CS 240E – Data Structures and Data Management (Enriched)

Module 3: Sorting, Average-case and Randomization

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Based on lecture notes by many previous cs240 instructors

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Outline

3. Sorting, Average-case and Randomization
   - Review and Outlook
   - Analyzing average-case run-time
   - Run-time on randomly chosen input
   - SELECTION and *quick-select*
   - Tips and Tricks for *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

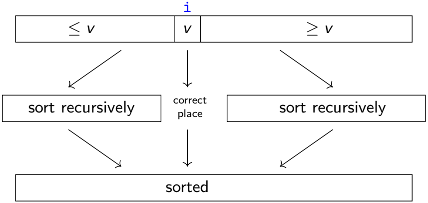
Outline

Review and Outlook

Recall the following well-known algorithm for SORTING:

quick-sort(A) `// array of size` n

1. **if** $n \leq 1$ **then return**
2. $i \leftarrow$ *partition*$(A,$ *choose-pivot*$(A))$
3. *quick-sort*$(A[0, 1, \ldots, i-1])$
4. *quick-sort*$(A[i+1, \ldots, n-1])$



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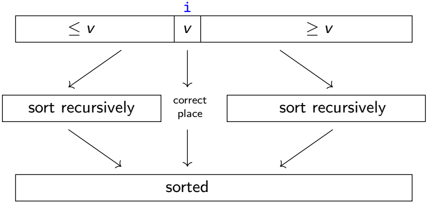

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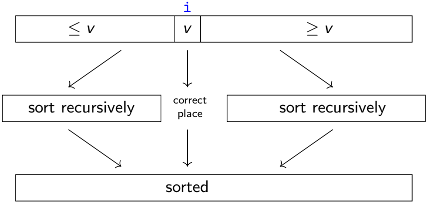

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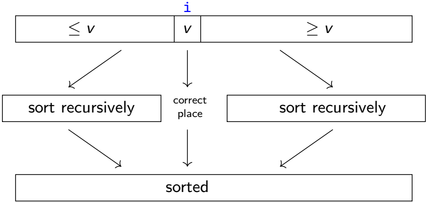

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Goal: Analyze *average-case* run-time via *randomization*.

Two detours needed:

- How does one analyze the average-case?
- And what does that have to do with randomization?

Outline

Average-case analysis

Recall definition of average-case run-time:

$$T^{\mathrm{avg}}_{\mathcal{A}}(n) = \sum_{\text{instance } I \text{ of size } n} T_{\mathcal{A}}(I) \cdot \big(\text{relative frequency of } I\big)$$

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For this module:

- Assume that the set $\mathcal{I}_n$ of size-n instances is finite (or can be mapped to a finite set in a natural way)
- Assume that all instances occur equally frequently

Then we can use the following *simplified formula*

$$T^{\text{avg}}(n) = \frac{\sum_{I:\text{size}(I)=n} T(I)}{\#\text{instances of size } n} = \frac{1}{|\mathcal{I}_n|} \sum_{I \in \mathcal{I}_n} T(I)$$

To learn how to analyze this, we will do simpler examples first.

A simple (contrived) example

silly-test(π, n)
π: a permutation of $\{0, \ldots, n-1\}$, stored as an array
1. **if** $\pi[0] = 0$ **then for** $j \leftarrow 1$ to n **do** print 'bad case'
2. **else** print 'good case'

$$T^{\text{avg}}(n) = \frac{1}{n!} \sum_{\pi \in \Pi_n} T(\pi) = \frac{1}{n!} \Big(\sum_{\substack{\pi \in \Pi_n \\ \text{in bad case}}} T(\pi) + \sum_{\substack{\pi \in \Pi_n \\ \text{in good case}}} T(\pi) \Big)$$

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- bad case $\Rightarrow$ run-time $\leq c \cdot n$ (for some constant $c > 0$)
- good case $\Rightarrow$ run-time $\leq c$

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$$T^{\text{avg}}(n) \leq \frac{1}{n!} \Big(\underbrace{\#\{\pi \in \Pi_n \text{ in bad case}\}} \cdot cn + \underbrace{\#\{\pi \in \Pi_n \text{ in good case}\}} \cdot c \Big)$$

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A simple (contrived) example

silly-test$(\pi,\ n)$
π: a permutation of $\{0,\dots,n{-}1\}$, stored as an array
1. **if** $\pi[0] = 0$ **then for** $j \leftarrow 1$ to n **do** print 'bad case'
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$$T^{\text{avg}}(n) \quad = \frac{1}{n!}\sum_{\pi\in\Pi_n} T(\pi) = \frac{1}{n!}\Big(\sum_{\substack{\pi\in\Pi_n\\ \text{in bad case}}} T(\pi) + \sum_{\substack{\pi\in\Pi_n\\ \text{in good case}}} T(\pi)\Big)$$

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$$\le \frac{1}{n!}\Big((n-1)!\cdot cn + n!\cdot c\Big) = \frac{1}{n}cn + c = 2c \in O(1)$$

A second (not-so-contrived) recursive example

```
all-0-test(w, n)
// test whether all entries of bitstring w[0..n−1] are 0
1.  if (n = 0) return true
2.  if (w[n−1] = 1) return false
3.  all-0-test(w, n−1)
```

(In real life, you would write this non-recursive.)

Define $T(w) = \#$ bit-comparisons (i.e., line 2) on input w. This is asymptotically the same as the run-time.

Worst-case run-time: Always go into the recursion until $n = 0$.
$T(n) = 1 + T(n-1) = 1 + 1 + \cdots + T(0) = n \in \Theta(n)$.

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Best-case run-time: Return immediately. $T(n) = 1 \in \Theta(1)$.

Average-case run-time?

Average-case run-time of *all-0-test*

$$T^{\text{avg}}(n) = \frac{1}{|\mathcal{B}_n|} \sum_{w \in \mathcal{B}_n} T(w). \qquad (\mathcal{B}_n = \{\text{bitstrings of length } n\}, |\mathcal{B}_n| = 2^n)$$

Recursive formula for one non-empty bitstring w:

$$T(w) = \begin{cases} 1 & \text{if } w[n-1] = 1 \\ 1 + T(\underbrace{w[0..n-2]}_{\text{length } n-1}) & \text{otherwise} \end{cases}$$

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Natural guess for the recursive formula for $T^{\text{avg}}(n)$:

$$T^{\text{avg}}(n) = \underbrace{\tfrac{1}{2}}_{\substack{\text{half have} \\ w[n-1]=1}} \cdot 1 + \underbrace{\tfrac{1}{2}}_{\substack{\text{half have} \\ w[n-1]=0}} (1 + T^{???}(n-1))$$

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Recursive formula for one non-empty bitstring w:

$$T(w) = \begin{cases} 1 & \text{if } w[n{-}1] = 1 \\ 1 + T(\underbrace{w[0..n{-}2]}_{\text{length } n-1}) & \text{otherwise} \end{cases}$$

Natural guess for the recursive formula for $T^{\text{avg}}(n)$:

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- This holds with $\leq$ (but is useless) if '???' is 'worst'.
- This is *not obvious* if '???' is 'avg'.

