CS240E S26	Midterm Review Session	June 19, 4:30-6:30PM

Note: This is a sample of problems designed to help prepare for the midterm exam. These problems do *not* encompass the entire coverage of the exam, and should not be used as a reference for its content.

Q1. Expected run-time analysis.

Consider the following pseudocode. What is its expected run-time?

```
int randomizedMystery(int n) {
   if (n<=1) return 0
   else {
      i = random(n)
      randomizedMystery(i)
   }
}
```

Q2. Amortized analysis.

Consider the following method to simulate a queue with two stacks S_1, S_2.

- initialize: initialize both S_1 and S_2 as empty stacks
- enqueue(x): do $S_1.push(x)$
- dequeue(): If S_2 is empty, do $S_2.push(S_1.pop())$ until S_1 is empty. Then return $S_2.pop()$.

Using a suitable potential function, argue that it is a potential function and show that all operations take constant amortized time.

Q3. Sorting an array.

Given an array $A[0...n-1]$ of numbers, show that if $A[i] \geq A[j]$ for all j with $j \leq i - \log n$, the array can be sorted in $o(n \log n)$ time.

Q4. Multi-Way Merge.

Given a set of k sorted arrays, where the combination of the k arrays has n elements in total, design a worst case $O(n \log k)$ time algorithm that produces a single sorted array containing all n elements.

Q5. Skip Lists.

Suppose you have a skip list with only three levels. The top level contains only the sentinels. The lowest level has $n+2$ keys: $-\infty, a_0, \cdots, a_{n-1}, \infty$, while the middle level contains $k+2$ keys including the sentinels. Assume k divides n. Suppose that the k entries are evenly spread out and the first entry corresponds to a_0.

1. What is the worst case time for a query? Give a tight bound involving k and n.

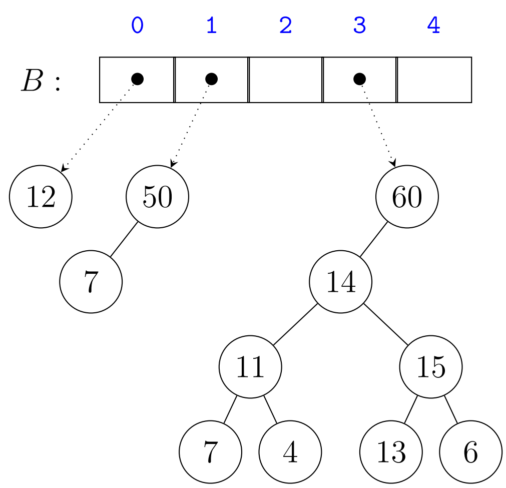
2. Given n, how should you choose k to minimise the worst case, and what does the worst case become in that case?

Q6. Lower bound.

Suppose you own n electrical devices. Each of them comes with a charger cable, which you tossed into a box when you got it. But now it is time to recharge the devices, and so you must find for each one the correct charger cable. For each device, exactly one charging cable is correct. The charging cables look similar enough that you cannot compare them amongst themselves. The only thing that you can do is plug a cable into a device, which will tell you whether the plug fits, or is too big, or is too small. Argue that any algorithm to find cables for all devices must use $\Omega(n \log n)$ such operations in the worst case.

Q7. Binomial Heap.

Consider the following binomial heap.

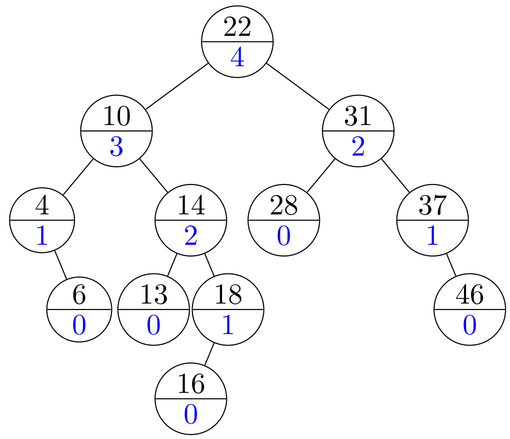


1. Show the resulting binomial heap when performing *insert*(40).

2. Show the resulting binomial heap when performing *deleteMax*.

Q8. AVL-trees.

