CS 240E – Data Structures and Data Management (Enriched)

Module 6: Dictionaries for special keys

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Based on lecture notes by many previous cs240 instructors

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Outline

6 Dictionaries for special keys

- Lower bound
- Improving binary search
- Interpolation Search
- Tries
  - Standard Tries
  - Variations of Tries
  - Compressed Tries
  - Multiway Tries

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Dictionary ADT: Implementations thus far

Realizations we have seen so far:

- **Balanced Binary Search trees** (AVL trees):
  $\Theta(\log n)$ search, insert, and delete (worst-case)
- **Skip lists**:
  $\Theta(\log n)$ search, insert, and delete (expected)
- Various other realizations sometimes faster on insert,
  but *search* always takes $\Omega(\log n)$ time.

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  $\Theta(\log n)$ search, insert, and delete (expected)
- Various other realizations sometimes faster on insert,
  but *search* always takes $\Omega(\log n)$ time.

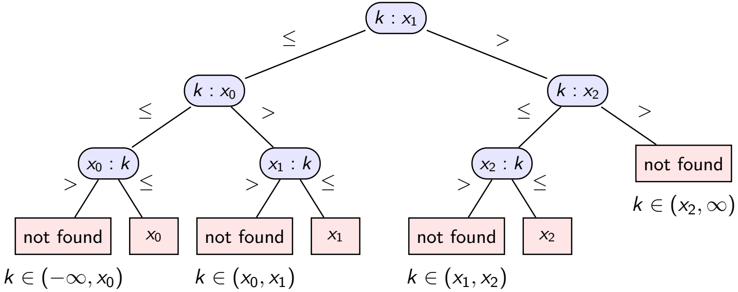
Question: Can one do better than $\Theta(\log n)$ time for *search*?
Answer: Yes and no! *It depends on what we allow*.

- No: Comparison-based searching lower bound is $\Omega(\log n)$.
- Yes: Non-comparison-based searching can achieve $o(\log n)$ (under restrictions!).

Lower bound for search

Theorem: Any *comparison-based* algorithm requires in the worst case $\Omega(\log n)$ comparisons to search among n distinct items.

Proof: Via decision tree for items $x_0, \ldots, x_{n-1}$ and search for k



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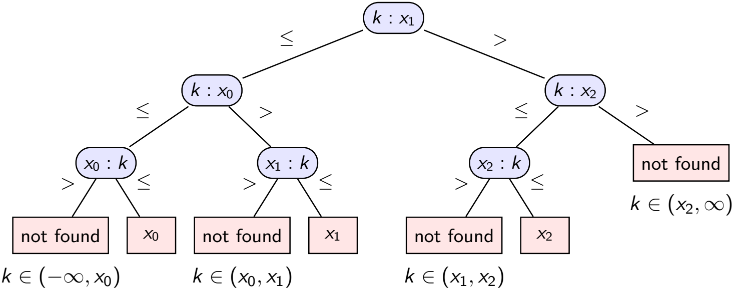

- How many possible outcomes are there?
- What does that tell us about the height of the decision tree?

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Matching the lower bound

- We can match lower bound asymptotically in a *sorted array*

```
binary-search($A, n, k$)                    // the CS136 version
$A$: Sorted array of size $n$, $k$: key
$\ell \leftarrow 0$, $r \leftarrow n-1$
while ($\ell \le r$)
    $m \leftarrow \lfloor \frac{\ell+r}{2} \rfloor$
    if ($A[m]$ equals $k$) then return “found at $A[m]$”
    else if ($A[m] < k$) then $\ell \leftarrow m+1$
    else $r \leftarrow m-1$
return “not found, but would be between $A[\ell-1]$ and $A[\ell]$”
```

- This uses $\approx 2\log n$ key-comparisons in worst-case.
  ($\le \lfloor \log n \rfloor + 1$ rounds, ≤ 2 key-comparisons per round)

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- This uses $\approx 2\log n$ key-comparisons in worst-case.
  ($\leq \lfloor \log n \rfloor + 1$ rounds, ≤ 2 key-comparisons per round)
- The lower bound can be improved to $\geq \lceil \log(2n) \rceil = \lceil \log n \rceil + 1$ key-comparisons (no details)
- **Goal:** Improve *binary-search* to use $\lceil \log n \rceil + 1$ key-comparisons.