Average-case run-time of *all-0-test*

$$T^{\text{avg}}(n) = \frac{1}{|\mathcal{B}_n|} \sum_{w \in \mathcal{B}_n} T(w)$$

$$=$$

$$= 1 + \frac{1}{2} T^{\text{avg}}(n-1)$$

Easy induction proof: $T^{\text{avg}}(n) \leq 2 \in O(1)$.

Average-case analysis and recursions

Why can't we always write 'avg' for '???' in $T^{\text{avg}}(n) = 1 + \frac{1}{2} T^{???}(n-1)$?

Consider the following (contrived) example:

silly-all-0-test(w, n)
w: array of size at least n that stores bits
1. **if** $(n = 0)$ **then return** `true`
2. **if** $(w[n-1] = 1)$ **then return** `false`
3. **if** $(n > 1)$ **then** $w[n-2] \leftarrow 0$ `// this is the only change`
4. *silly-all-0-test*$(w, n-1)$

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- Only one more line of code in each recursion, so same formula applies.
- But observe that now $T(w) = \begin{cases} 1 & \text{if } w[n-1] = 1 \\ n & \text{if } w[n-1] = 0 \end{cases}$.

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- Only one more line of code in each recursion, so same formula applies.
- But observe that now $T(w) = \begin{cases} 1 & \text{if } w[n-1] = 1 \\ n & \text{if } w[n-1] = 0 \end{cases}$.
- So $T^{\text{avg}}(n) = 1 + \frac{n}{2} \in \Theta(n)$. The "obvious" recursion did not hold.

Average-case analysis is highly non-trivial for recursive algorithms.

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Randomizations of algorithms

randomized-all-0-test$(w,\ n)$
w: array of size at least n that stores bits

```
if n = 0 return true
if (random(2)=0) then
        w[n−1] ← 1 − w[n−1]      // this is the only change
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In each recursion, we use the outcome $x \in \{0, 1\}$ of one coin toss.
We return without recursing if $x = w[n-1]$ (this has probability $\frac{1}{2}$).

Expected run-time of *randomized-all-0-test*

Let $T(w, R)$ be the # of bit-comparisons used on input w if the random outcomes are R.

- The random outcomes R consist of two parts $R = \langle x, R' \rangle$:
  - x: outcome of first coin toss
  - R': random outcomes (if any) for the recursions

  We have $\Pr(R) = \Pr(x) \cdot \Pr(R')$ (random choices are independent).

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- Recursive formula for one instance:

$$T(w, R) = T(w, \langle x, R' \rangle) = \begin{cases} 1 & \text{if } x = w[n-1] \\ 1 + T(w[0..n-2], R') & \text{otherwise} \end{cases}$$

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- Natural guess for the recursive formula for $T^{\exp}(n)$:

$$T^{\exp}(n) = \underbrace{\tfrac{1}{2}}_{\Pr(x=w[n-1])} \cdot 1 + \underbrace{\tfrac{1}{2}}_{\Pr(x \neq w[n-1])} (1 + T^{\exp}(n-1)) = 1 + \tfrac{1}{2} T^{\exp}(n-1)$$

Expected run-time of *randomized-all-0-test*

In contrast to average-case analysis, the natural guess usually is correct for the expected run-time.

Proof for *randomized-all-0-test*:

$$T^{\exp}(w) \;=\; \sum_R \Pr(R)\, T(w,R) =$$

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\begin{aligned}
T^{\exp}(w) &= \sum_{R} \Pr(R)\, T(w,R) = \sum_{x}\sum_{R'} \Pr(x)\Pr(R')\, T(w,\langle x, R'\rangle) \\
&=
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Proof for *randomized-all-0-test*:

$$T^{\exp}(w) = \sum_R \Pr(R)\, T(w,R) = \sum_x \sum_{R'} \Pr(x) \Pr(R')\, T(w, \langle x, R' \rangle)$$

$$=$$

Therefore $T^{\exp}(n) = \max_{w \in \mathcal{B}_n} T^{\exp}(w) \leq 1 + \frac{1}{2} T^{\exp}(n{-}1)$

Expected run-time of *randomized-all-0-test*

- We had $T^{\exp}_{\textit{rand-all-0-test}}(n) \leq 1 + \frac{1}{2}\,T^{\exp}_{\textit{rand-all-0-test}}(n-1)$
- We earlier had $T^{\text{avg}}_{\textit{all-0-test}}(n) \leq 1 + \frac{1}{2}\,T^{\text{avg}}_{\textit{all-0-test}}(n-1)$
- Same recursion $\Rightarrow$ same upper bound $\Rightarrow$ $T^{\exp}_{\textit{rand-all-0-test}}(n) \in O(1)$.

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Recall: *randomized-all-0-test* was very similar to *all-0-test*
(The only different was a random bitflip.)

Is it a coincidence that the two recursive formulas are the same?

Or does the expected time of a randomized version always have something to do with the average-case time?

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Is it a coincidence that the two recursive formulas are the same?

Or does the expected time of a randomized version always have something to do with the average-case time?

- Not in general! (It depends how we randomize.)
- Yes if the randomization is a *shuffle* (choose instance randomly).

Avg-case run-time via expected run-time

Consider the following randomization of a deterministic algorithm $\mathcal{A}$.

shuffle-$\mathcal{A}(n)$
1. Among all instances $\mathcal{I}_n$ of size n for $\mathcal{A}$, choose I randomly
2. $\mathcal{A}(I)$

(*shuffle-*$\mathcal{A}$ usually does not solve what $\mathcal{A}$ solves)

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- If we do not count the time for line 1:

$$T^{\text{avg}}_{\mathcal{A}}(n) = \frac{1}{|\mathcal{I}_n|} \sum_{I \in \mathcal{I}_n} T(I) = \sum_{I \in \mathcal{I}_n} Pr(I \text{ chosen}) \cdot T(I) = T^{\text{exp}}_{\textit{shuffle-}\mathcal{A}}(n)$$

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- So the average-case run-time of $\mathcal{A}$ is the same as this **run-time of $\mathcal{A}$ on randomly chosen input**.
- This gives us a different way to compute $T^{\text{avg}}_{\mathcal{A}}(n)$.

Avg-case run-time via expected run-time

Example: *all-0-test* (rephrased with for-loops):

```
shuffle-all-0-test(n)
for (i ← n−1; i ≥ 0; i--) do
        w[i] ← random(2)
for (i ← n−1; i ≥ 0; i--) do
        if (w[i] = 1) return false
return true
```

```
randomized-all-0-test(w, n)
for (i ← n−1; i ≥ 0; i--) do
        if (random(2)=0) then
                w[i] ← 1 − w[i]
        if (w[i] = 1) return false
return true
```

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 4.       if (w[i] = 1) return false
 5. return true
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randomized-all-0-test(w, n)
 1. for (i ← n−1; i ≥ 0; i--) do
 2.       if (random(2)=0) then
                w[i] ← 1 − w[i]
 3.       if (w[i] = 1) return false
 4. return true
```

- These algorithms are not quite the same.
  - Randomization outside respectively inside the for-loop.
- But this does not matter for the expected number of bit-comparisons.
  - Either way, at time of comparison the bit is 1 with probability $\frac{1}{2}$.

