CS 240 – Data Structures and Data Management

Module 1: Introduction and Asymptotic Analysis

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Based on lecture notes by many previous cs240 instructors

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Outline

1. Introduction and Asymptotic Analysis
   - CS240 Overview
   - Algorithm Design
   - Analysis of Algorithms I
   - Asymptotic Notation
   - Rules for asymptotic notation
   - Analysis of Algorithms Revisited
   - Example: Design and Analysis of *merge-sort*

Outline

Course objectives: What is this course about?

- Much of Computer Science is *problem solving*: Write a program that converts the given input to the expected output.
- When first learning to program, we emphasize *correctness*: does your program output the expected results?
- Starting with this course, we will also be concerned with *efficiency*: is your program using the computer's resources (typically processor time) efficiently?
- We will study efficient methods of *storing*, *accessing*, and *organizing* large collections of data.

  Motivating examples: Digital Music Collection, English Dictionary

  Typical operations include: *inserting* new data items, *deleting* data items, *searching* for specific data items, *sorting*.

Course objectives: What is this course about?

- We will consider various **abstract data types** (ADTs) and how to realize them efficiently using appropriate **data structures**.
- We will some problems in **data management** (sorting, pattern matching, compression) and how to solve them with efficient **algorithms**.
- There is a strong emphasis on mathematical analysis in the course.
- Algorithms are presented using pseudo-code and analyzed using order notation (big-Oh, etc.).

Course topics

1. background, big-Oh analysis
2. priority queues and heaps
3. efficient sorting, selection
4. binary search trees, AVL trees
5. skip lists
6. tries
7. hashing
8. quadtrees, kd-trees, range search
9. string matching
10. data compression
11. external memory

1 module $\approx$ 1 week per topic.

Required CS background

Topics covered in previous courses:

- arrays, linked lists
- strings
- stacks, queues
- abstract data types
- recursive algorithms
- binary trees
- basic sorting
- binary search
- binary search trees

Most are briefly reviewed in course notes, or consult any textbook (e.g. [Sedgewick,CLRS]).

Useful math facts

Logarithms:

- $y = \log_b(x)$ means $b^y = x$. e.g. $n = 2^{\log n}$.
- $\log(x)$ (in this course) means $\log_2(x)$
- $\log(x \cdot y) = \log(x) + \log(y)$, $\log(x^y) = y \log(x)$, $\log(x) \leq x$
- $\log_b(a) = \frac{\log_c a}{\log_c b} = \frac{1}{\log_a(b)}$, $a^{\log_b c} = c^{\log_b a}$
- $\ln(x) =$ natural log $= \log_e(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$

Factorial:

- $n! := n(n-1)(n-2) \cdots \cdot 2 \cdot 1 =$ # ways to permute n elements
- $\log(n!) = \log n + \log(n-1) + \cdots + \log 2 + \log 1 \in \Theta(n \log n)$

(We will define Θ soon.)

Probability:

- $E[X]$ is the expected value of X.
- $E[aX] = aE[X]$, $E[X+Y] = E[X] + E[Y]$ (linearity of expectation)

Useful sums

Arithmetic sequence:

$\sum_{i=0}^{n-1} i = ???$

Geometric sequence:

$\sum_{i=0}^{n-1} 2^i = ???$

Harmonic sequence:

$\sum_{i=1}^{n} \frac{1}{i} = ???$

A few more:

$\sum_{i=1}^{n} \frac{i}{2^i} = ???$

$\sum_{i=1}^{n} i^k = ???$

Useful sums

Arithmetic sequence:

$\sum_{i=0}^{n-1} i = \frac{(n-1)n}{2}$ $\qquad$ $\sum_{i=0}^{n-1}(a+di) = na + \frac{dn(n-1)}{2} \in \Theta(n^2)$ if $d \neq 0$.