Improving binary search

Main ingredient: Do only *one* comparison per round.

```
binary-search-optimized($A, n, k$)
$A$: Sorted array of size $n$, $k$: key
$\ell \leftarrow 0$, $r \leftarrow n-1$
while ($\ell < r$)
    $m \leftarrow\approx \frac{\ell+r}{2}$                // round up or round down?  TBD.
    if ($A[m] < k$) then $\ell \leftarrow m+1$
    else $r \leftarrow m$                               // this is different!
if ($k = A[\ell]$) then return "found at $A[\ell]$"
return "not found"
```

- **Non-trivial:** This terminates if we choose m the right way.

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```

- **Non-trivial:** This terminates if we choose m the right way.
- Actually show: $\underbrace{r^{new} - \ell^{new} + 1}_{size^{new}} \leq \frac{1}{2}(\underbrace{r - \ell + 1}_{size})$ (if rounded suitably)
    - This implies $size^{new} < size^{old}$ if $\ell < r$
    - This implies $\#rounds \leq \lceil \log n \rceil$

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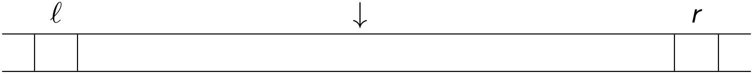
binary-search: Compare at index $\lfloor \frac{\ell+r}{2} \rfloor = \ell + \lceil \frac{1}{2}(r-\ell-1) \rceil$



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$\ell \leftarrow 0,\ r \leftarrow n-1$
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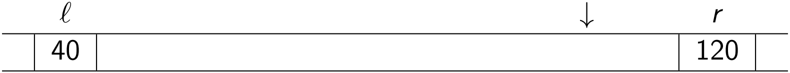

Question: If keys are *numbers*, where would you expect key $k = 100$?

Interpolation Search

- Code very similar to binary search, but compare at index

$$\ell + \left\lceil \frac{\overbrace{k - A[\ell]}^{\text{distance from left key}}}{\underbrace{A[r] - A[\ell]}_{\text{distance between left and right keys}}} \quad \underbrace{(r - \ell - 1)}_{\text{\# unknown keys in range}} \right\rceil$$

- Need a few extra tests to avoid crash during computation of m.

```
interpolation-search($A, n \leftarrow A.size, k$)
    $\ell \leftarrow 0$, $r \leftarrow n - 1$
    while ($\ell \leq r$)
        if ($k < A[\ell]$ or $k > A[r]$) return "not found"
        if ($k = A[r]$) then return "found at $A[r]$"
        $m \leftarrow \ell + \lceil \frac{k - A[\ell])}{A[r] - A[\ell]} \cdot (r - \ell - 1) \rceil$
        if ($A[m]$ equals $k$) then return "found at $A[m]$"
        else if ($A[m] < k$) then $\ell \leftarrow m + 1$
        else $r \leftarrow m - 1$
```

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	10	20	30	40	50	71	110	112	114	116	118	119	120

interpolation-search(A[0..13],14,71):

Interpolation Search Example

0	1	2	3	4	5	6	7	8	9	10	11	12	12
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ℓ								↑					r

interpolation-search(A[0..13],14,71):

- $\ell = 0,\ r = n - 1 = 13,\ m = \ell + \lceil \frac{71-0}{120-0}(13-0-1) \rceil = \ell + 8 = 8$

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- $\ell = 0,\ r = 7,\ m = \ell + \lceil \frac{71-0}{110-0}(7-0-1) \rceil = \ell + 4 = 4$

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- $\ell = 0,\ r = 7,\ m = \ell + \lceil \frac{71-0}{110-0}(7-0-1) \rceil = \ell + 4 = 4$
- $\ell = 5,\ r = 7,\ m = \ell + \lceil \frac{71-50}{110-50}(7-5-1) \rceil = \ell + 1 = 6$, found at $A[6]$

Interpolation Search Example

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- $\ell = 5,\ r = 7,\ m = \ell + \lceil \frac{71-50}{110-50}(7-5-1) \rceil = \ell + 1 = 6$, found at $A[6]$

If instead we had $A[6] = 72$:

- $\ell = 5 = r$, exit at line 3 with "not found"

Interpolation Search Second Example

0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	1500
ℓ	$\uparrow$									r

interpolation-search(A[0..10],10):

- $\ell = 0,\ r = n - 1 = 10,\ m = \ell + \lceil \frac{10-0}{1500-0}(10-0-1) \rceil = \ell + 1 = 1$

Interpolation Search Second Example

0	1	2	3	4	5	6	7	8	9	10
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		ℓ	↑							r

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- … in the worst case this can be very slow ($\Theta(n)$ time)

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- ... in the worst case this can be very slow ($\Theta(n)$ time)

But it works well on average:

- Can show (difficult): $\boxed{T^{\mathrm{avg}}(n) \leq T^{\mathrm{avg}}(\sqrt{n}) + \Theta(1).}$
- This resolves to $T^{\mathrm{avg}}(n) \in O(\log\log n)$.