Avg-case run-time via expected run-time

Example: *all-0-test* (rephrased with for-loops):

shuffle-all-0-test(n)
1. **for** ($i \leftarrow n{-}1$; $i \geq 0$; i−−) **do**
2. $\quad w[i] \leftarrow$ *random*(2)
3. **for** ($i \leftarrow n{-}1$; $i \geq 0$; i−−) **do**
4. $\quad$ **if** ($w[i] = 1$) **return** `false`
5. **return** `true`

randomized-all-0-test(w, n)
1. **for** ($i \leftarrow n{-}1$; $i \geq 0$; i−−) **do**
2. $\quad$ **if** (*random*(2)=0) **then**
   $\quad\quad w[i] \leftarrow 1 - w[i]$
3. $\quad$ **if** ($w[i] = 1$) **return** `false`
4. **return** `true`

- These algorithms are not quite the same.
  - Randomization outside respectively inside the for-loop.
- But this does not matter for the expected number of bit-comparisons.
  - Either way, at time of comparison the bit is 1 with probability $\frac{1}{2}$.

- So $T^{\text{avg}}_{\textit{all-0-test}}(n) = T^{\text{exp}}_{\textit{shuffle-all-0-test}}(n) = T^{\text{exp}}_{\textit{rand-all-0-test}}(n) \in O(1)$

  can be deduced without analyzing $T^{\text{avg}}_{\textit{all-0-test}}(n)$ directly.

Summary: Average-case run-time vs. expected run-time

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No!

average-case run-time	expected run-time
$\frac{1}{\lvert\mathcal{I}_n\rvert}\sum_{I\in\mathcal{I}_n} T(I)$	$\max_{I\in\mathcal{I}_n}\sum_{\text{outcomes } R} \Pr(R)\cdot T(I,R)$
average over instances	weighted average over random outcomes
(usually) applied to a deterministic algorithm	applied only to a randomized algorithm

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There is a relationship *only* if the randomization effectively achieves "choose the input instance randomly".

Outline

The SELECTION Problem

SELECTION problem: Given an array A of n numbers, and $0 \le k < n$, find the element that would be at position k of the sorted array.

(We also call this the element of **rank** k.)

0	1	2	3	4	5	6	7	8	9
30	60	10	0	50	80	90	10	40	70

select(3) should return 30.

SELECTION can be done with heaps in time $\Theta(n + k \log n)$.

Special case: MEDIANFINDING = SELECTION with $k = \lfloor \frac{n}{2} \rfloor$. With previous approaches, this takes time $\Theta(n \log n)$, no better than sorting.

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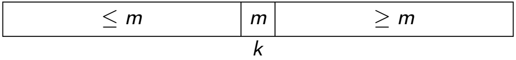
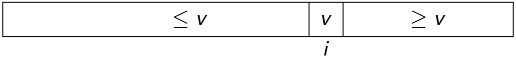
Question: Can we do selection in linear time?
Yes! We will develop algorithm *quick-select* below.

quick-select Algorithm

Goal: Find element m of rank k by rearranging A:



Recall: *partition* method (from *quick-sort*) achieves



Where is m if $i = k$? If $i < k$? If $i > k$?

quick-select(A, k)
A: array of size n, k: integer s.t. $0 \le k < n$

$p \leftarrow$ *choose-pivot*(A)
$i \leftarrow$ *partition*(A, p)
if $i = k$ **then return** $A[i]$
else if $i > k$ **then return** *quick-select*$(A[0 \ldots i{-}1], k)$
else if $i < k$ **then return** *quick-select*$(A[i{+}1 \ldots n{-}1], k - (i{+}1))$

Analysis of *quick-select*

Let $T(A,k)$ be the number of key-comparisons for *quick-select*(A,k).
partition uses n key-comparisons.
Write A' for rearranged A after *partition*, and i for the pivot-rank.

$$T(A,k)=\begin{cases} n & \text{if } i=k \\ n+T(A'[0..i-1],k) & \text{if } i>k \quad \text{(sub-array has size } i\text{)} \\ n+T(A'[i+1..n-1],k-i-1) & \text{if } i<k \quad (\ldots\text{size } n-i-1) \end{cases}$$

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Worst-case run-time:

- Sub-array always gets smaller, so $\leq n$ recursions $\Rightarrow O(n^2)$ time.
- This is tight: If pivot-rank is always $n-1$ and $k=0$
  $T^{\text{worst}}(n,0) \geq n + (n{-}1) + (n{-}2) + \cdots + 1 \in \Omega(n^2)$

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Average case run-time?

Average-case analysis of quick-select (sketch)

$$T(A,k) = \begin{cases} n & \text{if } i = k \\ n + T(A'[0..i{-}1], k) & \text{if } i > k \quad \text{(sub-array has size } i\text{)} \\ n + T(A'[i{+}1..n{-}1], k{-}i{-}1) & \text{if } i < k \quad (\ldots \text{size } n{-}i{-}1) \end{cases}$$

To obtain "obvious" recursive formula:

- Argue: $\frac{1}{n}$th of the inputs have pivot-rank i

  (Easy if we assume that instances are permutations.)

- Argue: the rank-index k does not matter for analysis.

  (Not obvious, but analysis works even if we take $\max_k$.)

- Argue: If A is "average", then A' is also "average". *Difficult!*

  (
  - Formally: Over all choices of A, we have equally many occurrences of each possibility of A'.
  - False for some implementations of *partition*. Correct if *partition* only compares to the pivot-value.

)

$$T^{\text{avg}}(n) \leq n + \frac{1}{n} \sum_{i=0}^{n-1} \max \left\{ T^{\text{avg}}(i), T^{\text{avg}}(n{-}i{-}1) \right\}$$

Randomizing quick-select: Shuffling

To avoid the difficult proof, use randomization instead.

Goal: Create a randomized version of *quick-select*.

- This will give a proof of the avg-case run-time of *quick-select*.
- This will be a better algorithm in practice.

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Goal: Create a randomized version of *quick-select*.

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- This will be a better algorithm in practice.

First idea: Shuffle the input, then do *quick-select*.

> *shuffle-quick-select*(A, k)
> 1. **for** $(j \leftarrow 1$ to $n-1)$ **do** *swap*$(A[j], A[$*random*$(j+1)])$ `// shuffle`
> 2. *quick-select*(A, k)

- **Assumption:** Shuffling (permuting) the input-array is equivalent to randomly choosing an input instance.
- So we know $T^{\text{avg}}_{\text{quick-select}}(n) = T^{\text{exp}}_{\text{shuffle-quick-select}}(n)$

(Recall: $T(\cdot)$ counts key-comparisons, so shuffling is free.)