Geometric sequence:

$\sum_{i=0}^{n-1} 2^i = 2^n - 1$ $\qquad$ $\sum_{i=0}^{n-1} a\, r^i = \begin{cases} a\dfrac{r^n-1}{r-1} & \in \Theta(r^{n-1}) & \text{if } r > 1 \\ na & \in \Theta(n) & \text{if } r = 1 \\ a\dfrac{1-r^n}{1-r} & \in \Theta(1) & \text{if } 0 < r < 1. \end{cases}$

Harmonic sequence:

$\sum_{i=1}^{n} \frac{1}{i} = ???$ $\qquad$ $H_n := \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + o(1) \in \Theta(\log n)$

A few more:

$\sum_{i=1}^{n} \frac{i}{2^i} = ???$ $\qquad$ $\sum_{i=1}^{n} \frac{i}{2^i} \in \Theta(1)$

$\sum_{i=1}^{n} i^k = ???$ $\qquad$ $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$ for $k \geq 0$

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Algorithms and problems: Review

Let us clarify a few more terms:

Problem: Description of possible input and desired output. Example: Sorting problem.

Problem **Instance:** One possible input for the specified problem.

Algorithm: *Step-by-step process* (can be described in finite length) for carrying out a series of computations, given an arbitrary instance I.

Solving a problem: An Algorithm $\mathcal{A}$ *solves* a problem Π if, for every instance I of Π, $\mathcal{A}$ computes a valid output for the instance I in finite time.

Program: A program is an *implementation* of an algorithm using a specified computer language.

In this course, our emphasis is on algorithms (as opposed to programs or programming). We do not use any particular computer language to describe them.

Algorithms and programs

Pseudocode: communicate an algorithm to another person.

In contrast, a program communicates an algorithm to a computer.

```
insertion-sort(A, n)
A: array of size n
    for (i ← 1; i < n; i++) do
        for (j ← i; j > 0 and A[j−1] > A[j]; j−−) do
            swap A[j] and A[j − 1]
```

- sometimes uses English descriptions, e.g. ‘swap’,
- omits obvious details, e.g. i is usually an integer
- has limited if any error detection, e.g. A is assumed initialized
- should be precise about exit-conditions, e.g. in loops
- should use good indentation and variable-names

Algorithms and programs

From problem Π to program that solves it:

1. Design an algorithm $\mathcal{A}$ that solves Π. → **Algorithm Design**
   A problem Π may have several algorithms. Design many!
2. Assess *correctness* and *efficiency* of each $\mathcal{A}$. → **Algorithm Analysis**
   Correctness → CS245 (here informal arguments are enough).
   Efficiency → later
3. If acceptable (correct and efficient), implement algorithm(s).
   For each algorithm, we can have several implementations.
4. If multiple acceptable algorithms/implementations, run experiments to determine best solution.

CS240 focuses on the first two steps.
The main point is to avoid implementing obviously-bad algorithms.

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Efficiency of algorithms

What do we mean by 'efficiency'?

- In this course, we are primarily concerned with the *amount of time* a program takes to run. → **Running Time**
- We also may be interested in the *amount of additional memory* the program requires. → **Auxiliary space**
- The amount of time and/or memory required by a program will usually depend on the given problem instance.
- So we express the time or memory requirements as a mathematical function of the instances (e.g. $T(I)$)
- But then aggregate over all instances $\mathcal{I}_n$ of size n (e.g. $T(n)$).
- Do we take $\max, \min, \text{avg}$? ($\rightarrow$ later)

Measuring efficiency of algorithms

What do we count as running time/space usage of an algorithm?

First option: *experimental studies*

- Write a program implementing the algorithm.
- Run the program with inputs of varying size and composition and measure time and space.
- Plot/compare the results.

There are numerous shortcomings:

- Implementation may be complicated/costly.
- Outcomes are affected by many factors: *hardware* (processor, memory), *software environment* (OS, compiler, programming language), and *human factors* (programmer).
- We cannot test all instances; what are good *sample inputs*?

