CS 240E – Data Structures and Data Management (Enriched)

Module 2: Priority Queues

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Based on lecture notes by many previous cs240 instructors

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Outline

- ADT Priority Queue
- Binary Heaps as PQ realization
- *PQ-sort* and *heap-sort*
- More PQ operations
- Meldable Heaps
- Detour: Randomized algorithms and their analysis
- Binomial Heaps

ADT Priority Queue

Priority Queue generalizes both ADT Stack and ADT Queue.

It is a collection of items (each having a **priority** or **key**) with operations

- *insert*: inserting an item tagged with a priority
- *delete-max*: removing and returning an item of *highest* priority.

We can have extra operations: *size*, *is-empty*, and *get-max*

This is a **maximum-oriented** priority queue. A **minimum-oriented** priority queue replaces operation *delete-max* by *delete-min*.

Applications:

- How would you simulate a stack with a priority queue?
- How would you simulate a queue with a priority queue?
- Other applications: typical todo-list, simulation systems, sorting

Using a Priority Queue to Sort

```
PQ-Sort($A[0..n-1]$)
1.  initialize $PQ$ to an empty priority queue
2.  for $i \leftarrow 0$ to $n-1$ do
3.      $PQ$.insert(an item with priority $A[i]$)
4.  for $i \leftarrow n-1$ down to $0$ do
5.      $A[i] \leftarrow$ priority of $PQ$.delete-max()
```

- Note: Run-time depends on how we implement the priority queue.
- Sometimes written as: $O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{delete-max})$

With two easy (but slow) realizations of priority queues:

- **Unsorted** array or list ($\rightsquigarrow$ *selection-sort*)
- **Sorted** array or list ($\rightsquigarrow$ *insertion-sort*)

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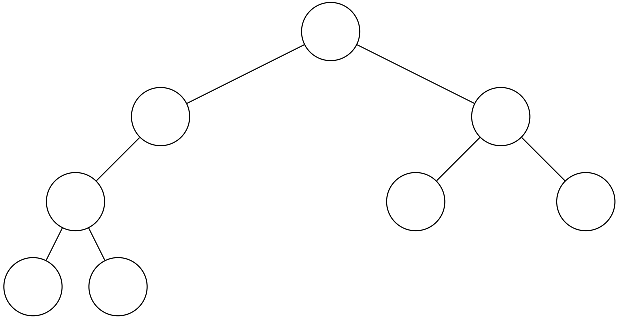
Better Realization: Binary Heap



Binary tree with

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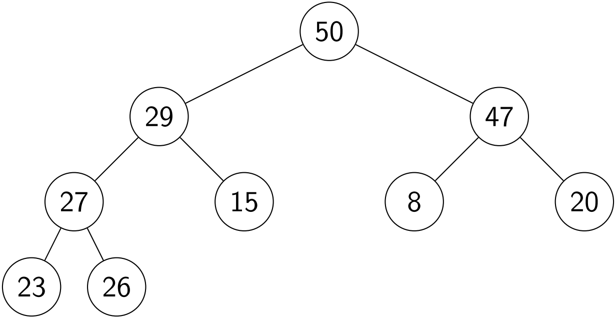
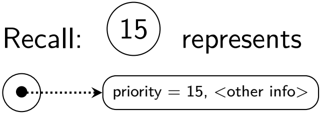


Binary tree with

1. structural property and
2. heap-order property.

Recall:



Heaps – Definition

A **heap** is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
2. **Heap-order Property:** For any node i, the key of the parent of i is larger than or equal to key of i.

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The full name for this is *max-oriented binary heap*.

Lemma: The height of a heap with n nodes is $\Theta(\log n)$.

Storing Heaps in Arrays

Heaps should *not* be stored as binary trees!

Let H be a heap of n items and let A be an array of size n. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

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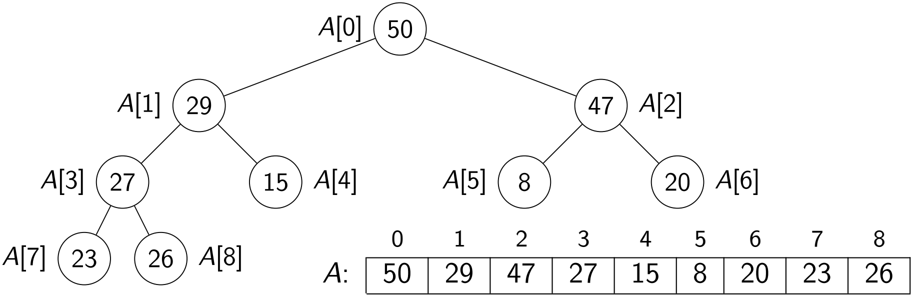

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A:	50	29	47	27	15	8	20	23	26

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insert in Heaps

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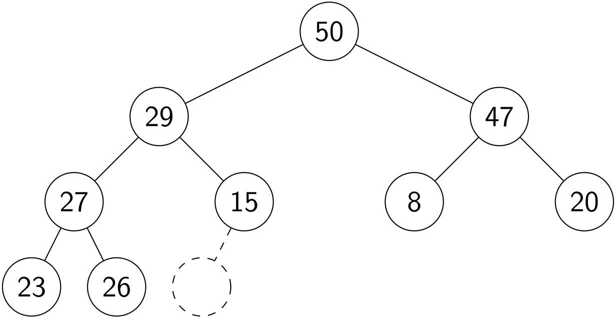

- By structural property: no choice where the new node can go.