Randomizing quick-select: Random Pivot

Second idea: Do the shuffling inside the recursion.
(Equivalently: Randomly choose which value is used for the pivot.)

randomized-quick-select(A, k)
1. swap $A[n-1]$ with $A[random(n)]$
2. $i \leftarrow$ *partition*$(A, n-1)$
3. **if** $i = k$ **then return** $A[i]$
4. **else if** $i > k$ **then**
5. **return** *randomized-quick-select*$(A[0 \ldots i-1], k)$
6. **else if** $i < k$ **then**
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    swap A[n−1] with A[random(n)]
    i ← partition(A, n−1)
    if i = k then return A[i]
    else if i > k then
        return randomized-quick-select(A[0 . . . i−1], k)
    else if i < k then
        return randomized-quick-select(A[i+1 . . . n−1], k − (i+1))
```

- $T^{\text{exp}}_{\textit{rand.-quick-select}}(n) = T^{\text{exp}}_{\textit{shuffle-quick-select}}(n)$.

(This is not completely obvious, but believable. No proof.)

Expected run-time of *randomized-quick-select*

Let $T(A, k, R) = \#$ key-comparisons of *randomized-quick-select* on input $\langle A, k \rangle$ if the random outcomes are R.

- Write random outcomes R as $R = \langle i, R' \rangle$ (where 'i' stands for 'the first random number was such that the pivot-rank is i')
- Observe: $\Pr(\text{pivot-rank is } i) = \frac{1}{n}$
- We recurse in an array of size i or $n-i-1$ (or not at all)

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- Recursive formula for one instance (and fixed $R = \langle i, R' \rangle$):

$$T(A, k, \langle i, R' \rangle) = n + \begin{cases} T(\text{ size-}i \text{ array}, k, R') & \text{if } i > k \\ T(\text{ size-}(n{-}i{-}1) \text{ array}, k{-}i{-}1, R') & \text{if } i < k \\ 0 & \text{otherwise} \end{cases}$$

Analysis of *randomized-quick-select*

Since the expected run-time uses the *worst-case instance*, the recursive formula can now be shown easily:

$$T^{\exp}(A,k) \left(\begin{aligned} &= \sum_R P(R)\cdot T(\langle A,k\rangle,R) = \sum_{i=0}^{n-1}\sum_{R'} P(i)\cdot P(R')\cdot T(\langle A,k\rangle,\langle i,R'\rangle) \\ &= \frac{1}{n}\sum_{i=0}^{k-1}\sum_{R'} P(R')\Big(n + T(\langle A'[i{+}1..n{-}1],k{-}i{-}1\rangle,R')\Big) \\ &\qquad + \underbrace{\frac{1}{n}\cdot n}_{i=k} \quad + \frac{1}{n}\sum_{i=k+1}^{n-1}\sum_{R'} P(R')\Big(n + T(\langle A'[0..i{-}1,k\rangle,R')\Big) \\ &= n + \frac{1}{n}\sum_{i=0}^{k-1}\sum_{R'} P(R')T(\langle A'[i{+}1..n{-}1],k{-}i{-}1\rangle,R') \\ &\qquad + \frac{1}{n}\sum_{i=k+1}^{n-1}\sum_{R'} P(R')T(\langle A'[0..i{-}1,k\rangle,R') \\ &= n + \frac{1}{n}\sum_{i=0}^{k-1}\underbrace{T^{\exp}(\langle A'[i{+}1..n{-}1],k{-}i{-}1\rangle)}_{\le T^{\exp}(n-i-1)} + \frac{1}{n}\sum_{i=k+1}^{n-1}\underbrace{T^{\exp}(\langle A'[0..i{-}1],k\rangle)}_{\le T^{\exp}(i)} \end{aligned} \right) \text{tedious but straightforward}$$

$$\le n + \frac{1}{n}\sum_{i=0}^{n-1}\max\{T^{\exp}(i),\,T^{\exp}(n{-}i{-}1)\}$$ *independent* of A, k

Analysis of *randomized-quick-select*

In summary, the expected run-time of *randomized-quick-select* satisfies:

$$T^{\exp}(n) \le n + \frac{1}{n}\sum_{i=0}^{n-1} \max\{T^{\exp}(i), T^{\exp}(n-i-1)\}$$

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Claim: This recursion resolves to $O(n)$.

Summary of Selection

- *randomized-quick-select* has expected run-time $\Theta(n)$.
  - The run-time bound is tight since *partition* takes $\Omega(n)$ time
  - If we're unlucky in the random numbers then the run-time is still $\Omega(n^2)$
- So the expected run-time of *shuffle-quick-select* is $\Theta(n)$.
- So the run-time of *quick-select* on randomly chosen input is $\Theta(n)$.
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- So the average-case run-time of *quick-select* is $\Theta(n)$.

- *randomized-quick-select* is generally the fastest solution to SELECTION.

- There exists a variation that solves SELECTION with worst-case run-time $\Theta(n)$, but it uses double recursion and is slower in practice. ($\rightarrow$ *cs*341, maybe)

Randomizing quick-sort

We analyze the avg-case run-time of *quick-sort* again via randomization.

randomized-quick-sort(A)
1. **if** $n \leq 1$ **then return**
2. $p \leftarrow$ *random*(n)
3. $i \leftarrow$ *partition*(A, p)
4. *randomized-quick-sort*$(A[0, 1, \ldots, i-1])$
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- $\Pr(\text{pivot has rank } i) = \frac{1}{n}$
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This implies

$$T^{\exp}(n) = \underbrace{\ldots \ = \ \ldots \ \leq \ \ldots}_{\text{long but straightforward}} = n + \frac{1}{n}\sum_{i=0}^{n-1}\Big(T^{\exp}(i) + T^{\exp}(n-i-1)\Big)$$

Expected run-time of *randomized-quick-sort*

$$T^{\exp}(n) \le n + \frac{1}{n}\sum_{i=0}^{n-1}\Big(T^{\exp}(i) + T^{\exp}(n-i-1)\Big) = n + \frac{2}{n}\sum_{i=1}^{n-1} T^{\exp}(i)$$

(since $T(0) = 0$)

Claim: $T^{\exp}(n) \in O(n \log n)$.

Proof:

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Summary of *quick-sort*

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- This implies (with the same detour through *shuffle-quick-sort*):

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- The auxiliary space is not good ($\Theta(n)$) but can be improved ($\rightsquigarrow$ later)

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- The auxiliary space is not good ($\Theta(n)$) but can be improved ($\rightsquigarrow$ later)
- There are numerous other tricks to improve *randomized-quick-select*
  - We will see some below.
- With these, this is in practice the fastest solution to SORTING (but *not* in theory).

Outline

3. Sorting, Average-case and Randomization
   - Review and Outlook
   - Analyzing average-case run-time
   - Run-time on randomly chosen input
   - SELECTION and *quick-select*
   - Tips and Tricks for *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

quick-sort with tricks