Running time of algorithms

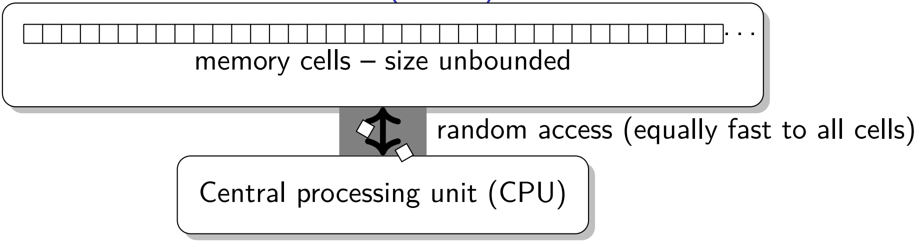
Better: theoretical analysis:

- Does not require implementing the algorithm (we work on *pseudo-code*).
- Is independent of the hardware/software environment (we work on an *idealized computer model*).
- Takes into account all input instances.

This is the approach taken in CS240.

We use experimental results only if theoretical analysis yields no useful results for deciding between multiple algorithms.

Random access machine (RAM) model



- Each **memory cell** stores one (finite-length) datum, typically a number, character, or reference.
  Assumption: cells are big enough to hold the items that we store.
- Any **access to a memory location** takes constant time.
  (We will revisit this assumption late in the course.)
- Any **primitive operation** takes constant time.
  (Add, subtract, multiply, divide, follow a reference, ...)
  Not primitive: $\sqrt{n}$, anything involving irrational numbers

These assumptions may not be valid for a "real" computer.

Running time and space

With this computer model, we can now formally define:

- The **running time** is the number of memory accesses plus the number of primitive operations.
- The **space** is the maximum number of memory cells ever in use.
- **Size(I)** of instance I is the number of memory cells that I occupies.

The real-life time and space is proportional to this.

We compare algorithms by considering the **growth rate**: What is the behaviour of algorithms as size n gets large?

- **Example 1:** What is larger, $100n$ or $10n^2$?

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- **Example 1:** What is larger, $100n$ or $10n^2$?
- **Example 2 (Matrix multiplication, approximately):** What is larger: $4n^3$, $300n^{2.807}$, or $10^{67}n^{2.373}$?

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To simplify comparisons, use **order notation** (big-O and friends).
Informally: ignore constants and lower order terms

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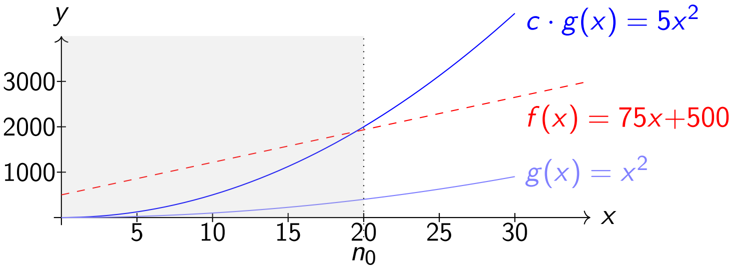
Order notation

Study relationships between *functions*.

Example: $f(x) = 75x + 500$ and $g(x) = x^2$ (e.g. $c = 5, n_0 = 20$)



***O*-notation:** $f(x) \in O(g(x))$ (f is *asymptotically upper-bounded* by g) if there exist constants $c > 0$ and $n_0 \geq 0$ s.t. $|f(x)| \leq c\,|g(x)|$ for all $x \geq n_0$.

In CS240: Parameter is usually an integer (write n rather than x).
$f(n), g(n)$ usually positive for sufficiently big n (omit absolute value signs).

