# CS 240E – Data Structures and Data Management (Enriched)

## Module 5: Other Dictionary Implementations

Tom Iagovet

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Spring 2026

*version 2026−06−09 01:33*

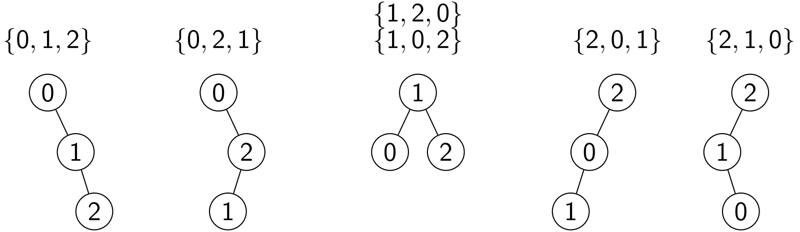
# Outline

5. Dictionaries with Lists revisited
    - Expected height of a BST
    - Treaps
    - Skip Lists
    - Biased Search Requests
    - Optimal Static Ordering
    - Optimal Static Binary Search Trees
    - Dynamic Ordering: MTF
    - MTF-heuristic in a BST
    - Splay Trees

# Expected height of BSTs

**Goal:** If the keys are nicely distributed, then using a BST without any extras will do.

**Details:** Assume that we *randomly* choose a permutation of $\{0, \ldots, n-1\}$ and build a binary search tree in this order:




**Theorem:** The expected height of the tree is $O(\log n)$.
**Proof:**

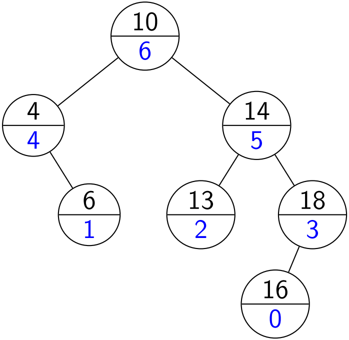
# Proof continued...

# Treaps

**Goal:** Build a binary search tree that acts as if it had been build in randomly picked insertion order.

**Idea:** Use binary search tree, but associate a priority with each key.

- Priorities are a permutation of $\{0, \ldots, n{-}1\}$.
- Permutation has been picked *randomly*
- All permutations should be equally likely.
- Priorities are *decreasing* when going downwards (similar to a heap).



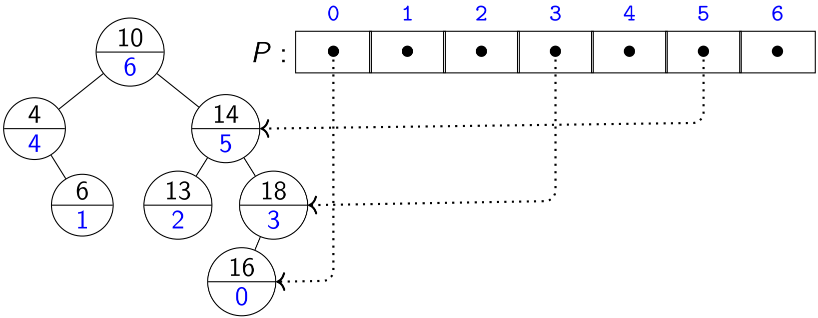

We call this a **treap** (= tree + heap).

# Treaps

**Observe:** The expected height of a treap is $O(\log n)$.

- Root-item has priority $n-1$.
- This is picked randomly, so proof for expected height of BST applies.

For updates, we need an array $P$ where $P[i]$ stores node with priority $i$.



# Treaps: Insertion

Consider adding a new KVP. What priority should it get?

- We need a random permutation of $\{0, \ldots, n-1\}$
  - Currently we had a random permutation of $\{0, \ldots, n-2\}$.

- Recall: *shuffle* creates a random permutation:

```
shuffle(A)
A: array of size n stores ⟨0, ...n−1⟩
1.   for i ← 1 to n − 1 do
2.        swap( A[i], A[random(i + 1)] )
```

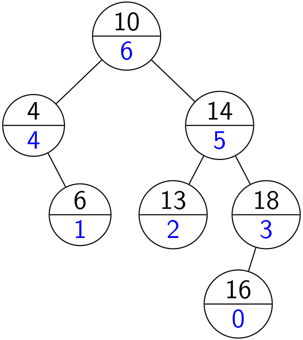
We imitate *shuffle*'s behaviour by *randomly* picking priority for new item.

