time is $\Theta(n)$ for *insert*.

In practice, double hashing seems the most popular, or cuckoo hashing if there are many more searches than insertions.

Outline

7. Dictionaries via Hashing
    - Hashing Introduction
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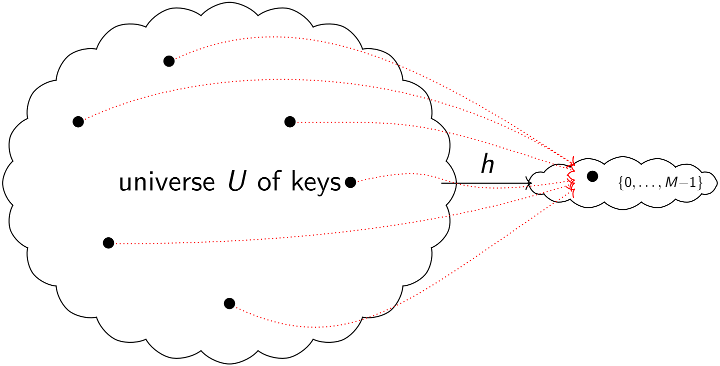
Hash functions

Every hash function *must* do badly for some inputs:

- If the universe is big enough ($|U| \geq M(n-1)+1$), then there are n keys that all hash to the same value.



- If we insert this set of keys, then we have $\Theta(n)$ run-time.

Choosing a good hash function

- Analysis works only under **uniform hashing assumption**: Hash function is randomly chosen among all possible hash-functions.
- Satisfying this is impossible: There are too many hash functions; we would not know how to look up $h(k)$.

Choosing a good hash function

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Two ways to compromise:

1. Deterministic: hope for good performance by choosing a hash-function that is
    - unrelated to any possible patterns in the data, and
    - depends on all parts of the key.
2. Randomized: Choose randomly among a limited set of functions.
    - But aim for $P(\text{two keys collide}) = \frac{1}{M}$ w.r.t. key-distribution.
    - This is enough to prove the expected run-time bounds for chaining

Deterministic hash functions

We saw two basic methods for integer keys:

- **Modular method**: $h(k) = k \bmod M$.
    - We should choose M to be a prime.
    - This means finding a suitable prime quickly when re-hashing.
    - This can be done in $O(M \log \log n)$ time (no details).

Deterministic hash functions

We saw two basic methods for integer keys:

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  - We should choose M to be a prime.
  - This means finding a suitable prime quickly when re-hashing.
  - This can be done in $O(M \log \log n)$ time (no details).
- **Multiplication method**: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$,
  for some floating-point number A with $0 < A < 1$.
  - Multiplying with A is used to scramble the keys.
    So A should be irrational to avoid patterns in the keys.
  - Experiments show that good scrambling is achieved when A is the golden ratio $\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618033988749.....$
  - We should use at least $\log |U| + \log |M|$ bits of A.

Carter-Wegman's universal hashing

Better idea: Choose hash-function randomly!

- Requires: all keys are in $\{0, \ldots, p-1\}$ for some (big) prime p.
- At initialization, and whenever we re-hash:
  - Choose $M < p$ arbitrarily, power of 2 is ok.
  - Choose (and store) two *random* numbers a, b
    - $b = \mathit{random}(p)$
    - $a = 1 + \mathit{random}(p-1)$ (so $a \neq 0$)
  - Use as hash-function $\boxed{h_{a,b}(k) = \big((ak+b) \bmod p\big) \bmod M}$
- $h(k)$ can be computed quickly.

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- $h(k)$ can be computed quickly.

Analysis of these **Carter-Wegman hash functions** (no details):

- Choosing h in this way does not satisfy uniform hashing assumption
- But can show: two keys collide with probability at most $\frac{1}{M}$.
- This suffices to prove the run-time bounds for hashing with chaining.

Multi-dimensional Data

What if the keys are multi-dimensional, such as strings?

Standard approach is to *flatten* string w to integer $f(w) \in \mathbb{N}$, e.g.

$$\begin{aligned} A \cdot P \cdot P \cdot L \cdot E \quad &\rightarrow \quad (65, 80, 80, 76, 69) \quad \text{(ASCII)} \\ &\rightarrow \quad 65R^4 + 80R^3 + 80R^2 + 76R^1 + 69R^0 \\ &\qquad \text{(for some radix } R\text{, e.g. } R = 255) \end{aligned}$$

We combine this with a modular hash function: $\boxed{h(w) = f(w) \bmod M}$

To compute this in $O(|w|)$ time without overflow, use Horner's rule and apply mod early. For exampe, $h(APPLE)$ is

$$\Bigg(\Bigg(\bigg(\bigg(\Big(\Big((65R+80) \bmod M\Big)R+80\Big) \bmod M\bigg)R+76\bigg) \bmod M\Bigg)R+69\Bigg) \bmod M$$

Hashing vs. Balanced Search Trees

Advantages of Balanced Search Trees

- $O(\log n)$ worst-case operation cost
- Does not require any assumptions, special functions, or known properties of input distribution
- Predictable space usage (exactly n nodes)
- Never need to rebuild the entire structure
- Supports ordered dictionary operations (successor, select, rank etc.)

Advantages of Hash Tables

- $O(1)$ operation cost (if hash-function random and load factor small)
- We can choose space-time tradeoff via load factor
- Cuckoo hashing achieves $O(1)$ worst-case for search & delete