Order Notation: Example 1

In order to prove that $2n^2 + 3n + 11 \in O(n^2)$ **from first principles** (i.e., directly from the definition), we need to find c and n_0 such that the following condition is satisfied:

$$2n^2 + 3n + 11 \leq c\, n^2 \text{ for all } n \geq n_0.$$

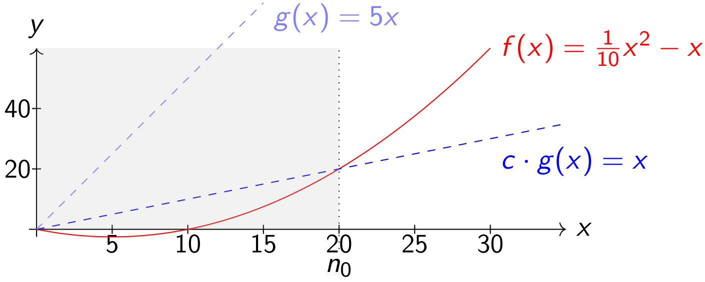
Many, but not all, choices of c and n_0 will work.

Asymptotic lower bound

- We have $2n^2 + 3n + 11 \in O(n^2)$.
- But we also have $2n^2 + 3n + 11 \in O(n^{10})$.
- We want a *tight* asymptotic bound.

Ω-notation: $f(x) \in \Omega(g(x))$ (f is *asymptotically lower-bounded* by g) if there exist constants $c > 0$ and $n_0 \geq 0$ s.t. $c\,|g(x)| \leq |f(x)|$ for all $x \geq n_0$.

Example: $f(x) = \frac{1}{10}x^2 - x$ and $g(x) = 5x$ (e.g. $c = \frac{1}{5}, n_0 = 20$)



Asymptotic lower bound

Example: Prove that $f(n) = 2n^2 + 3n + 11 \in \Omega(n^2)$ from first principles.

Example: Prove that $\frac{1}{2}n^2 - 5n \in \Omega(n^2)$ from first principles.

Asymptotic tight bound

Θ-notation: *$f(x) \in \Theta(g(x))$* (f is *asymptotically tightly-bounded* by g) if there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ such that

$$c_1 |g(x)| \leq |f(x)| \leq c_2 |g(x)| \text{ for all } x \geq n_0.$$

Equivalently: $f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

We also say that *the growth rates of f and g are the same*. Typically, $f(x)$ may be "complicated" and $g(x)$ is chosen to be a very simple function.

Example: Prove that $\log_b(n) \in \Theta(\log n)$ for all $b > 1$ from first principles.

Common growth rates

Commonly encountered growth rates in analysis of algorithms include the following:

- $\Theta(1)$ (*constant*),
- $\Theta(\log n)$ (*logarithmic*),
- $\Theta(n)$ (*linear*),
- $\Theta(n \log n)$ (*linearithmic*),
- $\Theta(n \log^k n)$, for some constant k (*quasi-linear*),
- $\Theta(n^2)$ (*quadratic*),
- $\Theta(n^3)$ (*cubic*),
- $\Theta(2^n)$ (*exponential*).

These are sorted in *increasing order* of growth rate.

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How do we define 'increasing order of growth rate'?

Growth rates and running time

It is interesting to see how the running time is affected when the size of the problem instance *doubles* (i.e., $n \to 2n$).

- constant complexity: $T(n) = c$
- logarithmic complexity: $T(n) = c \log n$
- linear complexity: $T(n) = cn$
- linearithmic $\Theta(n \log n)$: $T(n) = c\, n \log n$
- quadratic complexity: $T(n) = c\, n^2$
- cubic complexity: $T(n) = cn^3$
- exponential complexity: $T(n) = c\, 2^n$

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- linearithmic $\Theta(n \log n)$: $T(n) = c\, n \log n$ $\rightsquigarrow$ $T(2n) = 2T(n) + 2cn$.
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- quadratic complexity: $T(n) = c\, n^2$ $\quad\rightsquigarrow\quad T(2n) = 4T(n)$.
- cubic complexity: $T(n) = cn^3$
- exponential complexity: $T(n) = c\, 2^n$

Growth rates and running time

It is interesting to see how the running time is affected when the size of the problem instance *doubles* (i.e., $n \to 2n$).