- $p \leftarrow random(n)$ is in $\{0, \ldots, n-1\}$
- The item that prviously had priority $p$ now gets priority $n-1$.
- If this violates the heap-property, then rotate to fix it.

# Treaps insertion: Example

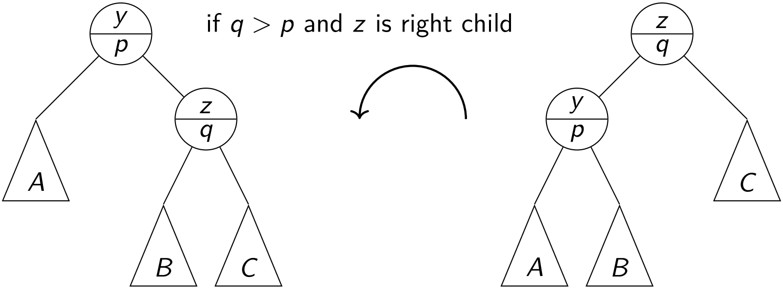
Example: *treap::insert*(17)
Randomly pick priority $5 \in \{0, \dots, 7\}$, and change priority of $P[5]$ to 7.

These priorities violate order-property.



# Fixing incorrect priorities with rotations

- In binary heaps, we fixed increased priorities via *fix-up* (swaps)
- Does not work here! We must also maintain the BST-order.
- Idea: Rotations maintain the BST-order and fix priorities.

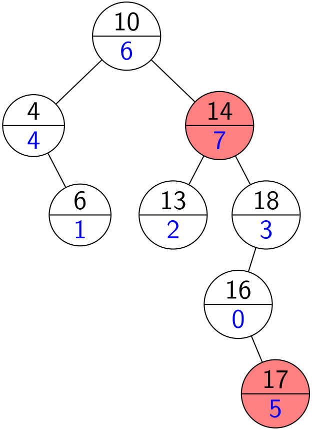



*treap::fix-up-with-rotations*($z$)

1. **while** ($y \leftarrow z$.*parent* is not `NULL` and $z$.*priority* $>$ $y$.*priority*) **do**
2.     **if** $z$ is the left child of $y$ **do** *rotate-right*($y$)
3.     **else** *rotate-left*($y$)

# Treaps insertions: Example

Example: *treap::insert*(17)
Fix *upper* violation first (why?)



# Treaps insertion: Code

Recall: $P[i]$ = node with priority $i$.

```
treap::insert(k, v)
    n ← P.size            // current size
    z ← BST::insert(k, v); n++
    p ← random(n)
    if p < n − 1 do
            z′ ← P[p], z′.priority ← n − 1, P[n − 1] ← z′
            fix-up-with-rotations(z′)
    z.priority ← p; P[p] ← z
    fix-up-with-rotations(z)
```

# Treaps: Summary

- Randomized binary search tree, so expected height is $O(\log n)$
- Achieves $O(\log n)$ expected time for *search* and *insert*
- *delete* can be handled similar (but even more exchanges)

But not particularly useful in practice

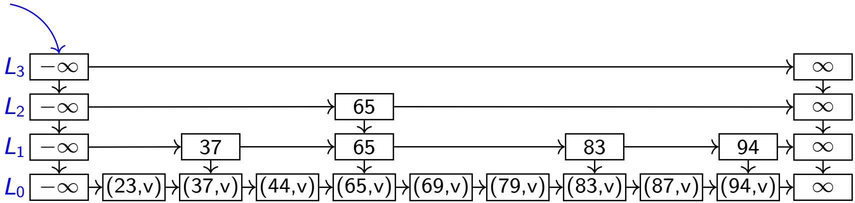
(except when priorities have meaning $\rightsquigarrow$ later)

- Large space overhead (parent-pointers, priorities, $P$)
- There are ways to avoid some of the space overhead, but in general randomized binary search trees are rarely used.
- We will now see a randomization that works better (but is not a binary search tree)

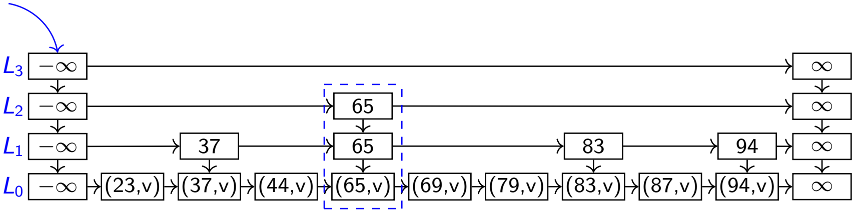
# Skip lists

A hierarchy of ordered linked lists (*levels*) $L_0, L_1, \cdots, L_h$:

- Each list $L_i$ contains the special keys $-\infty$ and $+\infty$ (**sentinels**)
- List $L_0$ contains the KVPs of $S$ in non-decreasing order.
  (The other lists store only keys and references.)
- Each list is a subsequence of the previous one, i.e.,
  $L_0 \supseteq L_1 \supseteq \cdots \supseteq L_h$
- List $L_h$ contains only the sentinels, all other lists contain at least one non-sentinel.