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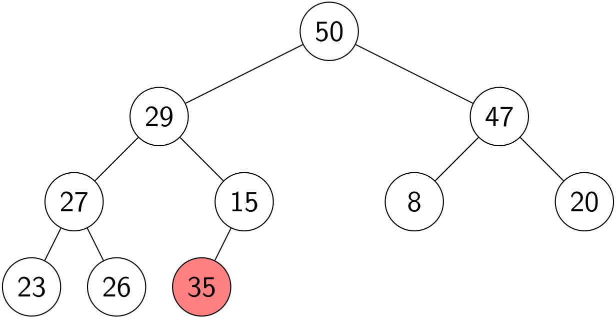

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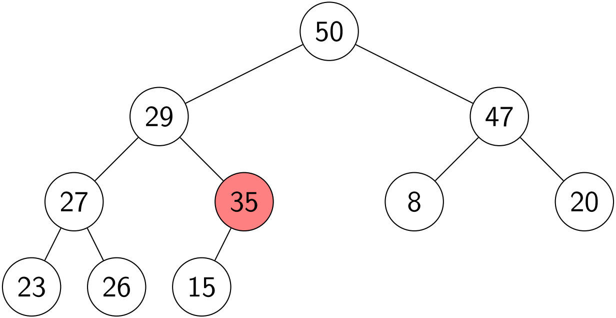

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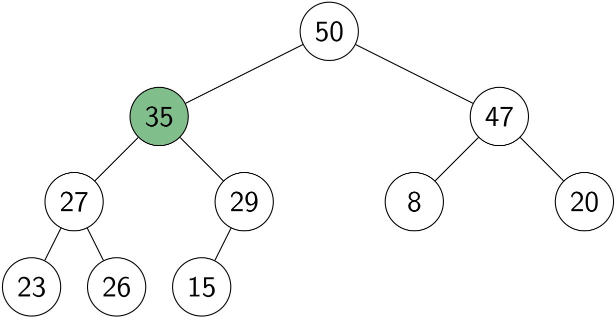