```
randomized-quick-sort-improved(A, n)
Initialize a stack S of index-pairs with { (0, n−1) }
while S is not empty
    (ℓ, r) ← S.pop()                          // avoid recursions
    while (r−ℓ+1 > 10) do                     // stop recursions early
        p ← ℓ + random(ℓ−r+1)
        i ← Hoare-partition(A, ℓ, r, p)       // use better routine
        if (i−ℓ > r−i) do                     // reduce aux. space
            S.push( (ℓ, i−1) )
            ℓ ← i+1                           // remove tail-recursion
        else
            S.push( (i+1, r) )
            r ← i−1
insertion-sort(A)
```

Hoare's Partition Routine

- *partition* is very easy to implement with lists or streams (exercise). This uses $O(n)$ auxiliary space and is rather slow.
- More challenging: partition **in place** (with $O(1)$ auxiliary space).
- **Idea**: Keep swapping the outer-most wrongly-positioned pairs.

i=-1	0	1	2	3	4	5	6	7	8	j=9
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- *partition* is very easy to implement with lists or streams (exercise). This uses $O(n)$ auxiliary space and is rather slow.
- More challenging: partition **in place** (with $O(1)$ auxiliary space).
- **Idea**: Keep swapping the outer-most wrongly-positioned pairs.

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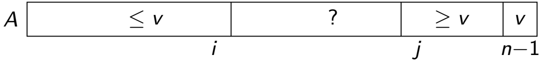
Hoare's In-Place Partition Routine

Loop invariant:



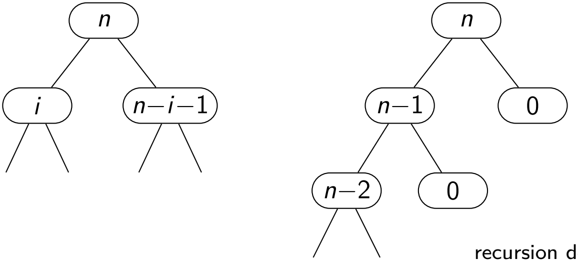
Hoare-partition(A, p)

A: array of size n, p: integer s.t. $0 \le p < n$

1. *swap*$(A[n-1], A[p])$
2. $i \leftarrow -1, \quad j \leftarrow n-1, \quad v \leftarrow A[n-1]$
3. **loop**
4. **do** $i \leftarrow i+1$ **while** $A[i] < v$
5. **do** $j \leftarrow j-1$ **while** $j > i$ and $A[j] > v$
6. **if** $i \ge j$ **then break** (goto 9)
7. **else** *swap*$(A[i], A[j])$
8. **end loop**
9. *swap*$(A[n-1], A[i])$
10. **return** i

Improvement ideas for quick-sort

- Every recursive call uses $O(1)$ auxiliary space to store a record.
- *quick-sort* has nested recursive calls. To analyze its auxiliary space, consider the **recursion tree** and analyze its height (**recursion depth**)
  - Write size of subproblem into each node.
  - If $n \geq 2$ then there are two subproblems, hence two children.



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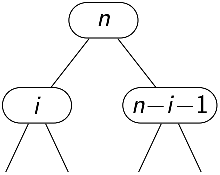

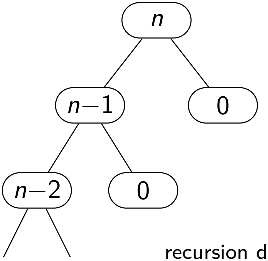

recursion depth can be $\Omega(n)$

- Recursion tree is also useful for analyzing the run-time:
  - On every level, the total number of key-comparisons is $\leq n$.
  - Can argue (later): On average, the height is $O(\log n)$.