- constant complexity: $T(n) = c$ $\leadsto$ $T(2n) = c$.
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- cubic complexity: $T(n) = cn^3$ $\leadsto$ $T(2n) = 8T(n)$.
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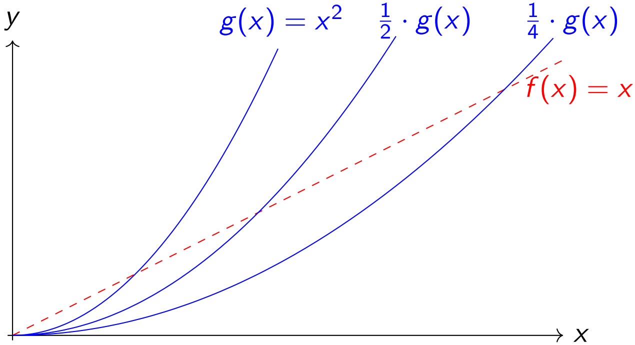
- cubic complexity: $T(n) = cn^3$ $\rightsquigarrow$ $T(2n) = 8T(n)$.
- exponential complexity: $T(n) = c\, 2^n$ $\rightsquigarrow$ $T(2n) = (T(n))^2/c$.

Strictly smaller asymptotic bounds

- We have $f(n) = n \in \Theta(n)$.
- How to express that $f(n)$ grows slower than n^2?



***o*-notation:** $f(x) \in o(g(x))$ (f is *asymptotically strictly smaller* than g) if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $|f(x)| \leq c\,|g(x)|$ for all $x \geq n_0$.

Strictly smaller/larger asymptotic bounds

Example: Prove that $n \in o(n^2)$ from first principles.

Strictly smaller/larger asymptotic bounds

Example: Prove that $n \in o(n^2)$ from first principles.

- Main difference between o and O is the quantifier for c.
- n_0 will depend on c, so it is really a function $n_0(c)$.
- We also say 'the growth rate of f is *less than* the growth rate of g'.
- Rarely proved from first principles (instead use limit-rule $\rightsquigarrow$ later).

ω-notation: $f(x) \in \omega(g(x))$ (f is *asymptotically strictly larger* than g) if for all constants $c > 0$, there exists a constant $n_0 \geq 0$ such that $|f(x)| \geq c\,|g(x)|$ for all $x \geq n_0$.

- Symmetric, the growth rate of f is *more than* the growth rate of g.

Order notation: Summary

.

O-notation: $f(x) \in O(g(x))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $|f(x)| \leq c\,|g(x)|$ for all $x \geq n_0$.

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Θ-notation: $f(x) \in \Theta(g(x))$ if there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ such that $c_1\,|g(x)| \leq |f(x)| \leq c_2\,|g(x)|$ for all $x \geq n_0$.

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The limit rule

Suppose that $f(x) > 0$ and $g(x) > 0$ for all $x \geq n_0$. Suppose that

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)} \quad \text{(in particular, the limit exists).}$$

Then

$$f(x) \in \begin{cases} o(g(x)) & \text{if } L = 0 \\ \Theta(g(x)) & \text{if } 0 < L < \infty \end{cases}$$

If the fraction tends to infinity then $f(x) \in \omega(g(x))$.

The required limit can often be computed using *l'Hôpital's rule*. Note that this result gives *sufficient* (but not necessary) conditions for the stated conclusion to hold.

Application 1: Logarithms vs. polynomials

Compare the growth rates of $f(n) = \log n$ and $g(n) = n$.

Now compare the growth rates of $f(n) = (\log n)^c$ and $g(n) = n^d$ (where $c > 0$ and $d > 0$ are arbitrary numbers).

Application 2: Polynomials

Let $f(n)$ be a polynomial of degree $d \geq 0$:

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

for some $c_d > 0$.