Consider the following AVL-tree.



1. Could there be an AVL-tree that (like this one) has height 4, but fewer nodes? If so, show such a tree. If not, why not?
2. Show the result after performing *insert*(17).
3. Show the result after performing *delete*(4).
4. Pick a node z and a child y of z such that performing a single rotation at z to bring y up yields again an AVL-tree.
5. Pick a node z and a child y and grandchild x of z such that performing a double rotation at z to bring x up yields again an AVL-tree.

Q9. Priority queues.

Our goal is to determine the output of a device that records a number x, and then prints x every $t(x)$ seconds.

More specifically, given an array t of n positive integers, the device will *print* the i-th item every $t[i]$ seconds. It is guaranteed that t is such that no two prints ever occur at the same time.

Give an algorithm to determine the first k prints in $O(kn\log(kn))$ time.

Q10. Asymptotics with rounding.

Prove or disprove: there exists a polynomial $p(n)$ such that

$$\lceil \log \log n \rceil ! \in O\left(p\left(n\right)\right).$$

Q11. Uniform random permutations.

Consider the following procedure for generating a permutation of the array A:

```
permute_by_cyclic(A[0..n-1]):
    B[0..n-1] = new array
    offset = random(n)
    for(i = 0..n-1):
        dest = (i + offset) % n
        B[dest] = A[i]
    return B
```

1. Show that each element $A[i]$ has a $1/n$ probability of going to any particular position in B.
2. Show that the resulting permutation is **not** uniformly random.

Q12. Interpolation Search.

Show the execution steps of interpolation-search$(A, 30)$ where $A = [1, 2, 4, 20, 29, 30, 40, 90, 100]$. (Recall that the pseudocode is this:

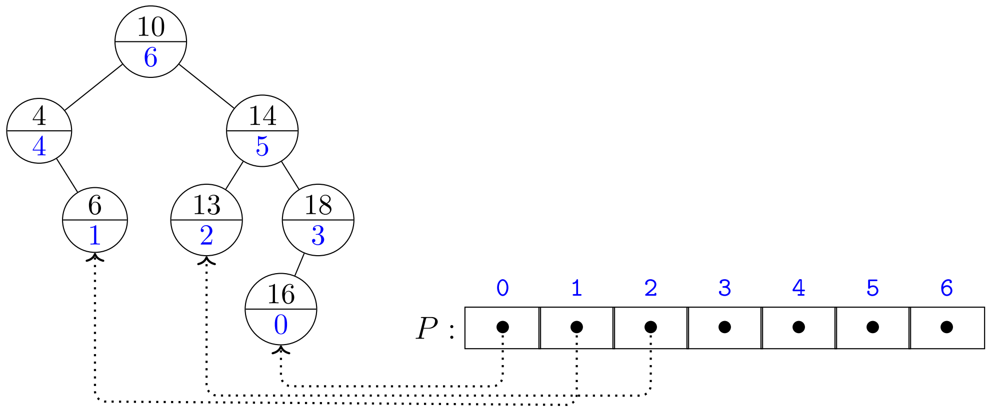
Algorithm 1: *interpolation-search*(A, n, k)

$\ell \leftarrow 0, r \leftarrow n-1$;
while $\ell \leq r$ **do**
  if $k < A[\ell]$ **then return** *"not found, would be between indices $\ell - 1$ and ℓ"*;
  if $k > A[r]$ **then return** *"not found, would be between indices r and $r+1$"*;
  if $k = A[r]$ **then return** *"found at index r"*;
  $m \leftarrow \ell + \lceil \frac{k - A[\ell]}{A[r] - A[\ell]} \cdot (r - \ell - 1) \rceil$;
  if $A[m] = k$ **then return** *"found at index m"*;
  else if $A[m] < k$ **then** $\ell \leftarrow m + 1$;
  else $r \leftarrow m - 1$;
end
`// Can argue that we never reach this point.`

)