# Skip lists




A few more definitions:

- *node* = entry in one list vs. *KVP* = one non-sentinel entry in $L_0$
- There are (usually) more *nodes* than *KVPs*
  Here # (non-sentinel) nodes = 14 vs. $n \leftarrow$ # KVPs = 9.
- *root* = topmost left sentinel is the only field of the skip list.
- Each node $p$ has references $p.after$ and $p.below$
- Each key $k$ belongs to a **tower** of nodes
  - Height of tower of $k$: maximal index $i$ such that $k \in L_i$
  - Height of skip list: maximal index $h$ such that $L_h$ exists

# Skip lists: Search

For each list, find **predecessor** (node before where $k$ would be).
This will also be useful for *insert*/*delete*.

*get-predecessors* $(k)$

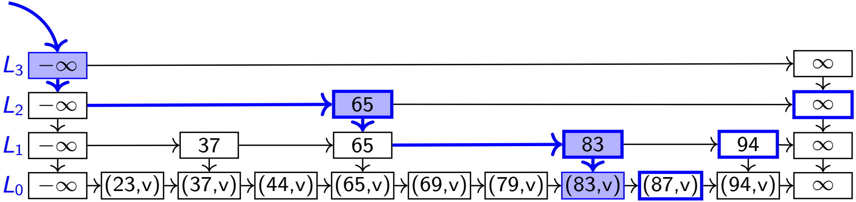
$p \leftarrow$ root
$P \leftarrow$ stack of nodes, initially containing $p$
**while** $p.below \neq$ `NULL` **do**
  $p \leftarrow p.below$
  **while** $p.after.key < k$ **do** $p \leftarrow p.after$
  $P.push(p)$
**return** $P$

*skipList::search* $(k)$

$P \leftarrow$ *get-predecessors*$(k)$
$p_0 \leftarrow P.top()$ // predecessor of $k$ in $L_0$
**if** $p_0.after.key = k$  **return** KVP at $p_0.after$
**else return** "not found, but would be after $p_0$"

# Skip list search: Example

**Example**: *search*(87)




key compared with $k$

added to $P$

path taken by $p$

Final stack returned:

| |
|---|
| (83,v) |
| 83 |
| 65 |
| $-\infty$ |

# Skip list: Deletion

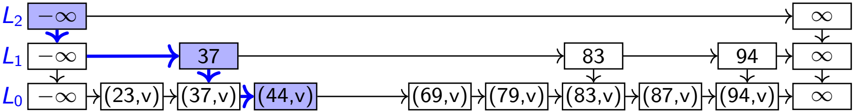
It is easy to remove a key since we can find all predecessors.
Then eliminate lists if there are multiple ones with only sentinels.

```
skipList::delete(k)
P ← get-predecessors(k)
while P is non-empty
      p ← P.pop()          // predecessor of k in some list
      if p.after.key = k
            p.after ← p.after.after
      else break           // no more copies of k

p ← left sentinel of the root-list
while p.below.after is the ∞-sentinel
      // top two lists have only sentinels, remove one
      p.below ← p.below.below
      p.after.below ← p.after.below.below
```

# Skip list deletion: Example

Example: *skipList::delete*(65)



# Skip lists: Insertion

*skipList::insert*$(k, v)$

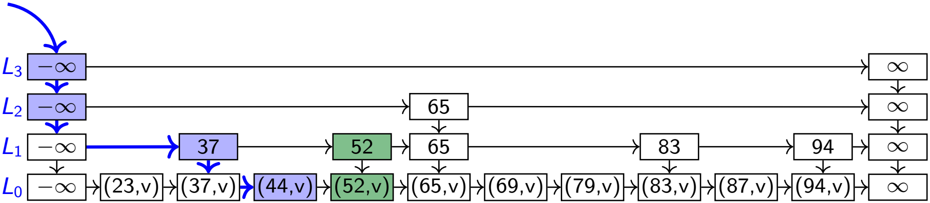
- There is no choice as to where to put the tower of $k$.
- Only choice: how tall should we make the tower of $k$?
  - Choose *randomly*! Repeatedly toss a coin until you get tails
  - Let $i$ the number of times the coin came up heads
  - We want key $k$ to be in lists $L_0, \ldots, L_i$, so $i \to$ *height* of tower of $k$

$$\mathbb{P}(\text{tower of key } k \text{ has height} \geq i) = \left(\tfrac{1}{2}\right)^i$$