Then $f(n) \in \Theta(n^d)$:

Example: Oscillating functions

Consider two oscillating functions f_1, f_2 for which $\lim_{n\to\infty} \frac{f_i(x)}{x}$ does not exist. Are they in $\Theta(n)$?

So no limit $\rightsquigarrow$ must use other methods to prove asymptotic bounds.

Algebra of order notations

Many rules are easily proved from first principle (exercise).

Identity rule: $f(n) \in \Theta(f(n))$

Transitivity:

- If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$.
- If $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$ then $f(n) \in \Omega(h(n))$.
- If $f(n) \in O(g(n))$ and $g(n) \in o(h(n))$ then $f(n) \in o(h(n))$.
- ...

Maximum rules: Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Then:

- $f(n) + g(n) \in O(\max\{f(n), g(n)\})$
- $f(n) + g(n) \in \Omega(\max\{f(n), g(n)\})$

Key proof-ingredient: $\max\{f(n), g(n)\} \leq f(n)+g(n) \leq 2\max\{f(n), g(n)\}$

Relationships between order notations

- $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$

- $f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \notin \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \notin O(g(n))$

Example: Fill the following table with TRUE or FALSE:

		Is $f(n) \in \ldots(g(n))$?			
$f(n)$	$g(n)$	o	O	Ω	ω
$\log n$	$\sqrt{n}$				

Asymptotic notation and arithmetic

- Normally, we say $f(n) \in \Theta(g(n))$ because $\Theta(g(n))$ is a set.
- Avoid doing arithmetic with asymptotic notations.
  Do **not** write $O(n) + O(n) = O(n)$.
  (CS136 allowed you to be sloppy here. CS240 does not, mostly because it can go badly wrong with recursions.)
- Instead, when you do arithmetic, replace '$\Theta(f(n))$' by '$c \cdot f(n)$ for some constant $c > 0$'
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- There are some (very limited) exceptions:
  - $f(n) = n^2 + \Theta(n)$ means "$f(n)$ is n^2 plus a linear term"
    - nicer to read than "$n^2 + n + \log n$"
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  - Similarly $f(n) = n^2 + o(1)$ means "n^2 plus a vanishing term."

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Techniques for run-time analysis

- Goal: Use asymptotic notation to simplify run-time analysis.
- Running time of an algorithm depends on the *input size* n.

```
print-pairs(A, n)
for i ← 0 to n − 1 do
    for j ← 0 to i − 1 do
        print 'the next pair is {A[i], A[j]}'
```

- Identify *primitive operations* that require $\Theta(1)$ time.
  (For doing arithmetic, assume they require c time for some $c > 0$.)
- The complexity of a loop is expressed as the *sum* of the complexities of each iteration of the loop.
- Nested loops: start with the innermost loop and proceed outwards. This gives *nested summations*.

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For *print-pairs*: The run-time is $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} c$.

Techniques for run-time analysis

Two general strategies are as follows.

Strategy I: Use Θ-bounds *throughout the analysis* and obtain a Θ-bound for the complexity of the algorithm.
For *print-pairs*:

Strategy II: Prove a O-bound and a *matching* Ω-bound *separately*.
Use upper bounds (for O) and lower bounds (for Ω) early and frequently.
This may be easier because upper/lower bounds are easier to sum.
For *print-pairs*:

Complexity of algorithms

- Algorithm can have different running times on two instances of the same size.

insertion-sort(A, n)
A: array of size n

```
for (i ← 1; i < n; i++) do
    for (j ← i; j > 0 and A[j−1] > A[j]; j−−) do
        swap A[j] and A[j − 1]
```

Let $T_{\mathcal{A}}(I)$ denote the running time of an algorithm $\mathcal{A}$ on instance I.

Study this value for the worst-possible, best-possible and 'typical' (average) instance I.