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insert in Heaps

- Place the new key at the first free leaf
- Use *fix-up* to restore heap-order.

```
insert($x$)
    $\ell \leftarrow$ last()+1
    $A[\ell] \leftarrow x$          // assume dynamic array used
    increase size     // size:  stored by PQ
    fix-up($A, \ell$)
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i: an index corresponding to a node of the heap
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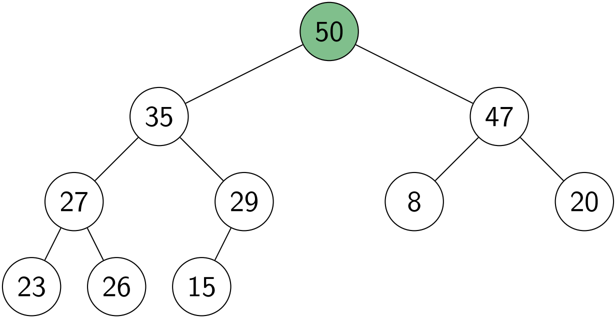
Time: $O(\text{height of heap}) = O(\log n)$ (and this is tight).

(Correctness may seem obvious, but is actually non-trivial.)

delete-max in Heaps



- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a *fix-down*:

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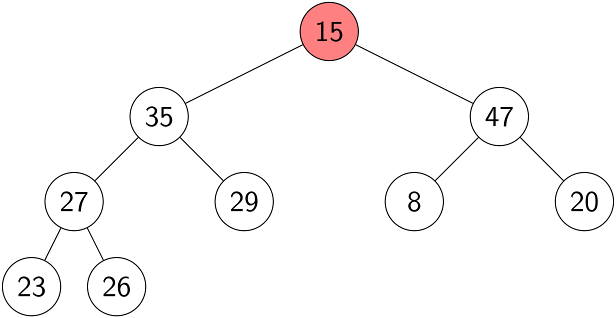

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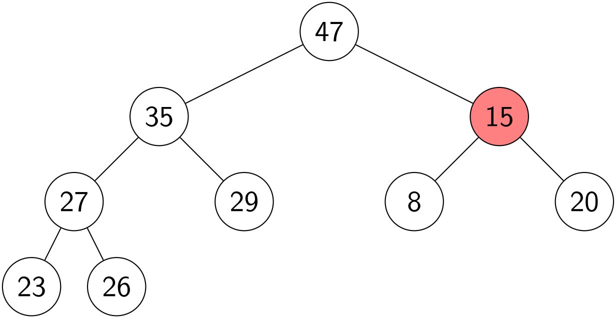

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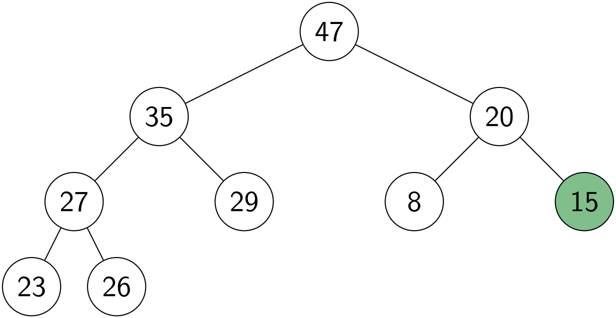

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```
fix-down($A, i$)
A: an array that stores a heap of size $n$
i: an index corresponding to a node of the heap
while $i$ is not a leaf do
    $j \leftarrow$ left child of $i$        // find child with larger key
    if ($i$ has right child and $A$[right child of $i$].key $> A[j]$.key)
        $j \leftarrow$ right child of $i$
    if $A[i]$.key $\geq A[j]$.key break
    swap $A[j]$ and $A[i]$
    $i \leftarrow j$
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Sorting using heaps

- Recall: Any priority queue can be used to *sort* in time

$$O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{delete-max})$$

- Using the binary-heaps implementation of PQs, we obtain:

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PQ-sort-with-heaps(A)
initialize H to an empty heap
for $i \leftarrow 0$ to $n-1$ do
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⇝ *PQ-sort* using heaps takes $O(n \log n)$ time (and this is tight).

- Can improve this with two simple tricks → **heap-sort**
  1. Can use the same array for input and heap. ⇝ *$O(1)$ auxiliary space!*
  2. Heaps can be built faster if we know all input in advance.

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Solution 2: Using *fix-downs* instead:

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heapify(A)
A: an array
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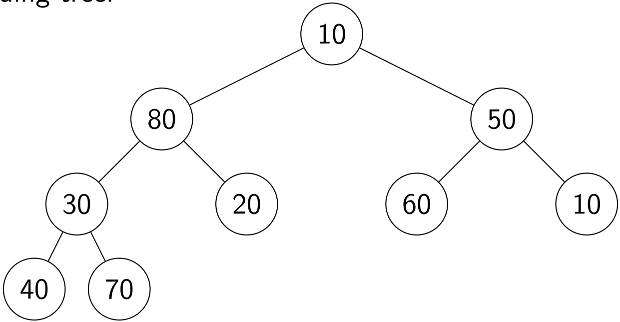
Show: *heapify* has run-time $\Theta(n)$.

heapify example

	0	1	2	3	4	5	6	7	8
A :	10	80	50	30	20	60	10	40	70

Corresponding tree:



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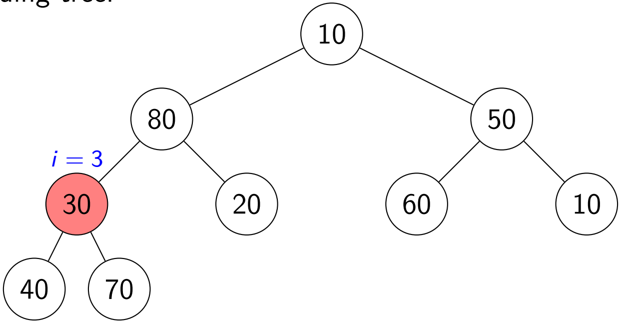

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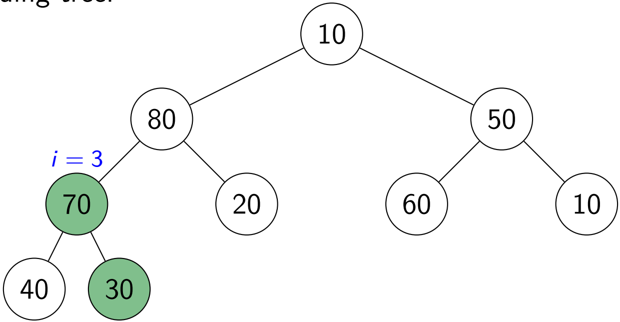

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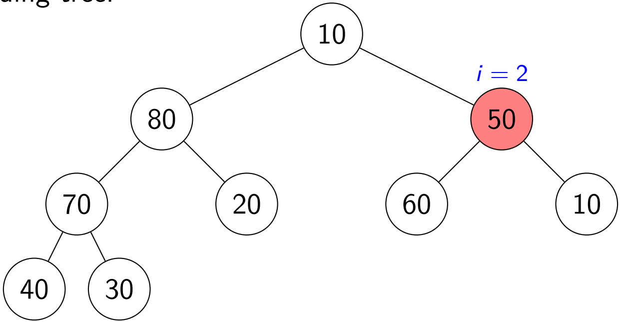

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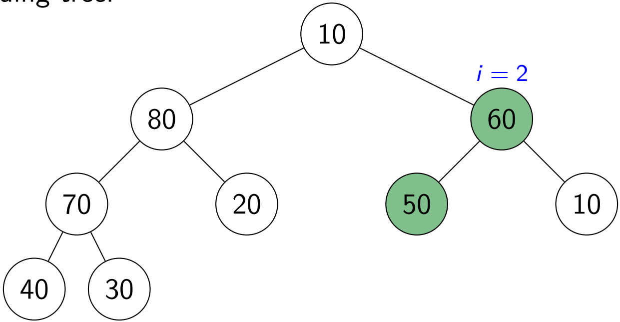

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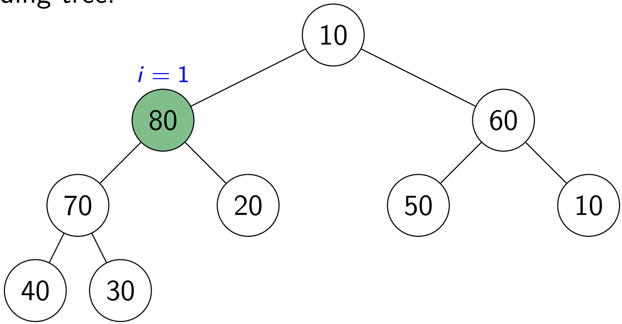

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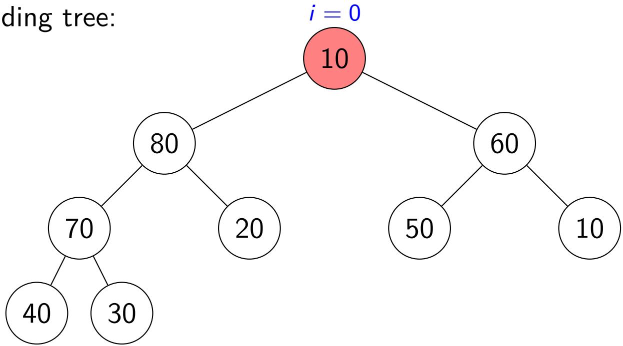

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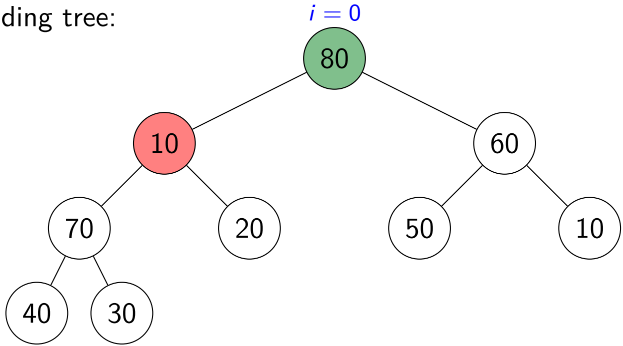

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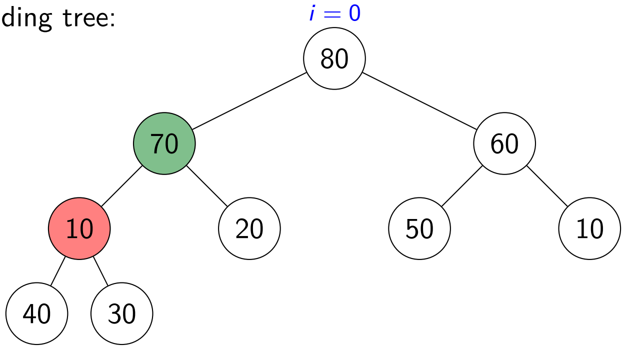

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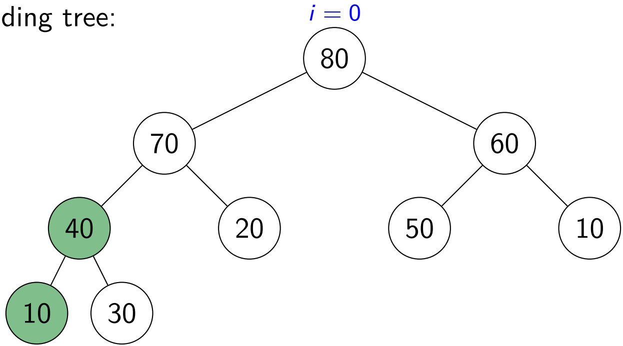

heapify run-time: proof

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More Priority Queues Operations

Binary Heaps are a good realization for *insert* and *delete-max*.
What if we want *more* operations for a priority queue (PQ)?

increase-key(v, k), *decrease-key*(v, k)

- Change key of node v to k if k is bigger/smaller than v.*key*
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- Given: two priority queues P_1, P_2 of size n_1 and n_2.
- Want: One priority queue P that contains all their items
- **Outlook:** Three approaches (where $n = n_1 + n_2$):
  - Merge binary heaps. $O(\log^3 n)$ worst-case time (no details)
  - **Meldable heaps**: heap-order-property but no structural property. $O(\log n)$ *expected* run-time for all operations.
  - **Binomial heaps**: different structural and order property. $O(\log n)$ *worst-case* run-time for all operations.

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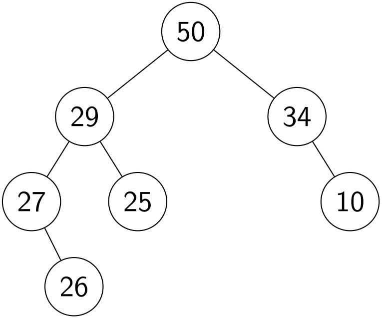
Meldable Heaps

- Priority queue stored as a binary tree
- Heap-order-property: Parent no smaller than child.
- No structural property; any binary tree is allowed.



- *Tree-based:* Store items at nodes with references to *left*/*right* child

  (Array-based implementations must use $\Omega(n)$ time for merge—why?)

PQ-operations in Meldable Heaps

Both *insert* and *delete-max* can be done by *reduction* to *merge*.

$P.\mathit{insert}(x)$:

- Create a 1-node meldable heap P' that stores x.
- Merge P' with P.

$P.\mathit{delete\text{-}max}()$:

- Stash item that is at root.
- Let P_ℓ and P_r be left and right sub-heap of root.
- Update $P \leftarrow \mathit{merge}(P_\ell, P_r)$
- Return stashed item.

Both operations have run-time $O(\mathit{merge})$.

Merging Meldable Heaps

- Idea: Merge heap with smaller root into other one, *randomly* choose into which sub-heap to merge.