  - This gives another proof of $O(n \log n)$ average-case run-time.

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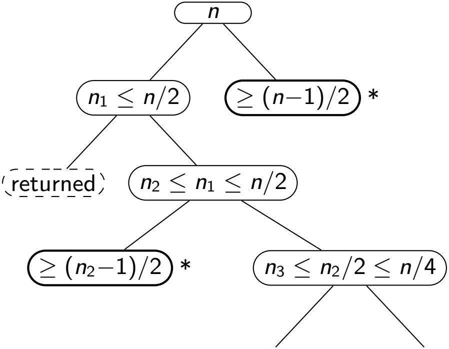
- Either halved the size (or better).
- Or the sibling is done $\Rightarrow$ not on stack



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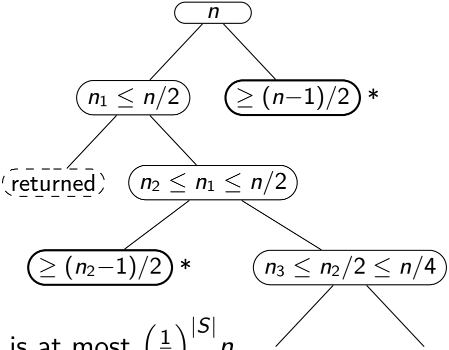

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At all times, the current problem size is at most $\left(\frac{1}{2}\right)^{|S|} n$.

$\Rightarrow$ At all times, $|S| \leq \log n$.

Outline

Lower bounds for sorting

We have seen many sorting algorithms:

Sort	Running time	Analysis
selection-sort	$\Theta(n^2)$	worst-case
insertion-sort	$\Theta(n^2)$ $\Theta(n)$	worst-case best-case
merge-sort	$\Theta(n \log n)$	worst-case
heap-sort	$\Theta(n \log n)$	worst-case
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Question: Can one do better than $\Theta(n \log n)$ running time?
Answer: Yes and no! *It depends on what we allow*.

- No: Comparison-based sorting lower bound is $\Omega(n \log n)$.
- Yes: Non-comparison-based sorting can achieve $O(n)$ (under restrictions!). ($\rightarrow$ later)

Lower bound for sorting in the comparison model

All algorithms so far are **comparison-based**: Data is accessed only by

- comparing two elements (a *key-comparison*)
- moving elements around (e.g. copying, swapping)

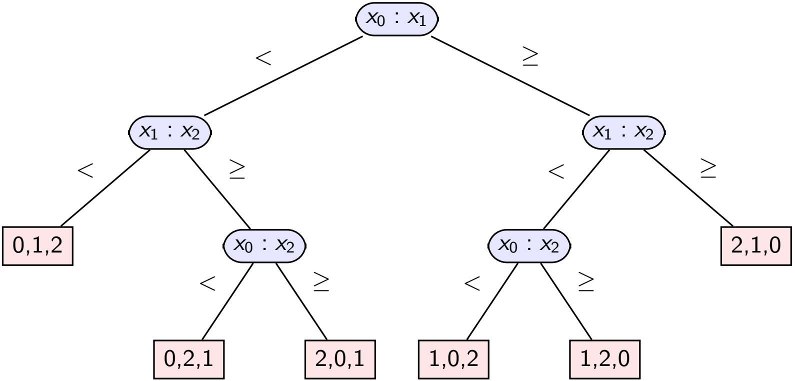
Theorem. Any *comparison-based* sorting algorithm requires in the worst case $\Omega(n \log n)$ comparisons to sort n distinct items.

Proof.

Decision trees

Any comparison-based algorithms can be expressed as **decision tree**.

To sort $\{x_0, x_1, x_2\}$:



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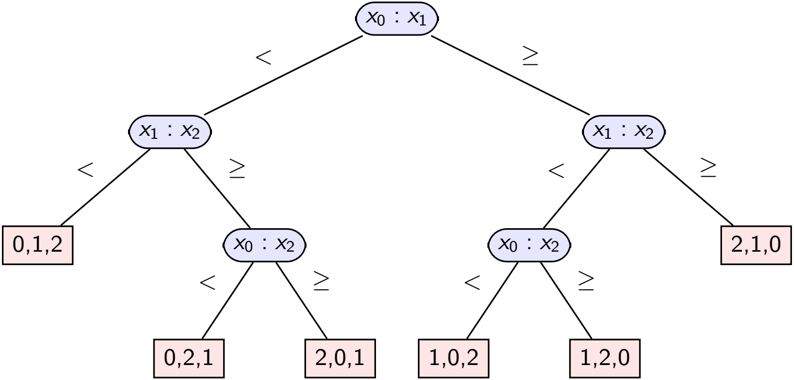
To sort $\{x_0, x_1, x_2\}$:

Example: $\{x_0{=}4, x_1{=}2, x_2{=}7\}$



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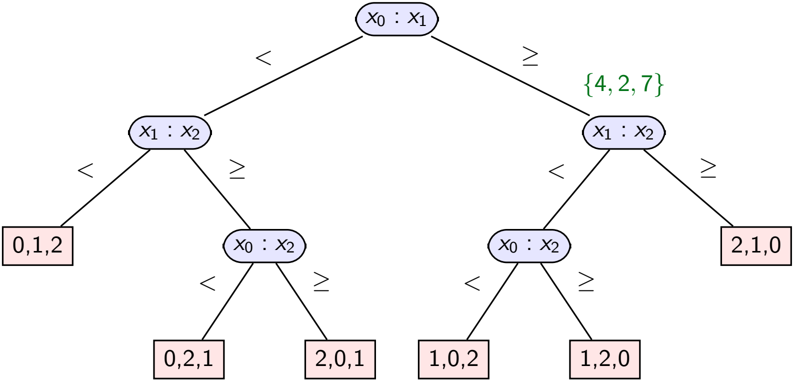

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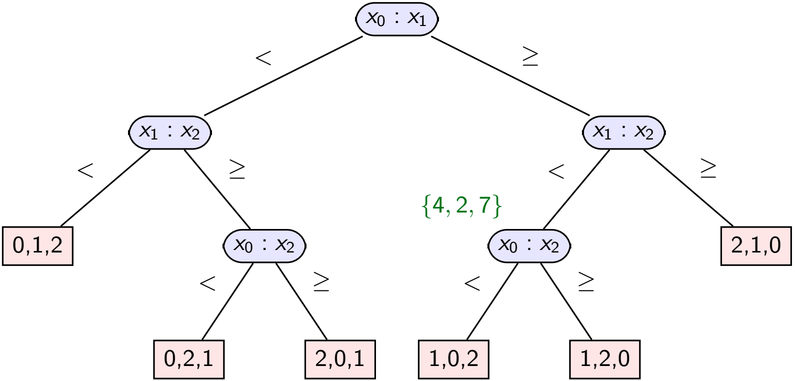

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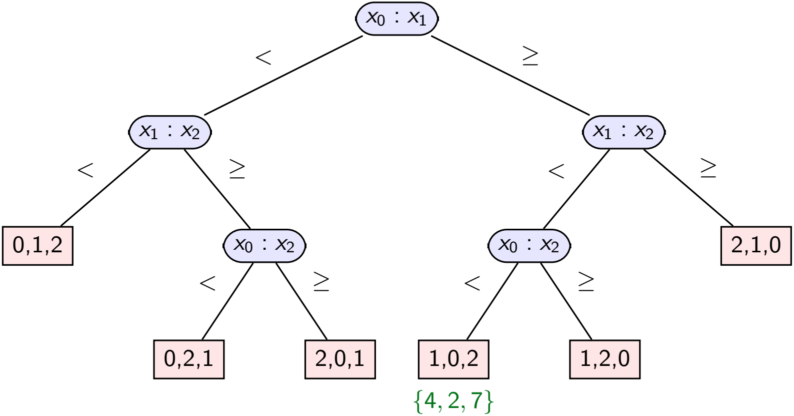


Output: $\{4, 2, 7\}$ has sorting permutation $\langle 1, 0, 2\rangle$

(i.e., $x_1{=}2 \leq x_0{=}4 \leq x_2{=}7$)

Outline

3. Sorting, Average-case and Randomization
   - Review and Outlook
   - Analyzing average-case run-time
   - Run-time on randomly chosen input
   - SELECTION and *quick-select*
   - Tips and Tricks for *quick-sort*
   - Lower Bound for Comparison-Based Sorting
   - Non-Comparison-Based Sorting