- Before we can insert, we must check that these lists exist.
  - Add sentinel-only lists, if needed, until height $h$ satisfies $h > i$.
- Then do the actual insertion.
  - Use *get-predecessors*$(k)$ to get stack $P$.
  - The top $i$ items of $P$ are the predecessors $p_0, p_1, \cdots, p_i$ of where $k$ should be in each list $L_0, L_1, \cdots, L_i$
  - Insert $(k, v)$ after $p_0$ in $L_0$, and $k$ after $p_j$ in $L_j$ for $1 \leq j \leq i$

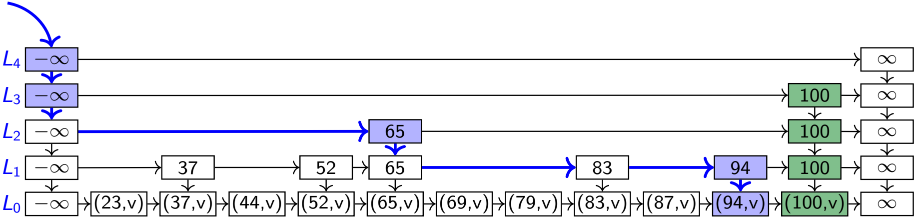
# Skip list insertion: Example

Example: *skipList::insert*(52, $v$)
Coin tosses: H,T $\Rightarrow i = 1$
Have $h = 3 > i \Rightarrow$ no need to add lists
*get-predecessors*(52)
Insert 52 in lists $L_0, \ldots, L_i$



# Skip list insert: Example 2

Example: *skipList::insert*(100, $v$)
Coin tosses: H,H,H,T $\Rightarrow i = 3$
*Height increase*
*get-predecessors*(100)
Insert 100 in lists $L_0, \ldots, L_i$



# Skip lists: Space

**Claim:** The expected number of non-sentinels is $O(n)$.

- Set $X_k$ = tower height of key $k$. Recall $\mathbb{P}(X_k \geq i) = \left(\frac{1}{2}\right)^i$.
- Define $|L_i|$ = #non-sentinels in $L_i$. Observe $|L_i| = \sum_k \chi_{(X_k \geq i)}$.

$$\left(\textbf{Indicator-variable } \chi_Z = \begin{Bmatrix} 1 & \text{if } Z \text{ is true} \\ 0 & \text{otherwise} \end{Bmatrix} \text{ has } E[\chi_Z] = P(Z \text{ is true}).\right)$$

- What is $E[|L_i|]$?

- What is $E[\#\text{non-sentinels}] = \sum_{i=0}^{h} E[|L_i|]$?

# Skip lists: Height

**Claim:** The expected height $h$ is $O(\log n)$.

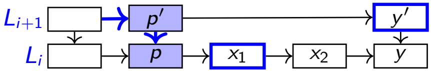
- Observe: $h = \sum_{i \geq 0} I_i$ where $I_i = \chi_{(|L_i| \geq 1)} = \begin{cases} 1 & \text{if } L_i \text{ has non-sentinels} \\ 0 & \text{otherwise} \end{cases}$
- Observe: $I_i \leq \min\{1, |L_i|\}$, so $E[I_i] \leq \min\{1, E[|L_i|]\}$.
- What is $E[h] = \sum_{i \geq 0} E[I_i]$?

Therefore the expected space is $O(n)$.

# Skip lists: Getting predecessors

**Claim:** *skipList::get-predecessors* has $O(\log n)$ expected run-time.

- How often do we *drop down* (execute $p \leftarrow p.below$)? *height*.
- How often do we *step forward* (execute $p \leftarrow p.after$)?
  - We immediately drop down in $L_h$, so consider $L_i$ for $i < h$.