```
meldableHeap::merge($r_1, r_2$)
$r_1, r_2$: roots of two heaps (possibly NULL)
returns root of merged heap
    if $r_1$ is NULL return $r_2$
    if $r_2$ is NULL  return $r_1$
    if $r_1.key < r_2.key$   swap($r_1, r_2$)
    // now $r_1$ has max-key and becomes the root.
    randomly pick one child $c$ of $r_1$
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We will see: The **expected run-time** is $O(\log n)$.

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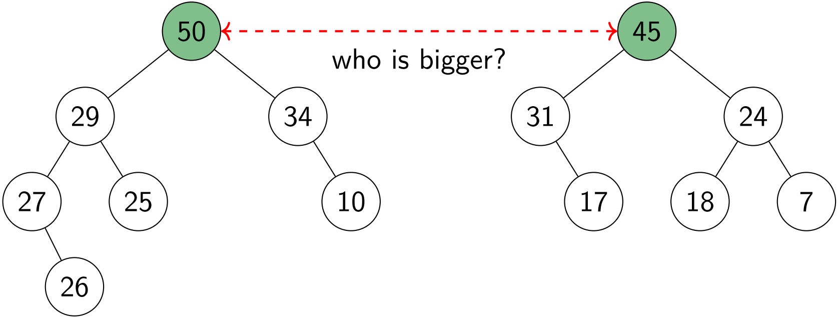

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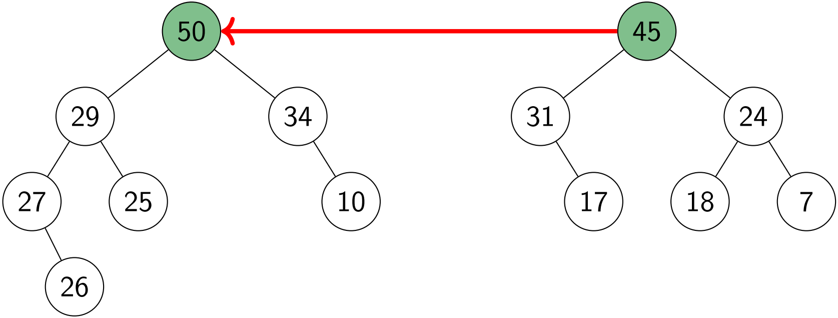