Complexity of algorithms

Worst-case (best-case) complexity of an algorithm: The *worst-case (best-case) running time* of an algorithm $\mathcal{A}$ is a function $T : \mathbb{Z}^+ \to \mathbb{R}$ mapping n (the input size) to the *longest (shortest)* running time for any input instance of size n:

$$T_{\mathcal{A}}^{\text{worst}}(n) = \max_{I \in \mathcal{I}_n} \{ T_{\mathcal{A}}(I) \}$$

$$T_{\mathcal{A}}^{\text{best}}(n) = \min_{I \in \mathcal{I}_n} \{ T_{\mathcal{A}}(I) \}$$

To prove a lower bound on the worst-case run-time: Pick one especially bad example, and bound its run-time (using Ω-notation).

Average-case complexity of an algorithm: The average-case running time of an algorithm $\mathcal{A}$ is a function $T : \mathbb{Z}^+ \to \mathbb{R}$ mapping n (the input size) to the *average* running time of $\mathcal{A}$ over all instances of size n:

$$T_{\mathcal{A}}^{avg}(n) = \sum_{I \in \mathcal{I}_n} T_{\mathcal{A}}(I) \cdot (\text{relative frequency of } I)$$

O-notation and complexity of algorithms

Goal in cs240: For a problem, find an algorithm that solves it and whose tight bound on the worst-case running time has the smallest growth rate.

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  No! The worst-case run-time of $\mathcal{A}_1$ may only be achieved on some instances. Possibly $\mathcal{A}_1$ is better on most instances.

  Also, the hidden constants may be so large that $\mathcal{A}_1$ is better on all but unrealistically big n.

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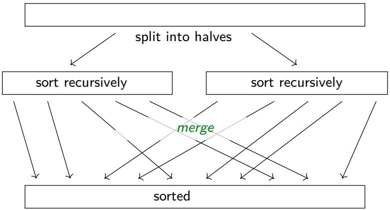
Explaining the solution of a problem

To give an algorithm that 'solves a problem', we usually do four steps. We illustrate this here on *merge-sort*.

Step 1: Describe the overall idea

Input: Array A of n integers

1. We split A into two subarrays A_L and A_R that are roughly half as big.
2. *Recursively* sort A_L and A_R
3. After A_L and A_R have been sorted, use a function *merge* to merge them into a single sorted array.



Explaining the solution of a problem

Step 2: Give pseudo-code or detailed description.

Idea for merging: Always extract from each sub-array the value that is smaller and append it to the output.

merge$(A, \ell, m, r, S \leftarrow \texttt{NULL})$

A is an array, $A[\ell..m]$ is sorted, $A[m+1..r]$ is sorted

if S is `NULL` **then** initialize it with same size as A `// tmp-array`

copy $A[\ell..r]$ into $S[\ell..r]$

$(i_L, i_R) \leftarrow (\ell, m+1)$; `// start-indices of subarrays`

for $(k \leftarrow \ell; k \leq r; k{+}{+})$ **do** `// fill-index for result`

    if $(i_L > m)$ $A[k] \leftarrow S[i_R{+}{+}]$

    else if $(i_R > r)$ $A[k] \leftarrow S[i_L{+}{+}]$

    else if $(S[i_L] \leq S[i_R])$ $A[k] \leftarrow S[i_L{+}{+}]$

    else $A[k] \leftarrow S[i_R{+}{+}]$

Explaining the solution of a problem

Step 2: Give pseudo-code or detailed description.

```
merge-sort(A, n)
A: array of size n
if (n ≤ 1) then return
else
        m = ⌊(n − 1)/2⌋
        merge-sort(A[0..m], m + 1)
        merge-sort(A[m + 1..n−1], n − m − 1)
        merge(A, 0, m, n−1)
```

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```

Two tricks to reduce constant in the run-time and auxiliary space:

- Do not pass array A by value, instead indicate the range of the array that needs to be sorted.
- *merge* needs an auxiliary array S. Allocate this only *once*.