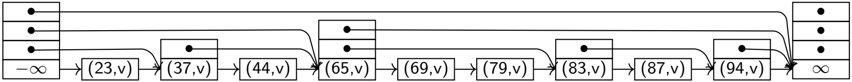



  - Key insight: $x_1$ did *not* exist in $L_{i+1}$ (else would go forward there)
  - So the tower of $x_1$ *exactly* ended with $L_i$. What is the probability?

- So *search*, *insert*, *delete* have $O(\log n)$ expected run-time

# Skip lists: Summary

- $O(n)$ expected space, all operations take $O(\log n)$ expected time.
- Lists make it easy to implement. We can also easily add more operations (e.g. *successor*, *merge*,...)
- As described they are no better than randomized binary search trees.
- But there are numerous improvements on the space:
  - Can save links (hence space) by implementing towers as array.




  - Biased coin-flips to determine tower-heights give smaller expected space
  - With both ideas, expected space is $< 2n$ (less than for a BST).

# Biased search request

**Thus far:** All keys are assumed to be equal.

- Any key $k$ is equally often the key used when searching.

**In reality:** Some keys are more frequently accessed than others (**access**: insertion or successful search)

- 80/20 rule: 80% of outcomes result from 20% of causes.
- Rule of **temporal locality**: A recently accessed item is likely to be accessed soon again.
- How can we handle such **biased search requests**?

**Intuition:** Frequently accessed items should be near the **front** (place where we first search in the data structure).

- Two scenarios: Do we know the access distribution beforehand or not?

# Optimal static ordering

**Scenario:** We know access distribution, and want the best order of a list.

**Example:**

| key | A | B | C | D | E |
|---|---|---|---|---|---|
| frequency of access | 2 | 8 | 1 | 10 | 5 |

- Order $\boxed{D} \to \boxed{B} \to \boxed{E} \to \boxed{A} \to \boxed{C}$ seems better than order $\boxed{A} \to \boxed{B} \to \boxed{C} \to \boxed{D} \to \boxed{E}$
- How do we formalize "better"?
- Define **access probability** of key $k = \frac{\text{frequency of } k}{\text{total number of accesses}}$
- Analyze (for any fixed order of keys) the

$$\textbf{expected access cost} = \sum_{i \geq 1} i \cdot (\text{access-probability of key at position } i)$$

(This is proportional to the (weighted) average-case time for *search*.)

# Optimal static ordering

**Example**:

| key | A | B | C | D | E |
|---|---|---|---|---|---|
| frequency of access | 2 | 8 | 1 | 10 | 5 |
| access-probability | $\frac{2}{26}$ | $\frac{8}{26}$ | $\frac{1}{26}$ | $\frac{10}{26}$ | $\frac{5}{26}$ |

- Order $\boxed{A} \to \boxed{B} \to \boxed{C} \to \boxed{D} \to \boxed{E}$ has expected access cost $\frac{2}{26} \cdot 1 + \frac{8}{26} \cdot 2 + \frac{1}{26} \cdot 3 + \frac{10}{26} \cdot 4 + \frac{5}{26} \cdot 5 = \frac{86}{26} \approx 3.31$

- Order $\boxed{D} \to \boxed{B} \to \boxed{E} \to \boxed{A} \to \boxed{C}$ is better! $\frac{10}{26} \cdot 1 + \frac{8}{26} \cdot 2 + \frac{5}{26} \cdot 3 + \frac{2}{26} \cdot 4 + \frac{1}{26} \cdot 5 = \frac{66}{26} \approx 2.54$

**Claim:** Over all possible static orderings, we minimize the expected access cost by sorting by non-increasing access-probability.

**Proof:**

- Consider any other ordering. How can we improve its access cost?

# Optimal static BSTs

- Biased requests are very common in practice.
- We should re-visit all data structures in this light.
- Natural question:

  *What is the best binary search tree*
  *(presuming we know access probabilities)?*

**Can show:**

- The "obvious" greedy-strategy

  (put highest-frequency item at root, build subtrees recursively)

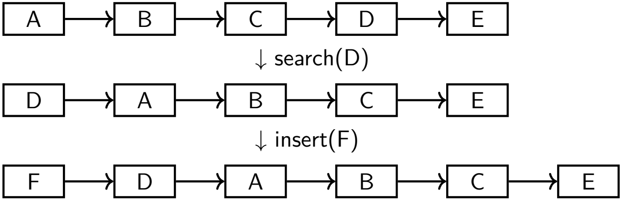
  does *not* always give the best binary search tree.
- There *is* an algorithm to find the best binary search tree, but it uses an advanced algorithm design technique ("dynamic programming") that will be seen in CS341.