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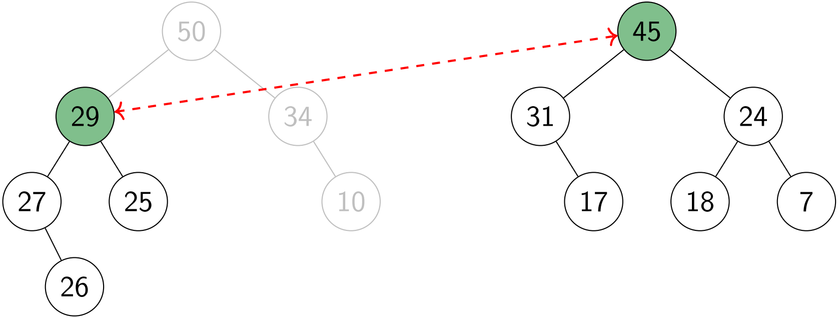

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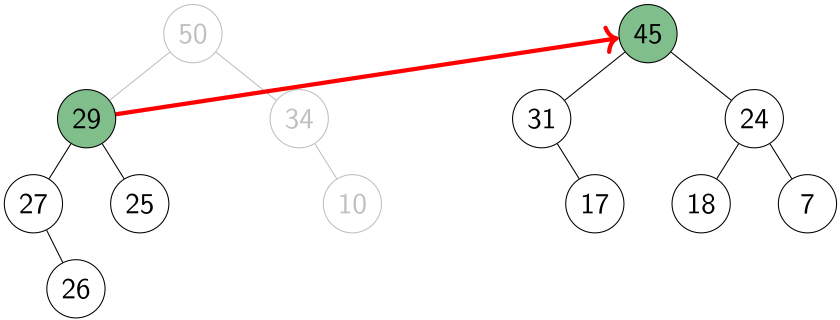

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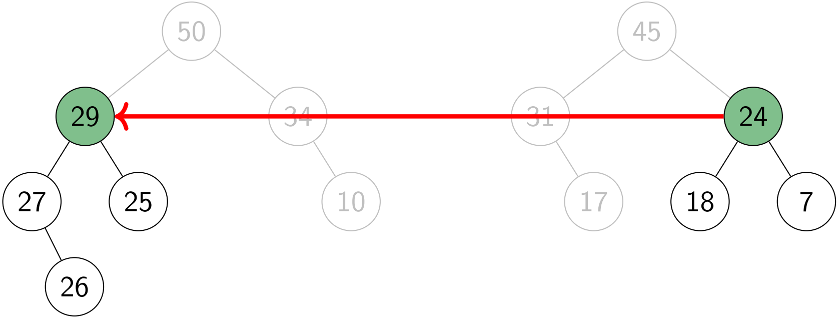

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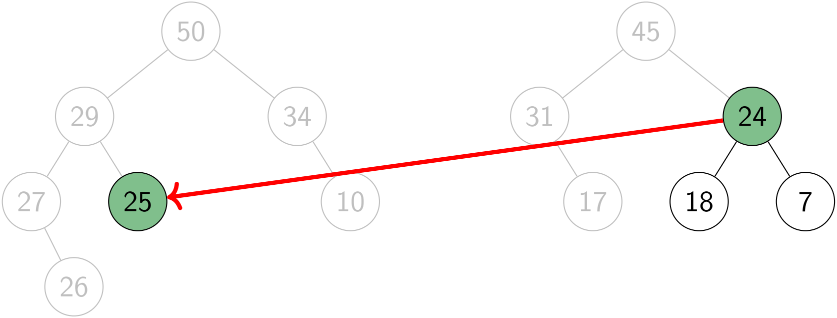

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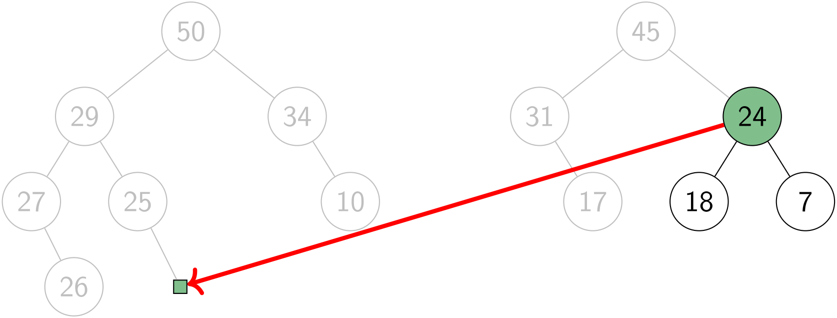

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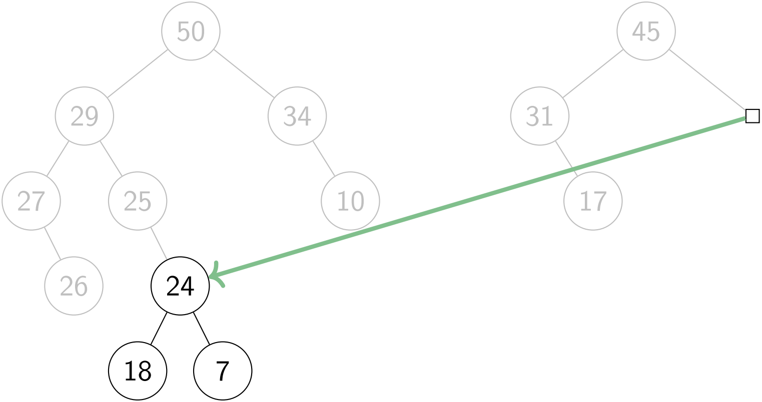

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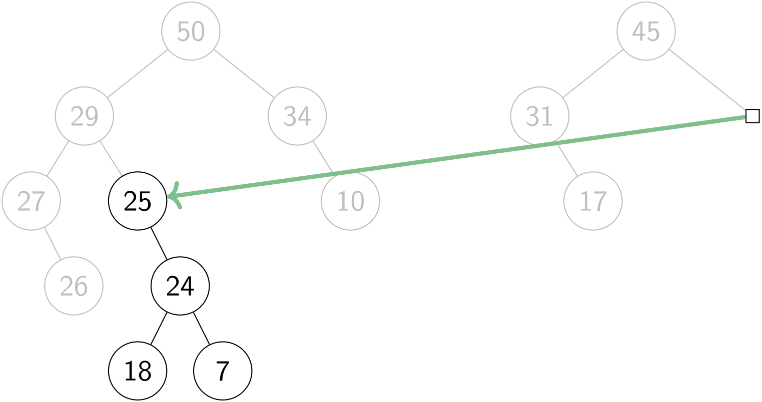