Non-Comparison-Based Sorting

- Assume keys are numbers in base R (R: **radix**)
  - So all digits are in $\{0, \ldots, R-1\}$
  - $R = 2, 10, 128, 256$ are the most common, but R need not be constant

Example ($R = 4$):

123	230	21	320	210	232	101

- Assume all keys have the same number w of digits.
  - Can achieve after padding with leading 0s.
  - In typical computers, $w = 32$ or $w = 64$, but w need not be constant

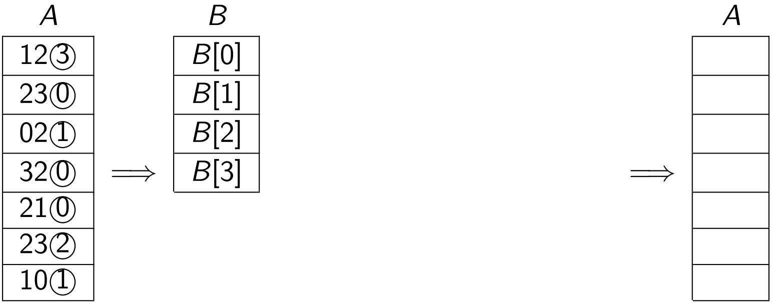
Example ($R = 4$):

123	230	021	320	210	232	101

- Can sort based on individual digits.
  - How to sort 1-digit numbers?
  - How to sort multi-digit numbers based on this?

(Single-digit) *bucket-sort*

Sort array A by last digit:



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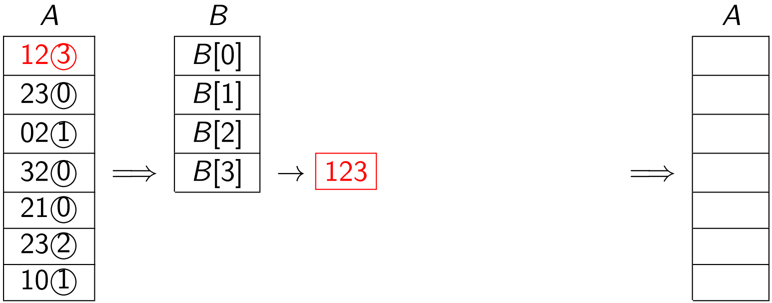

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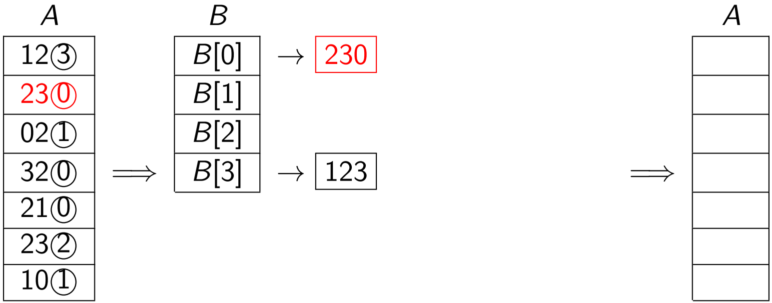

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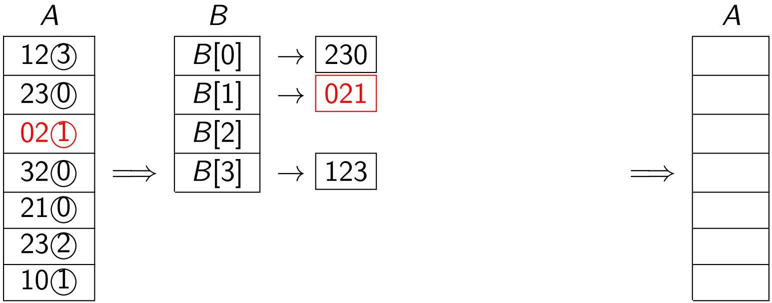

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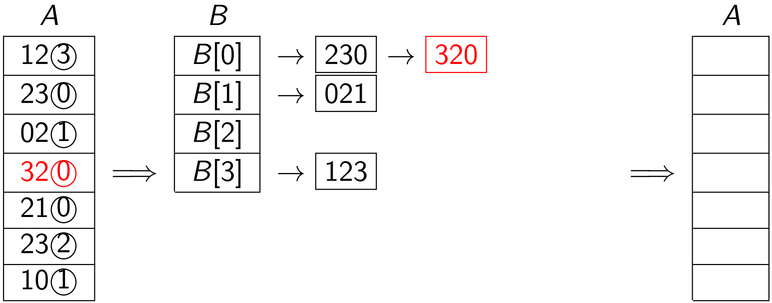

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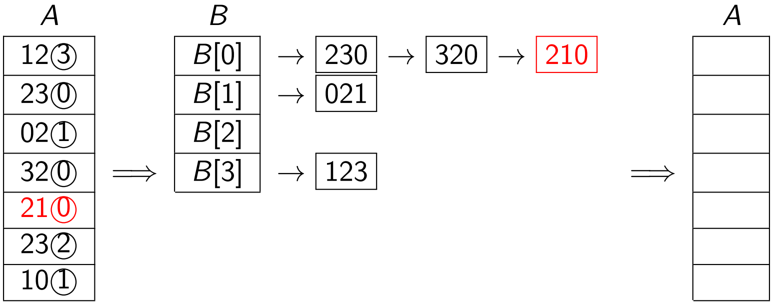

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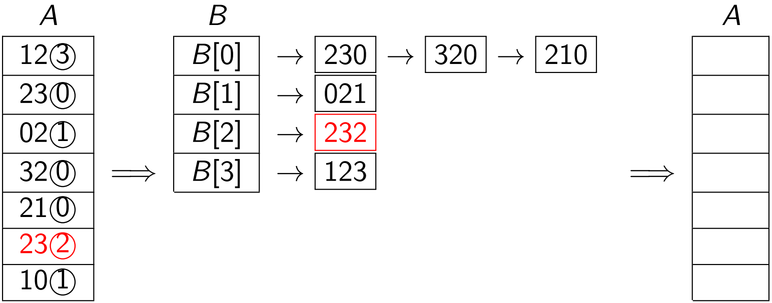

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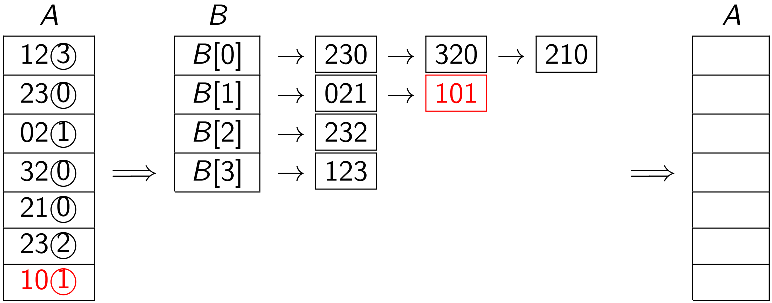

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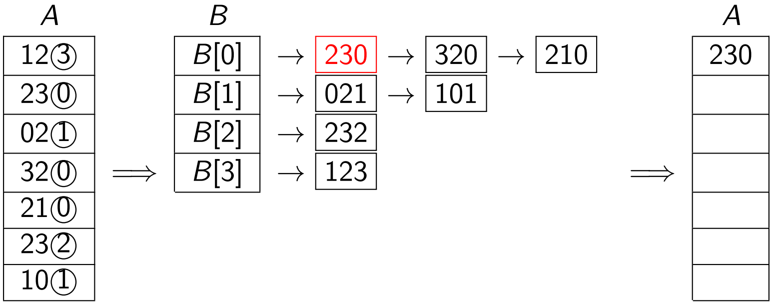

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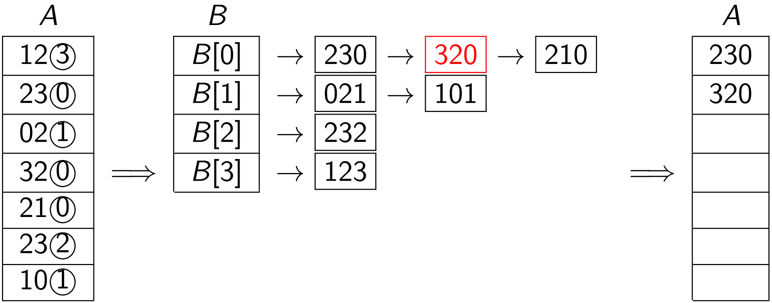

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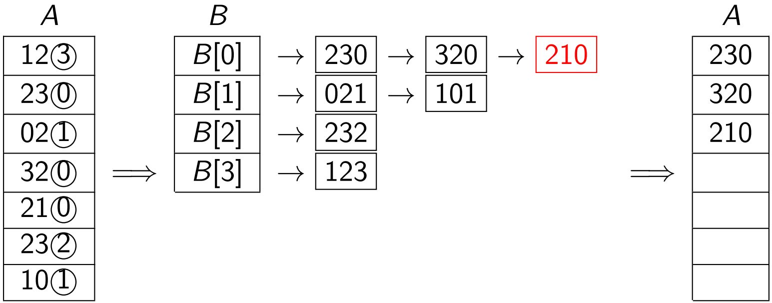

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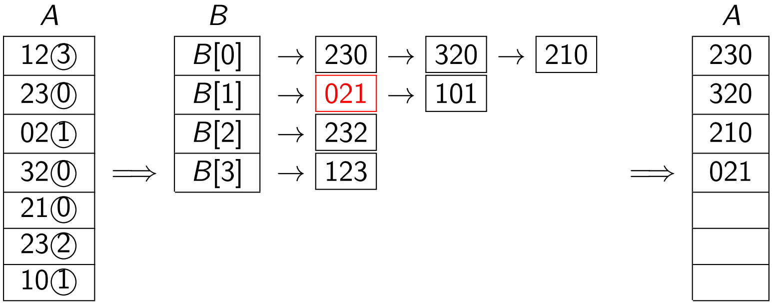

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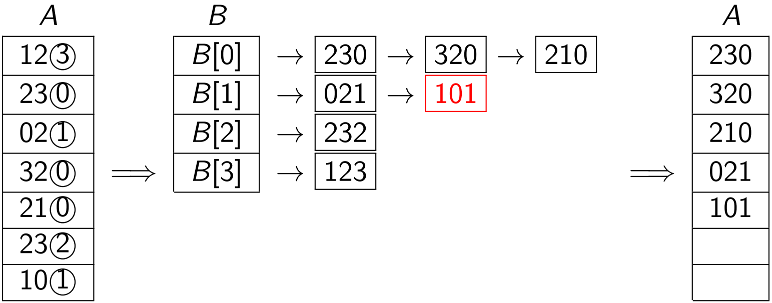

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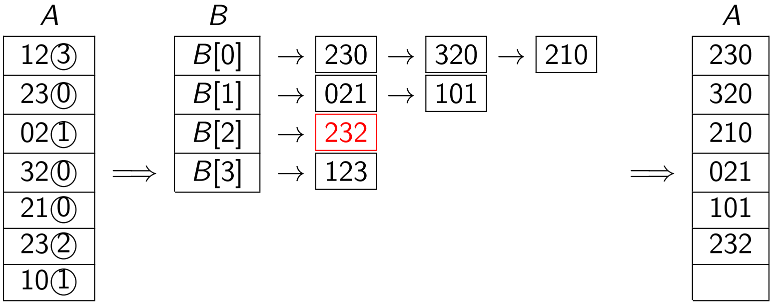

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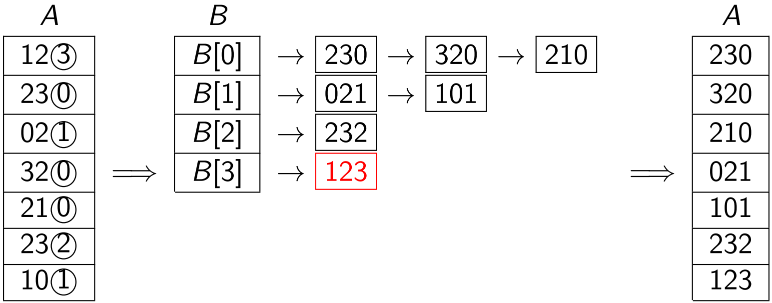

(Single-digit) *bucket-sort*

bucket-sort$(A, n,$ *sort-key*$(\cdot))$
A: array of size n
sort-key$(\cdot)$: maps items of A to $\{0, \ldots, R-1\}$
Initialize an array $B[0...R-1]$ of empty queues (**buckets**)
for $i \leftarrow 0$ to $n-1$ **do**
    Append $A[i]$ at end of $B[$*sort-key*$(A[i])]$
$i \leftarrow 0$
for $j \leftarrow 0$ to $R-1$ **do**
    while $B[j]$ is non-empty **do**
        move front element of $B[j]$ to $A[i{+}{+}]$

- In our example *sort-key*$(A[i])$ returns the last digit of $A[i]$

(Single-digit) *bucket-sort*