Improving Interpolation Search

- Proving $T^{\mathrm{avg}}(n) \leq T^{\mathrm{avg}}(\sqrt{n}) + \Theta(1)$ is very complicated.
  - (Switch to analyze run-time on randomly chosen input.
  - Study expected error, i.e., distance between index of k and where we probed.
  - Argue that error is in $O(\sqrt{n})$ in first round.
  - Argue that error is in $O(\frac{1}{2^i} n)$ after i rounds.
  - Study the martingale formed by the errors in the rounds.
  - Argue that its expected length is $O(\log \log n)$.)

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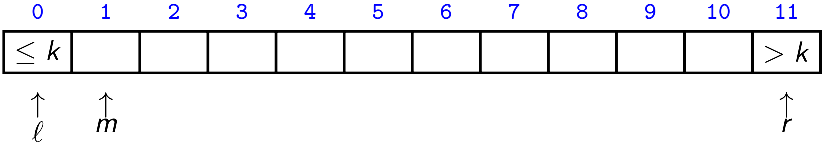
    - Study the martingale formed by the errors in the rounds.
    - Argue that its expected length is $O(\log \log n)$.
- Instead: Define a variant of *interpolatation-search*
    - Better worst-case run-time.
    - Easier to analyze.
- Idea: *Force* the sub-array to have size $\sqrt{n}$
- To do so, search for suitable sub-array with repeated **probes** (comparison between array-entry and search-key)
- Crucial question: how many probes are needed?

Improving Interpolation Search



- First probe at m as before.

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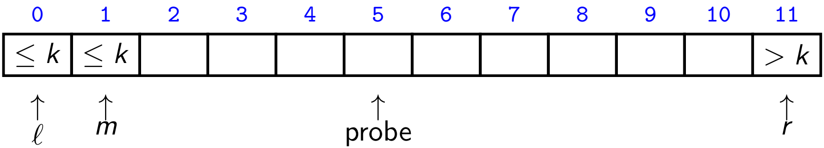

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- If $A[m] \leq k$, probe rightward.
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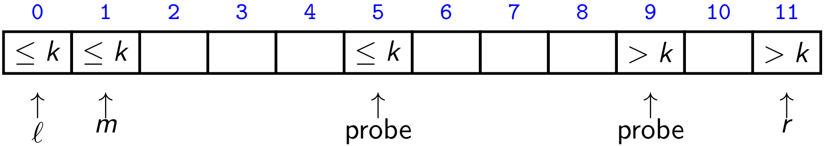

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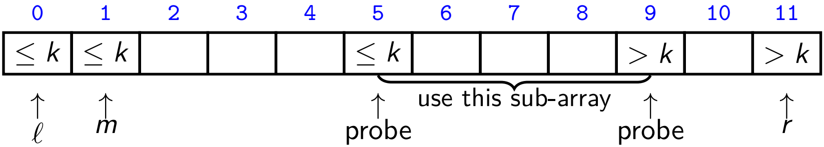
Improving Interpolation Search

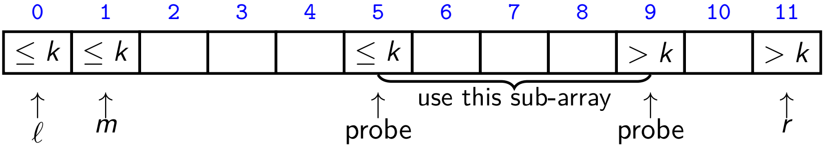


0	1	2	3	4	5	6	7	8	9	10	11
$\leq k$	$\leq k$				$\leq k$				$> k$		$> k$
$\uparrow$ ℓ	$\uparrow$ m				$\uparrow$ probe				$\uparrow$ probe		$\uparrow$ r

use this sub-array

- First probe at m as before.
- If $A[m] \leq k$, probe rightward.
- Probes always go $\lceil \sqrt{N} \rceil$ indices rightward (where $N = r - \ell - 1 = \#$ unknown keys, here $N = 10$)
- Continue probing until $> k$ or out-of-bounds
- Recurse in the only sub-array where k can be. It has $\leq \lceil \sqrt{N} \rceil - 1$ unknown keys.