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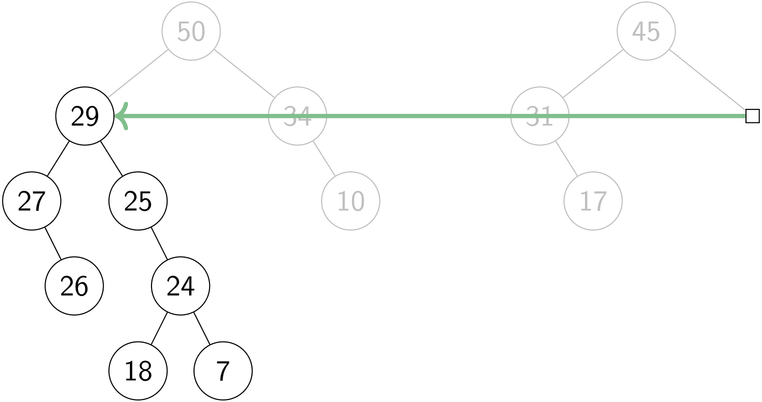

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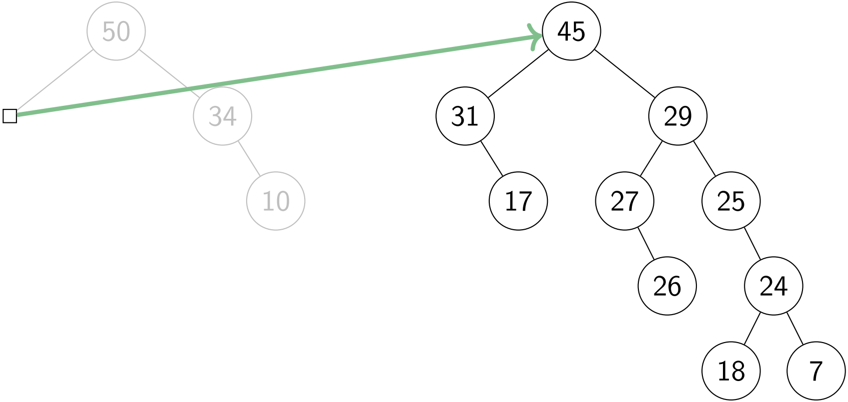

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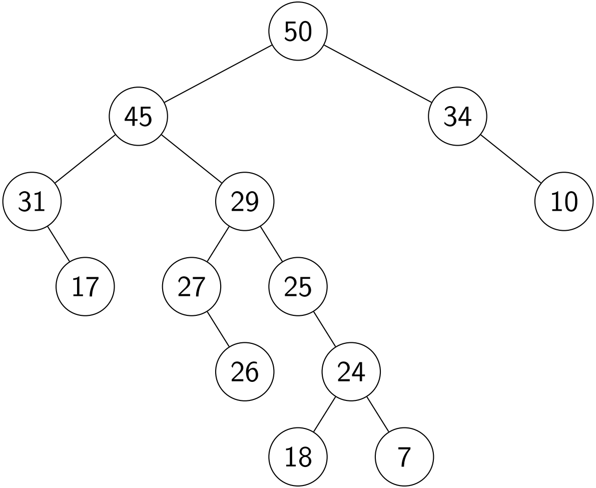

Outline

- ADT Priority Queue
- Binary Heaps as PQ realization
- *PQ-sort* and *heap-sort*
- More PQ operations
- Meldable Heaps
- Detour: Randomized algorithms and their analysis
- Binomial Heaps

Randomized algorithms

- A **randomized algorithm** is one which relies on some random numbers in addition to the input.

  $\left(\right.$Computers cannot generate randomness. We assume that there exists a *pseudo-random number generator (PRNG)*, a deterministic program that uses an initial value or *seed* to generate a sequence of seemingly random numbers. The quality of randomized algorithms depends on the quality of the PRNG!$\left.\right)$

- Doing randomization is often a good idea if an algorithm has bad worst-case time but seems to perform much better on most instances.

- **Goal:** Shift the dependency of run-time from what we can't control (the input) to what we *can* control (the random numbers).

  No more bad instances, just unlucky numbers.

Expected run-time

The run-time of the algorithm now depends on the random numbers.

Define $T_{\mathcal{A}}(I, R)$ to be the run-time of a randomized algorithm $\mathcal{A}$ for an instance I and the sequence R of outcomes of random trials.

The **expected run-time $T^{exp}(I)$ for instance I** is the expected value:

$$T^{exp}(I) = \mathbf{E}[T(I,R)] = \sum_{R} T(I,R) \cdot \Pr(R)$$

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We can still have good luck or bad luck, so occasionally we also discuss the very worst that could happen, i.e., $\max_I \max_R T(I,R)$.

Analysis of *merge* in a meldable heap

Observe: *merge* does two *random downward walks* in a binary tree.

- Let $T(I, R)$ = length of random downward walk in tree I when random outcomes are R.
- As usual: $T^{\exp}(n) = \max_{|I|=n} \sum_R Pr(R)T(I, R)$.

Theorem: $T^{\exp}(n) \in O(\log n)$.

Proof:

So *merge* (and also *insert* and *delete-max*) takes $O(\log n)$ expected time.

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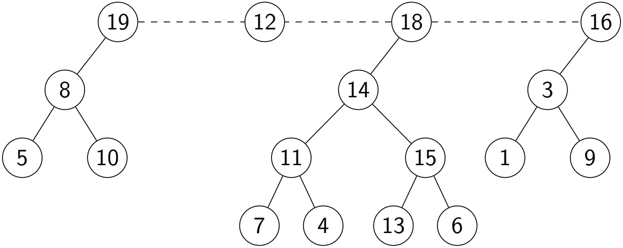
Binomial Heaps

Very different structure from binary heaps and meldable heaps:



- List L of binary trees.
- Each binary tree is a **flagged tree**:
  Complete binary tree T plus root r that has T as left subtree
  - Flagged tree of height h has 2^h nodes.
  - So $h \leq \log n$ for all flagged trees.
- Order-property: Nodes in *left* subtree have no-smaller keys.
  (No restrictions on nodes in the right subtree.)