- No details covered in CS240.

# Dynamic ordering: MTF

**Scenario:** We do *not know the access probabilities* ahead of time.

- **Idea:** modify the order dynamically, i.e., while we are accessing.
- Rule of thumb (**temporal locality**): A recently accessed item is likely to be used soon again.
- **Move-To-Front heuristic** (MTF): Upon a successful search, move the accessed item to the front of the list




- We can also do MTF on an array, but should then insert and search from the *back* so that we have room to grow.

# Dynamic ordering: Other ideas

There are other heuristics we could use:

- **Transpose heuristic**: Upon a successful search, swap the accessed item with the item immediately preceding it

  A → B → C → D → E

  ↓ search(D)

  A → B → D → C → E

  ↓ insert(F)

  F → A → B → D → C → E

  Here the changes are more gradual.

- **Frequency-count heuristic**: Keep counters how often items were accessed, and sort in non-decreasing order.
  Works well in practice, but requires auxiliary space.

# Biased search requests: Summary

- We are unlikely to know the access-probabilities of items, so optimal static order is mostly of theoretical interest.
- For any dynamic reordering heuristic, some sequence will defeat it (have $\Theta(n)$ access-cost for each item).
- MTF and Frequency-count work well in practice.
- For MTF, can also prove theoretical guarantees.

  (
  - MTF is an *online* algorithm: Decide based on incomplete information.
  - Compare it to the best *offline* algorithm (has complete information).
  - Here, best offline-algorithm builds optimal static ordering.
  - **Can show:** MTF is "2-competitive": $\mathit{cost}(MTF) \leq 2 \cdot \mathit{cost}(OPT)$.

  )
- There is very little overhead for MTF and other strategies; they should be applied whenever unordered lists or arrays are used ($\rightarrow$ Hashing, text compression).

# MTF-heuristic for binary search trees

What does ‘move-to-front’ mean in a binary search tree?

- Front $=$ the place that is easiest to access
- In a binary search tree, that’s the root.

$\Rightarrow$ After every access, bring item to the root of BST

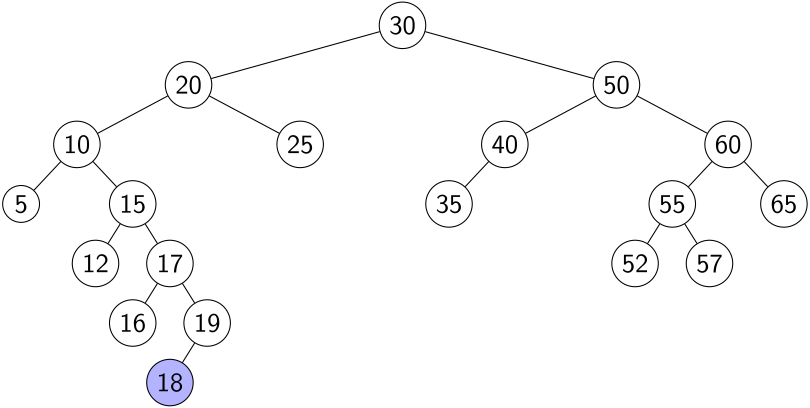
- But: order-property must be maintained!

$\Rightarrow$ Use *rotations*!

(This should remind you of treaps.)

# MTF-heuristic for binary search trees

**Example**: *BST-MTF::search*(18)




This should work well, but we can do better by moving two level at a time.

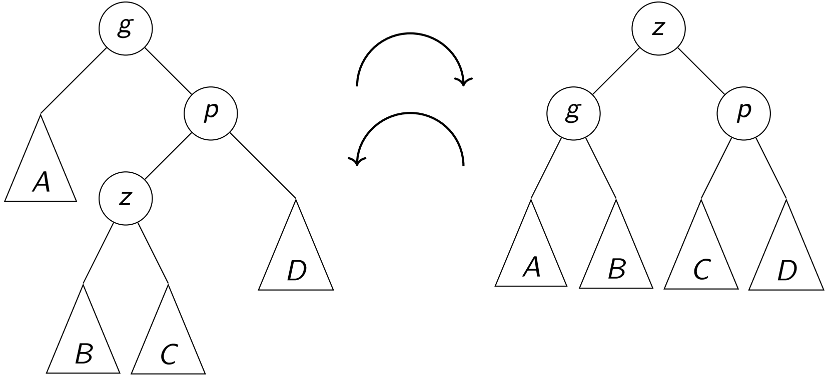
# Splay trees

Splay tree overview:

- Binary search tree
- *No* extra information (such as height, balance, size) needed at nodes
- As for MTF-heuristic:
  After each operation, bring lowest accessed node $z$ to the root.
- In contrast to MTF-heuristic:
  move $z$ up two layers at a time when possible.
  - Let $p$ and $g$ be parent and grand-parent of $z$.
  - Study relative order of $g$-$p$-$z$
  - If $p$ is middle key: Use double-rotation = **zig-zag rotation**
  - Otherwise: Use **zig-zig rotation** (new!)