Improving Interpolation Search



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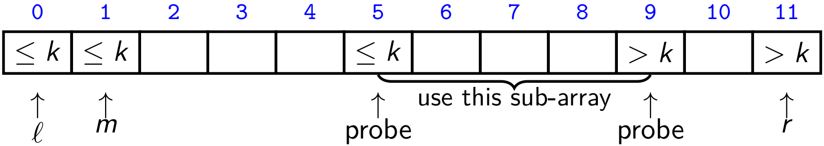

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- Continue probing until $> k$ or out-of-bounds
- Recurse in the only sub-array where k can be.
  It has $\leq \lceil \sqrt{N} \rceil - 1$ unknown keys.
- Observe: $\#\{\text{probes in this round}\} \leq \sqrt{N}$

Improving Interpolation Search

```
Interpolation-search-better$(A, n, k)$
$A$: sorted array of size $n$, $k$: key
if $(k < A[0]$ or $k > A[n-1])$ return "not found"
if $(k = A[n-1])$ return "found at index $n-1$"
$\ell \leftarrow 0$, $r \leftarrow n-1$                         // have $A[\ell] \le k < A[r]$
while $(N \leftarrow (r-\ell-1) \ge 1)$
    $m \leftarrow \ell + \lceil \frac{k-A[\ell])}{A[r]-A[\ell]} \cdot (r-\ell-1) \rceil$
    if $(A[m] \le k)$                                  // probe rightward
        for $h = 1, 2, \ldots$
            $\ell \leftarrow m + (h-1)\lceil\sqrt{N}\rceil$, $r' \leftarrow \min\{r, m + h\lceil\sqrt{N}\rceil\}$
            if $(r' = r$ or $A[r'] > k)$ then $r \leftarrow r'$ and break
    else ...                              // symmetrically probe leftward
if $(k = A[\ell])$ return "found at index $\ell$"
else return "not found"
```

Analysis of *interpolation-search-improved*

- Let $T(n)$ be the total number of probes if there were n unknown keys.
- $T(n) \leq T(\sqrt{n}) + \sqrt{n}$ since sub-array has $\leq \sqrt{n}$ unknowns.
- This resolves to $O(\sqrt{n})$

(see table, or prove $T(n) \leq 2\sqrt{n} + O(1)$ for $n \geq 16$.)

Result: The worst-case run-time of *interpolation-search-improved* is in $O(\sqrt{n})$.

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Result: The worst-case run-time of *interpolation-search-improved* is in $O(\sqrt{n})$.

Average-case run-time?

- Rephrase: If array-entries are chosen uniformly at random, what is the expected number of probes per found?

Analysis of *interpolation-search-improved*

Claim: The number of rounds is $\lceil \log\log n \rceil + O(1)$ in worst case.

- Key ingredient: $\log\log\sqrt{n} = \log\log n - 1$.

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Claim: The number of rounds is $\lceil \log \log n \rceil + O(1)$ in worst case.

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Claim: Expected number of probes per round is at most 2.5.

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Claim: Expected number of probes per round is at most 2.5.

(Proof later, study consequences first)

- $\#\text{probes} \leq \#(\text{rounds}) * \#(\text{probes per round})$
  $\leq 2.5\lceil \log\log n \rceil + O(1)$ on average
- **Result:** The average-case run-time of *interpolation-search-improved* is in $O(\log\log n)$.

Analysis of *interpolation-search-improved*

Claim: The number of rounds is $\lceil \log\log n \rceil + O(1)$ in worst case.

- Key ingredient: $\log\log\sqrt{n} = \log\log n - 1$.

Claim: Expected number of probes per round is at most 2.5.

(Proof later, study consequences first)

- $$\begin{aligned} \#\text{probes} \leq &\quad \#(\text{rounds}) * \#(\text{probes per round}) \\ \leq &\quad 2.5\lceil \log\log n \rceil + O(1) \text{ on average} \end{aligned}$$
- **Result:** The average-case run-time of *interpolation-search-improved* is in $O(\log\log n)$**.**

Fewer probes than *binary-search-optimized*'s $\lceil \log n \rceil + 1$ even for small n.

Expected number of probes

Recall: $E[\#\text{probes}] = \sum_{i \geq 0} i \cdot P(\#\text{probes} = i) = \sum_{i \geq 1} P(\#\text{probes} \geq i)$.

Expected number of probes

Recall: $E[\#\text{probes}] = \sum_{i\geq 0} i \cdot P(\#\text{probes} = i) = \sum_{i \geq 1} P(\#\text{probes} \geq i)$.