Explaining the solution of a problem

Step 2: Give pseudo-code or detailed description.

merge-sort$(A, n, \ell \leftarrow 0, r \leftarrow n-1, S \leftarrow \texttt{NULL})$
A: array of size n, $0 \le \ell \le r \le n-1$

1. **if** S is NULL **then** initialize it as array $S[0..n-1]$
2. **if** $(r \le \ell)$ **then**
3.   **return**
4. **else**
5. $m = \lfloor (r+\ell)/2 \rfloor$
6.   *merge-sort*(A, n, ℓ, m, S)
7.   *merge-sort*$(A, n, m+1, r, S)$
8.   *merge*(A, ℓ, m, r, S)

- This would be much better for an efficient implementation.
- But the idea is much harder to understand.
- CS240 pseudocode will often prefer clarity over improved constants.

Analysis of *merge-sort*

Step 3: Argue correctness.

- Typically state loop-invariants, or other key-ingredients, but no need for a formal (CS245-style) proof by induction.
- Sometimes obvious enough from idea-description and comments.

Step 4: Analyze the run-time.

- First analyze work done outside recursions.
- If applicable, analyze subroutines separately.
- If there are recursions: how big are the subproblems?
  The run-time then becomes a recursive function.

Let $T(n)$ denote the time to run *merge-sort* on an array of length n.

1. (initialize array) takes time $\Theta(n)$
2. (recursively call *merge-sort*) takes time $T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor)$
3. (call *merge*) takes time $\Theta(n)$

The run-time of *merge-sort*

- The **recurrence relation** for $T(n)$ is as follows (constant factor c replaces Θ):

$$T(n) = \begin{cases} T\left(\lceil \frac{n}{2} \rceil\right) + T\left(\lfloor \frac{n}{2} \rfloor\right) + c\,n & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

- The following is the corresponding **sloppy recurrence** (it has floors and ceilings removed):

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

- When n is a power of 2, then the exact and sloppy recurrences are *identical* and can easily be solved by various methods. E.g. prove by induction that $T(n) = cn\log(2n) \in \Theta(n\log n)$.
- It is possible to show that $T(n) \in \Theta(n\log n)$ *for all n* by analyzing the exact recurrence.

Order notation and arithmetic revisited

Recall: You should not intermix order notation and arithmetic.

- Writing $O(n) + O(n) = O(n)$ is very bad style.
- It even occasionally leads to *incorrect* results.
- Example: What is wrong with the following proof?

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- Example: What is wrong with the following proof?

Claim (false!): If $T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$ then $T(n) \in O(n)$.

"**Proof**": Use induction on n.

- In the base case $(n = 1)$ we have $T(n) = c \in O(1) = O(n)$.
- Assume the claim holds for all n' with $n' < n$.
- Step: We have

$$T(n) = 2T\left(\tfrac{n}{2}\right) + cn \overset{IH}{\in} 2O\left(\tfrac{n}{2}\right) + O(n) = O(n) + O(n) = O(n)$$

Some recurrence relations

Recursion	resolves to	example
$T(n) \leq T(n/2) + O(1)$	$T(n) \in O(\log n)$	binary-search
$T(n) \leq 2T(n/2) + O(n)$	$T(n) \in O(n \log n)$	merge-sort
$T(n) \leq 2T(n/2) + O(\log n)$	$T(n) \in O(n)$	heapify (*)
$T(n) \leq cT(n-1) + O(1)$ for some $c < 1$	$T(n) \in O(1)$	avg-case analysis (*)
$T(n) \leq 2T(n/4) + O(1)$	$T(n) \in O(\sqrt{n})$	range-search (*)
$T(n) \leq T(\sqrt{n}) + O(\sqrt{n})$	$T(n) \in O(\sqrt{n})$	interpol. search (*)
$T(n) \leq T(\sqrt{n}) + O(1)$	$T(n) \in O(\log \log n)$	interpol. search (*)

- Once you know the result, it is (usually) easy to prove by induction.
- These bounds are tight if the upper bounds are tight.
- Many more recursions, and some methods to find the result, in CS341.

(*) These may or may not get used later in the course.