Binomial Heap Operations

- *insert*: Reduce to *merge* as before.
- *delete-max*: Bottleneck is *finding* the maximum.
    - At each flagged tree, root contains the maximum of tree.
    - Search roots in $L \Rightarrow O(|L|)$ time.
    - (Removal also will be non-trivial $\rightsquigarrow$ later)
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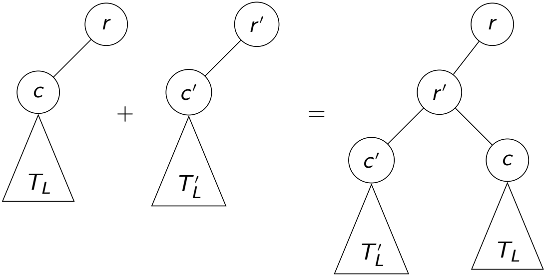
- **Proper binomial heap:** No two flagged trees have the same height.
- **Observation:** A proper binomial heap has $|L| \leq \log n + 1$.
  - The flagged tree of largest height h has $h \leq \log n$.
  - Can have only one flagged tree of each height in $\{0, \ldots, h\}$.

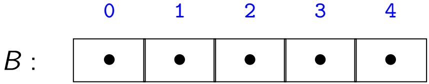
Making Binomial Heaps Proper

- Goal: Given a binomial heap, make it proper.
- Need subroutine: combine two flagged trees of the same height. This can be done in constant time: If $r.\mathit{key} \geq r'.\mathit{key}$:



- Idea: Do this whenever two flagged trees have same height.
- With this, *make-proper* can be implemented in $O(|L| + \log n)$ time.

Making Binomial Heaps Proper



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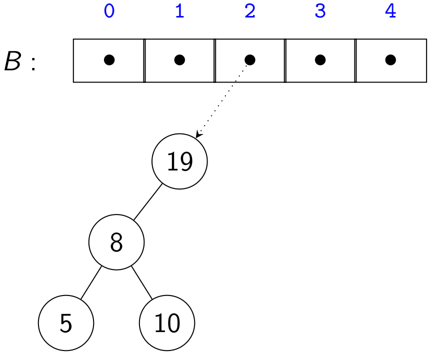

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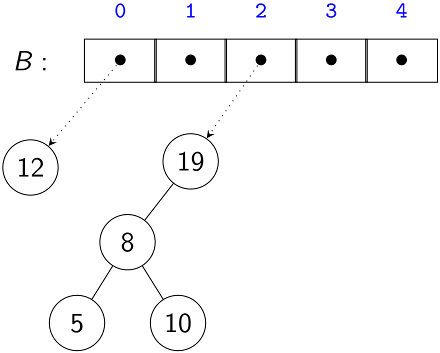

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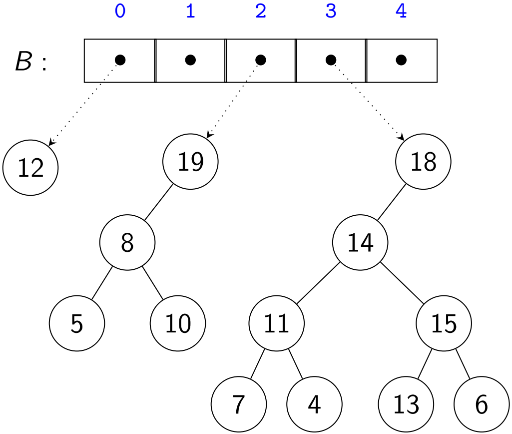

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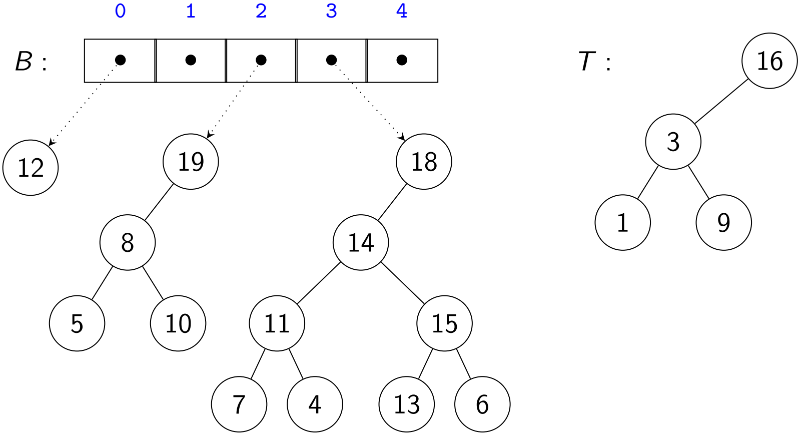

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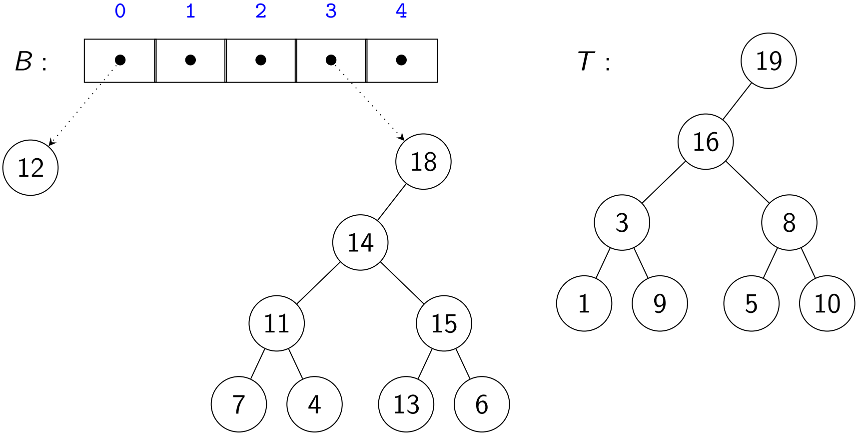