```
bucket-sort(A, n, sort-key(·))
A: array of size n
sort-key(·) : maps items of A to {0, ..., R−1}
    Initialize an array B[0...R − 1] of empty queues (buckets)
    for i ← 0 to n−1 do
        Append A[i] at end of B[sort-key(A[i])]
    i ← 0
    for j ← 0 to R − 1 do
        while B[j] is non-empty do
            move front element of B[j] to A[i++]
```

- In our example *sort-key*($A[i]$) returns the last digit of $A[i]$
- *bucket-sort* is **stable**: equal items stay in original order.
- Run-time $\Theta(n + R)$, auxiliary space $\Theta(n + R)$
- It is possible to replace the lists by arrays $\rightsquigarrow$ *count-sort* (no details).

Most-significant-digit(MSD)-radix-sort

Sort array of w-digit radix-R numbers recursively:
sort by 1st digit, then each group by 2nd digit, etc.

①23
②32
⓪21
③20
②10
②30
①01

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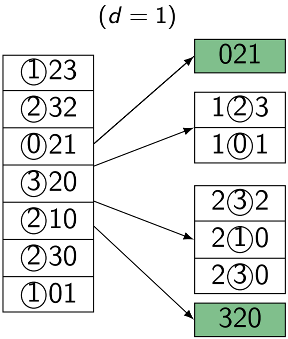

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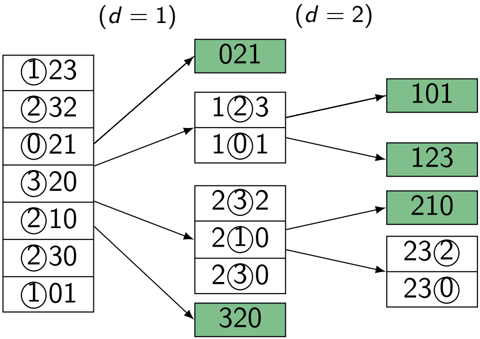

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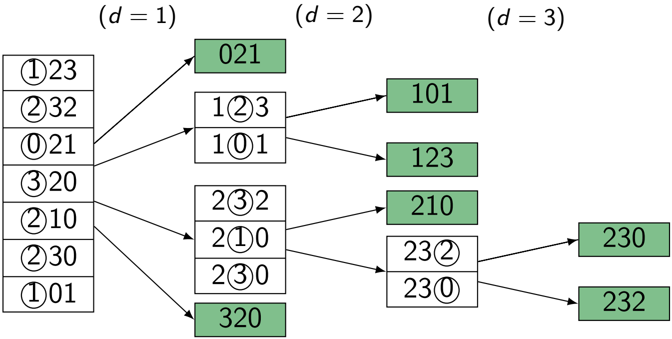

MSD-radix-sort

MSD-radix-sort$(A, n, \quad d \leftarrow 1)$
A: array of size n, contains w-digit radix-R numbers
if $(d \leq w$ and $(n > 1))$
    bucket-sort(A, n,'return dth digit of $A[i]$')
    $\ell \leftarrow 0$ `// find sub-arrays and recurse`
    for $j \leftarrow 0$ to $R-1$
        Let $r \geq \ell - 1$ be maximal s.t. $A[\ell..r]$ have dth digit j
        MSD-radix-sort$(A[\ell..r], r-\ell+1, d+1)$
        $\ell \leftarrow r + 1$

Analysis:

- $\Theta(w)$ levels of recursion in worst-case.
- $\Theta(n)$ subproblems on most levels in worst-case.
- $\Theta(R + \text{(size of sub-array)})$ time for each *bucket-sort* call.

$\Rightarrow$ Run-time $\Theta(wnR)$ — slow. Many recursions and allocated arrays.

Least-significant-digit(LSD)-radix-sort

LSD-radix-sort(A, n)
A: array of size n, contains m-digit radix-R numbers
for $d \leftarrow$ least significant to most significant digit **do**
    bucket-sort$(A$, n, 'return dth digit of $A[i]$')

12(3)		2(3)0		(1)01		021
23(0)		3(2)0		(2)10		101
02(1)	$(d = 3)$ $\Longrightarrow$	2(1)0	$(d = 2)$ $\Longrightarrow$	(3)20	$(d = 1)$ $\Longrightarrow$	123
32(0)		0(2)1		(0)21		210
21(0)		1(0)1		(1)23		230
23(2)		2(3)2		(2)30		232
10(1)		1(2)3		(2)32		320

- Loop-invariant: A is sorted w.r.t. digits $d, \ldots, w$ of each entry.
- **Time cost**: $\Theta(w(n + R))$ **Auxiliary space**: $\Theta(n + R)$

Summary

- Sorting is an important and *very* well-studied problem
- Can be done in $\Theta(n \log n)$ time; faster is not possible for general input
- *heap-sort* is the only $\Theta(n \log n)$-time algorithm we have seen with $O(1)$ auxiliary space.
- *merge-sort* is also $\Theta(n \log n)$, selection & insertion sorts are $\Theta(n^2)$.
- *quick-sort* is worst-case $\Theta(n^2)$, but often the fastest in practice
- *bucket-sort* and *radix-sort* achieve $o(n \log n)$ if the input is special

- Randomized algorithms can eliminate "bad cases"
- Best-case, worst-case, average-case can all differ.
- Often it is easier to analyze the run-time on randomly chosen input rather than the average-case run-time.