- So must analyze $P(\#\text{probes} \geq i)$.
  - For $i = 1, 2$, use $P(\#\text{probes} \geq i) \leq 1$
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  - $idx(k)$: index of rightmost item that is $\leq k$
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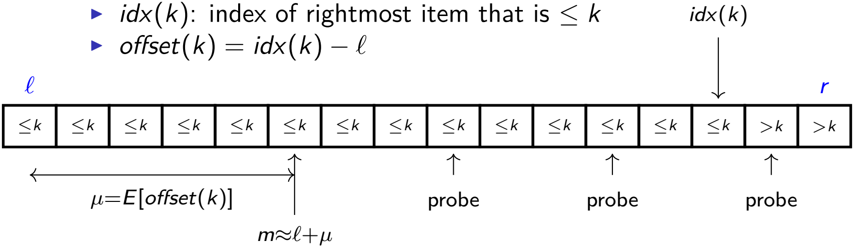

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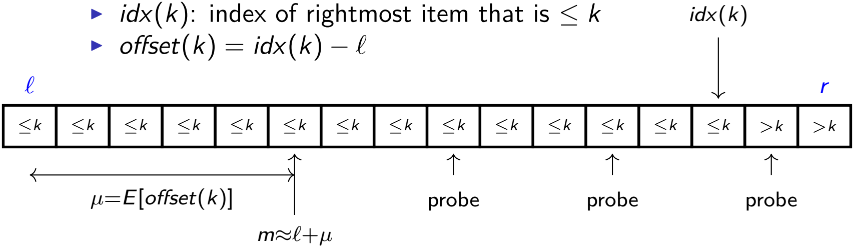

- $E[offset(k)] = ???$. $V(offset(k)) \leq ???$.
  And how do they relate to $P(\#\text{probes} \geq i)$?

Outline

6. Dictionaries for special keys
   - Lower bound
   - Improving binary search
   - Interpolation Search
   - Tries
     - Standard Tries
     - Variations of Tries
     - Compressed Tries
     - Multiway Tries

Words (review)

Scenario: Keys in dictionary are *words*. Need brief review.

Words (= strings): sequences of characters over alphabet Σ

$$\{\texttt{be}, \texttt{bear}, \texttt{beer}\}$$

- Typical alphabets: $\{0, 1\}$ (→bitstrings), ASCII, $\{C, G, T, A\}$
- Stored in an array: $w[i]$ gets ith character (for $i = 0, 1. \dots$)

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- $w.\mathit{size} = |w| = \#$ non-sentinel characters: $|\texttt{be\$}|$=2.

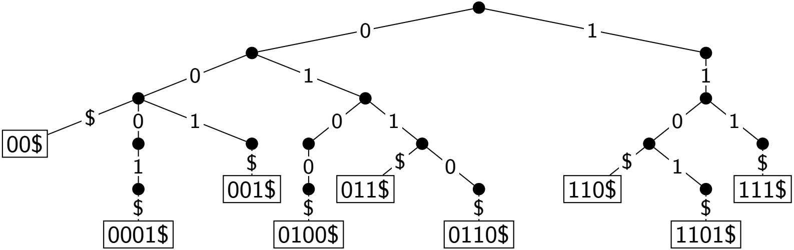
Should know:

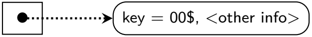
- prefix, suffix, substring
- Sort words **lexicographically**: $\texttt{be\$} <_{\text{lex}} \texttt{bear\$} <_{\text{lex}} \texttt{beer\$}$
  This is different from sorting numbers: $\texttt{001\$} <_{\text{lex}} \texttt{010\$} <_{\text{lex}} \texttt{1\$}$

Tries: Introduction

Trie (also know as **radix tree**): A dictionary for bitstrings.

- Comes from retrieval, but pronounced "try"
- A tree based on *bitwise comparisons*: Edge labelled with corresponding bit
- Similar to *radix sort*: use individual bits, not the whole key
- Due to end-sentinels, all key-value pairs are at leaves.