**Goal:** This has amortized run-time $O(\log n)$.

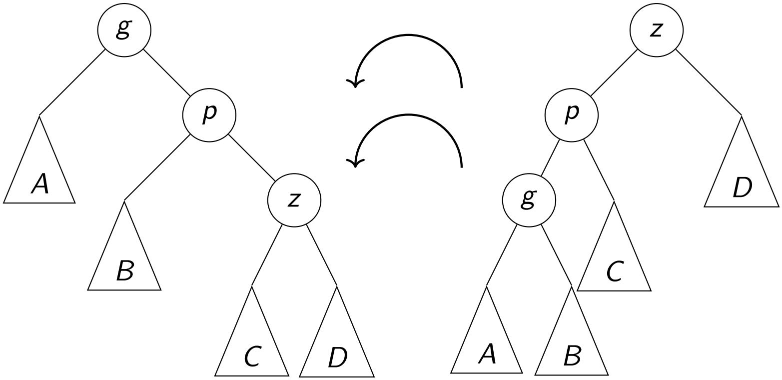
# Zig-zag rotation = double rotation



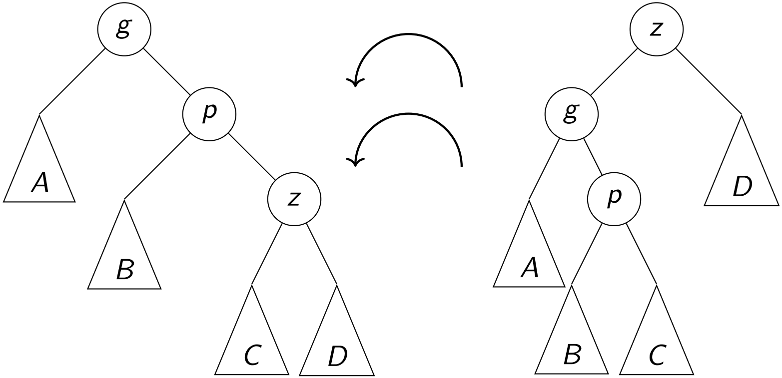

Exactly the same as the double rotation for AVL-trees; only the names of nodes are different.

# Zig-zig rotation

New rotation: Rotate at the grand-parent *before* rotating at the parent.



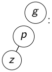
# Compare to what MTF does

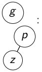



- Both operations bring $z$ two levels higher.
- But using the zig-zig rotation allows to do amortized analysis.

# Splay tree: Operations

*SplayTree::insert*$(k, v)$

$z \leftarrow$ *BST::insert*$(k, v)$
**while** ($z$ is not the root)
    $p \leftarrow z.parent$
    **if** ($z$ is the left child of $p$)
        **if** ($p$ is the root) *rotate-right*$(p)$
        **else** $g \leftarrow p.parent$
        **case** 

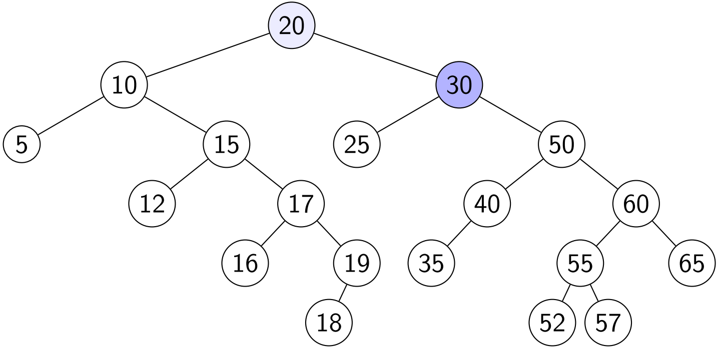
        : `// Zig-zig rotation`
            *rotate-right*$(g)$
            *rotate-right*$(p)$
        

        : `// Zig-zag rotation`
            *rotate-right*$(p)$
            *rotate-left*$(g)$
    **else** … `// symmetric , z is right child`

*search* and *delete* similar (rotate the lowest visited node up).