Q13. Treaps

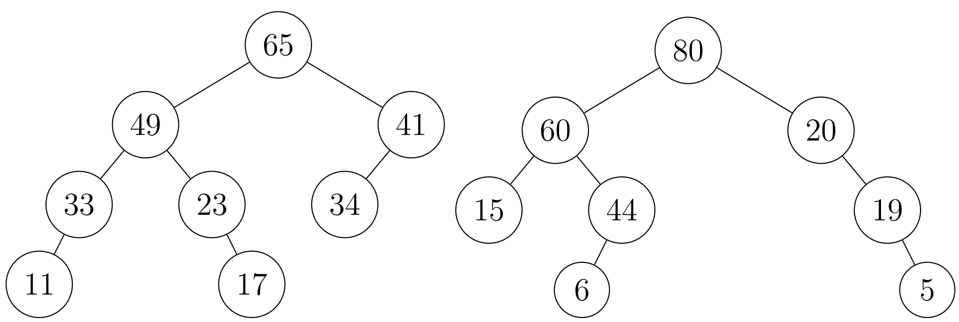
Consider the following treap.



Show the result of doing *insert*(12) when the randomly chosen priority is 4.

Q14. Meldable Heaps

Merge the two following meldable heaps with the following randomization (L means left, R means right): LLRRLRLR (you may not need all of them.)



Q15. Radix Sort

Let A$[0..n-1]$ be an unsorted array that stores integers, where each entry is in the range $[0, n5)$. Entries in A are not necessarily unique. Design an algorithm to test whether there are indices i and j such that $|A[i] - A[j]| = 10$. The worst-case run-time and the auxiliary space must be in $O(n)$.

Remark: You are not allowed to use hashing for this question. We have not even covered hashing yet, but even if we had, its worst-case run-time is **not fast enough**.

The following are harder problems than what is expected on the midterm. However, they only use concepts taught in class, and may be useful for review of in-class techniques.

BQ1. Root-leaf path.

We are given a complete binary tree of height $h > 0$ with nodes numbered $0, \ldots, n-1$ in level order:

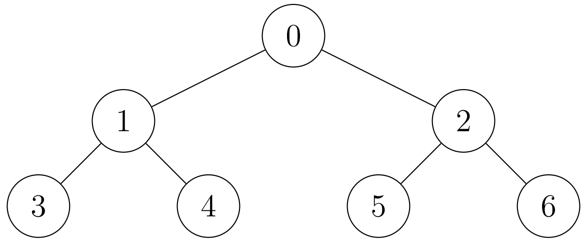


Figure 1: Example tree.

Every node has a flag which is either *true* or *false*. Initially, all flags are *false*.

We will be *dropping balls* one by one from the root of the tree. Each time a ball is dropped it first visits an internal node, an either follows to the left subtree, or to the right subtree. If the node's flag's current value is *false*, then ball goes to the left; and if the value is *true*, the ball goes to the right. In both cases, the ball switches this flag's value. The ball stops when it reaches a leaf node.

Give an algorithm to determine the number of the leaf node at which the i-th ball stops in $O(h)$ time. Note that the runtime of the algorithm is independent of i.

BQ2. Batch processing.

We are given a two-dimensional grid that contains n cells. Initially, all cells except one are white. We perform n operations, each of which first calculates the minimum distance from a given white cell to a black cell, and then paints the white cell black.

Assume that we have an $O(n)$ algorithm *closest*(w) which computes a two-dimensional array C such that

$$C[x][y] = \text{ the distance to a nearest black cell from the white cell } (x, y).$$

Give an $O(n\sqrt{n})$ algorithm to process all n operations.

BQ3. A special element.

Note: this problem is hard in a way that is not the point of CS240E.

(a) We are given integers $x_1, \ldots, x_n$, not necessarily in sorted order. Give an algorithm to find an index m such that $\sum_i |x_i - x_m|$ is minimized in $O(n)$ expected. Can you do $O(n)$ worst case as well?

(b) Suppose we don't know what the integers are, except that they are within the range $[0..u]$, and only have a magical function *sum-diffs(x)* which tells us $\sum_i |x_i - x|$ in $O(1)$ time. Can you find the x which minimizes *sum-diffs(x)* in $O(\log u)$ time? (Note that x need not be an element in the array.)