(Reminder: A more accurate picture would be )

Tries: Search

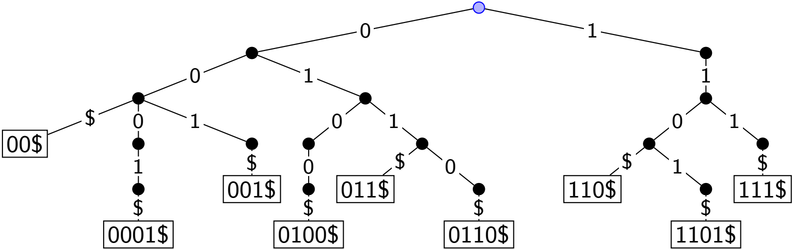
- Follow links that corresponds to current bits in w
- Repeat until no such link or w found at a leaf

Similar as for skip lists, we find search-path separately first.

```
Trie::get-path-to($w$)
Output: Stack with all ancestors of where $w$ would be stored
$P \leftarrow$ empty stack; $z \leftarrow root$; $d \leftarrow 0$; $P.push(z)$
while $d \le |w|$
    if $z$ has a child-link labelled with $w[d]$
        $z \leftarrow$ child at this link; $d$++; $P.push(z)$
    else break
return $P$
```

Tries: Search Example

Example: Trie::search(011$)



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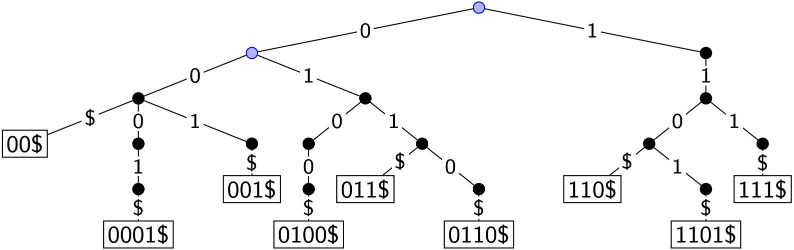

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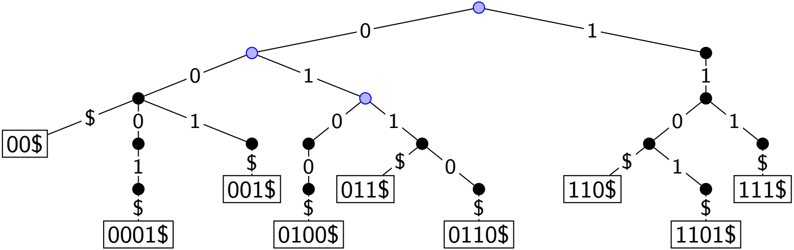

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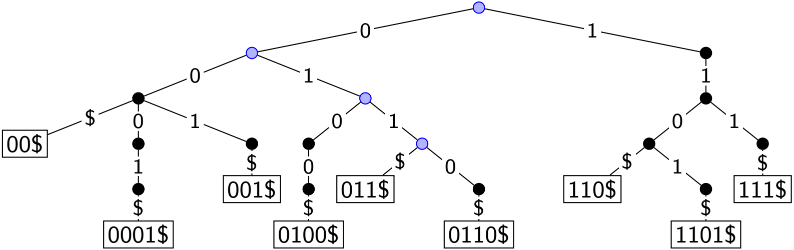

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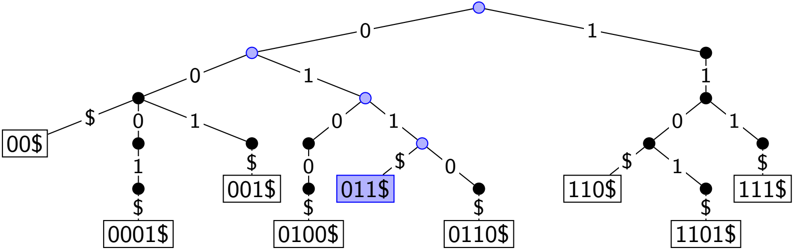
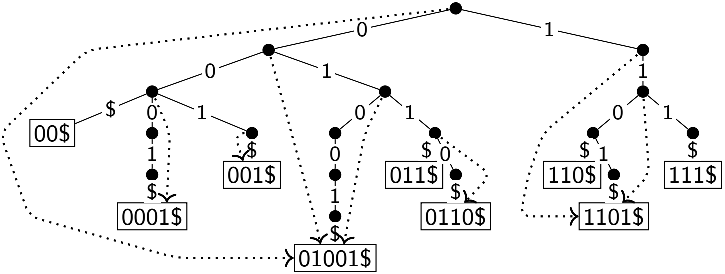


```
Trie::search(w)
    P ← get-path-to(w), z ← P.top
    if (z is not a leaf) then
        return "not found, would be in sub-trie of z"
    return key-value pair at z
```

Tries: Leaf-references

For later applications of tries, we want another search-operation:

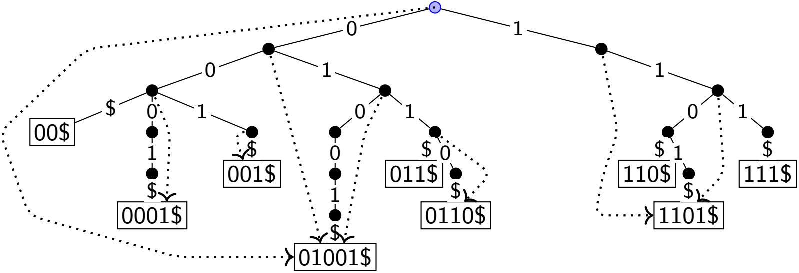
- *prefix-search*(w): Find word w' in trie for which w is a prefix.
- Testing whether w' exists is easy (how?)
- To find w' quickly, we need **leaf-references**
  - Every node z stores reference z.*leaf* to a leaf in subtree
  - **Convention:** store leaf with longest word



(not all leaf-references are shown)

Tries: Prefix-Search Example

Example: Trie::prefix-search(11$)



Tries: Prefix-Search Example

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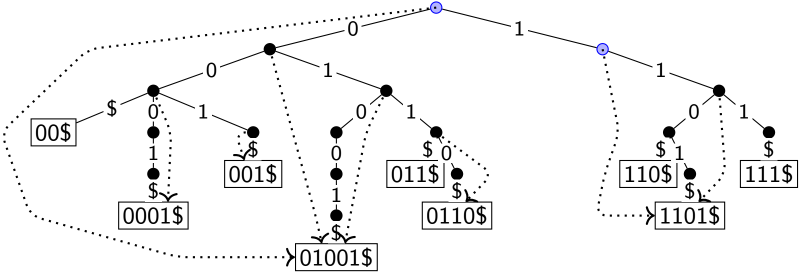

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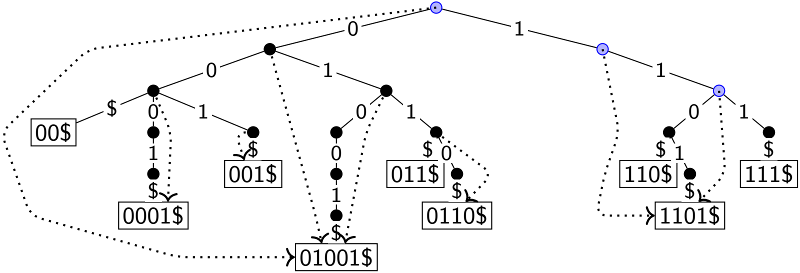
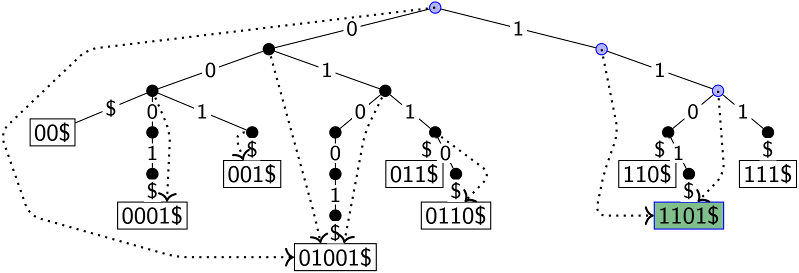


Tries: Prefix-Search Example

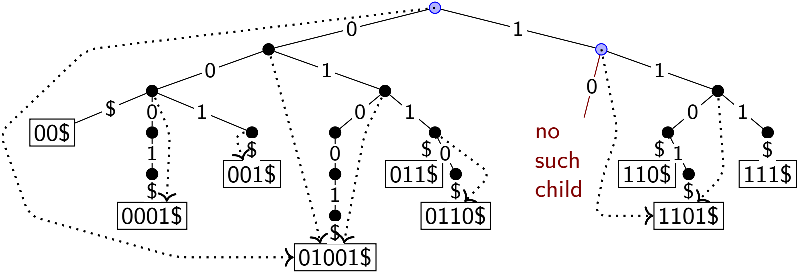
Example: Trie::prefix-search(11$)



```
Trie::prefix-search(w)
1.  P ← get-path-to(w)
2.  if number of nodes on P is w.size or less
3.        return "no extension of w found"
4.  return P.top().leaf
```

Tries: Prefix-Search Example

Example: Trie::prefix-search(10$)



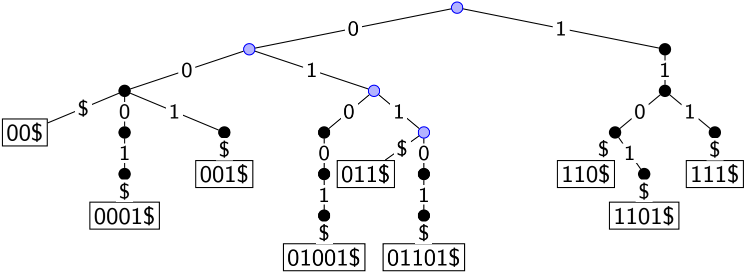
- Word 10$ has size 2.
- *get-path-to*(w) returns stack with two nodes.
- We need more than $w.\mathit{size}$ nodes on P to have an extension.