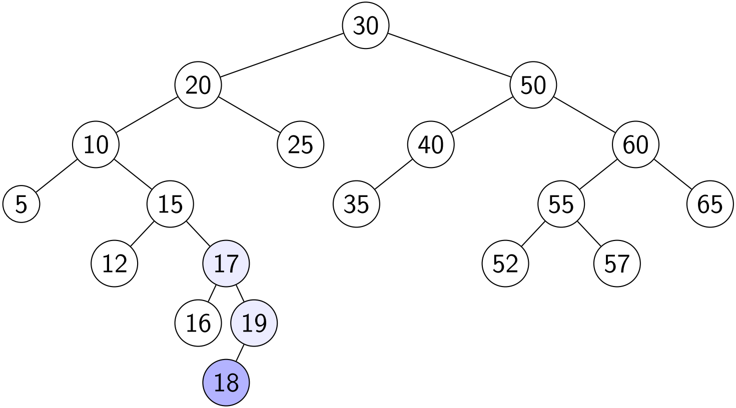
# Splay tree: Search

**Example**: *SplayTree::search*(30)



# Splay tree: Search

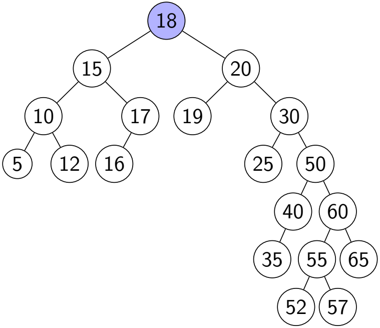
**Example**: *SplayTree::search*(18)



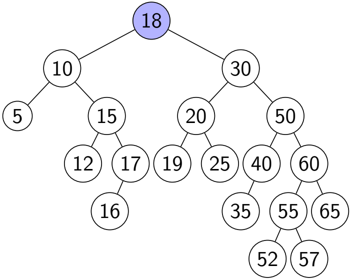
# Zig-zig rotations vs. single rotations

Compare the resulting trees:

Splay trees (zig-zig rotations):




With MTF (single rotations):

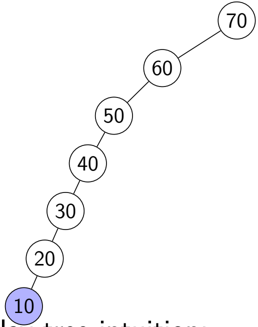



This is *not* more balanced, why do we apply zig-zig-rotations?

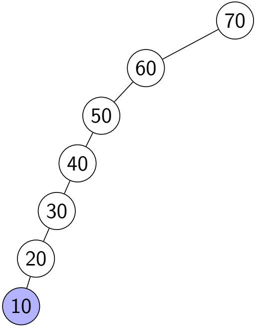
# Zig-zig rotations vs. single rotations

Compare the result for a different initial tree:

Splay trees (zig-zig rotations):




With MTF (single rotations):



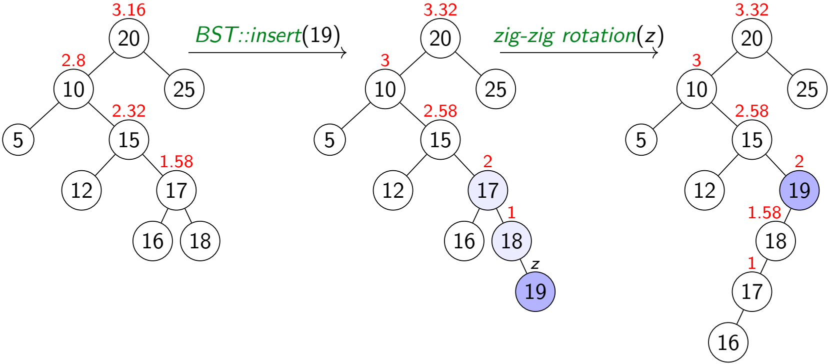

Splay tree intuition:

- For any node on search-path, the depth (roughly) halves
- For all nodes, the depth increases by at most 2

# Splay trees: Analysis

**Theorem:** In a splay tree, all operations take $O(\log n)$ amortized time.

**Proof:** Use potential function $\Phi(t) = \sum_x \log\big(\text{size of subtree at } x\big)$




Now compute amortized time of operations (highly non-trivial).

# Proof continued...

# Proof continued...

# Proof continued...

# Proof continued...