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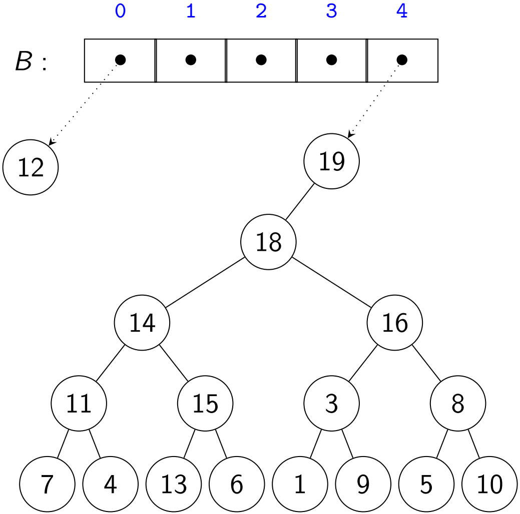


Making Binomial Heaps Proper

```
binomialHeap::make-proper()
n ← size of the binomial heap
compute $\ell \leftarrow \lfloor \log n \rfloor$
$B \leftarrow$ array of size $\ell + 1$, initialized all-NULL
$L \leftarrow$ list of flagged trees
while $L$ is non-empty do
    $T \leftarrow L.pop()$, $h \leftarrow T.height$
    while $T' \leftarrow B[h]$ is not NULL do
        if $T.root.key < T'.root.key$ do swap $T$ and $T'$
        // combine $T$ with $T'$
        $T'.right \leftarrow T.left$, $T.left \leftarrow T'$, $T.height \leftarrow h+1$
        $B[h] \leftarrow$ NULL, $h$++
    $B[h] \leftarrow T$
// copy $B$ back to list
for ($h = 0$; $h \leq \ell$; $h$++) do
    if $B[h] \neq$ NULL do $L.append(B[h])$
```

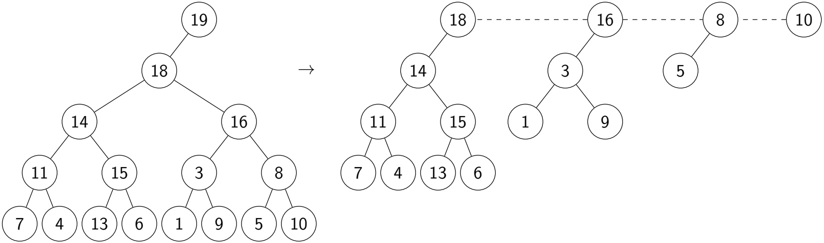
Binomial Heap Operations

- **Idea:** Make binomial heap proper after *every* operation.
  - ⇒ L *always* has length $O(\log n)$
  - ⇒ Each *make-proper* takes $O(\log n)$ time

- *merge*: $O(\log n)$ worst-case time.
  - Concatenate the two lists.
  - Call *make-proper*.
- *find-max*: $O(\log n)$ worst-case time.
  - Find maximum root in $O(|L|)$ time
- *insert*: $O(\log n)$ worst-case time
  - Create new flagged tree with one node, add to L.
  - Call *make-proper*.
- *delete-max*
  - Find maximum as in *find-max*
  - Now how do we remove it?

delete-max in a binomial heap

- Say the maximum key is at root of flagged tree T
- Split $T \setminus \{\text{root}\}$ into into flagged trees $T_1, \ldots, T_k$

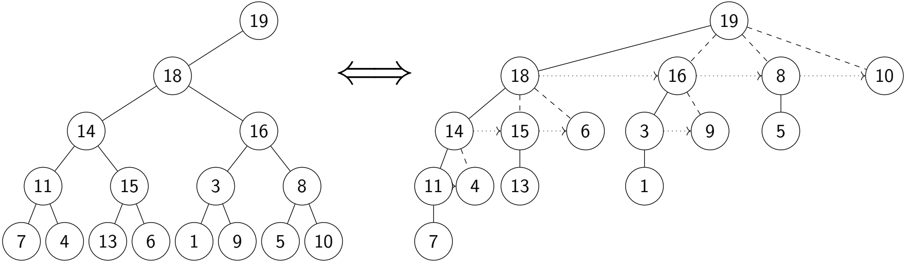


- Remove T from L, create new binomial heap M' with $\{T_1, \ldots, T_k\}$
- Have $k \leq \log n \Rightarrow O(\log n)$ worst-case time.
- Apply *merge* to M' and existing binomial heap

Summary: All operations have $O(\log n)$ worst-case run-time.

Why these weird conventions?

- Flagged trees, heap-order property, *delete-max* seem unintuitive.
- These are actually very intuitive if one knows **left-child-right-sibling conversion** from binary trees to multi-way trees.



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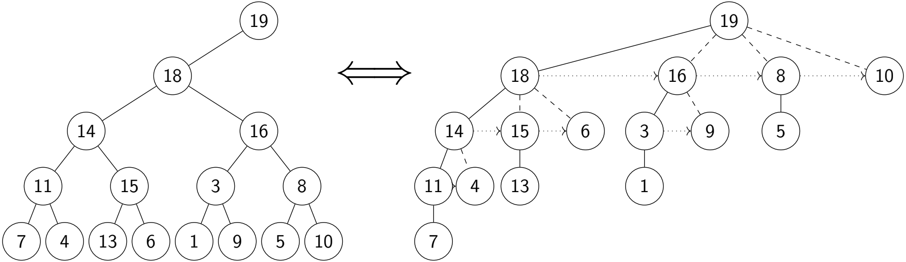

- Flagged tree of height d ⇔ d subtrees, of heights $d-1, d-2, \ldots, 0$
- Binomial-heap order property ⇔ standard heap-order property
- *delete-max* ⇔ create new heap with all children of the root