Tries: Insert

Trie::insert(w)

- $P \leftarrow$ *get-path-to*(w) gives ancestors that exist already,
- Expand the trie from P.*top*() by adding necessary nodes that correspond to extra bits of w.
- Update leaf-references (also at P if w is longer than previous leaves)

Example: *insert*(011101$)



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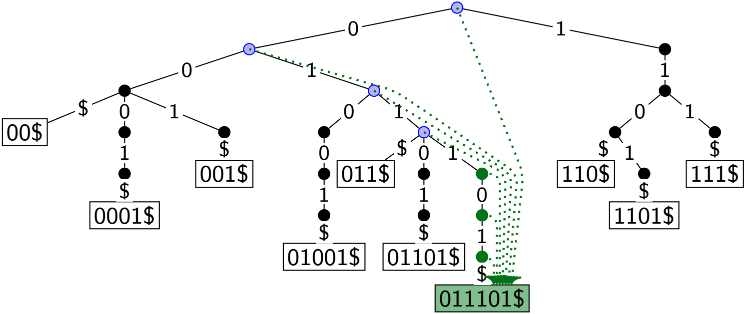


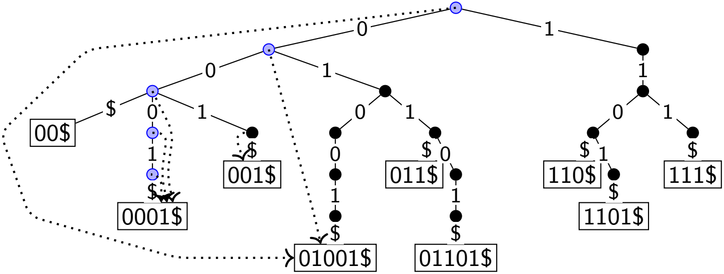
(only updated leaf-references are shown)

Tries: Delete

Trie::delete(w)

- $P \leftarrow$ *get-path-to*(w) gives all ancestors.
- Let ℓ be the leaf where w is stored
- Delete ℓ and nodes on P until ancestor has two or more children.
- Update leaf-references on rest of P.
  (If $z \in P$ referred to ℓ, find new $z.$*leaf* from other children.)

Example: *trie::delete*(0001$)



(only some leaf-references are shown)

Tries: Delete

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- $P \leftarrow$ *get-path-to*(w) gives all ancestors.
- Let ℓ be the leaf where w is stored
- Delete ℓ and nodes on P until ancestor has two or more children.
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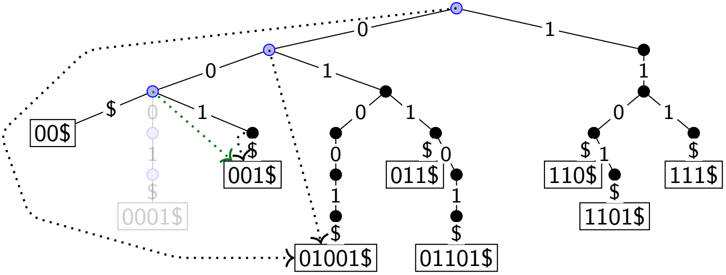
  (If $z \in P$ referred to ℓ, find new z.*leaf* from other children.)

Example: *trie::delete*(0001\$)



(only some leaf-references are shown)

Binary Tries summary

search(w), *prefix-search*(w), *insert*(w), *delete*(w) all take time $\Theta(|w|)$.

- Search-time is *independent* of number n of words stored in the trie!
- Search-time is small for short words.

The trie for a given set of words is unique

(except for order of children and ties among leaf-references)

Binary Tries summary

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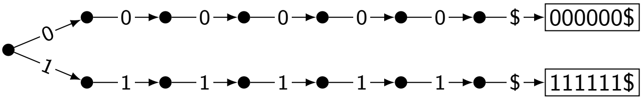
Disadvantages:

- Tries can be wasteful with respect to space.

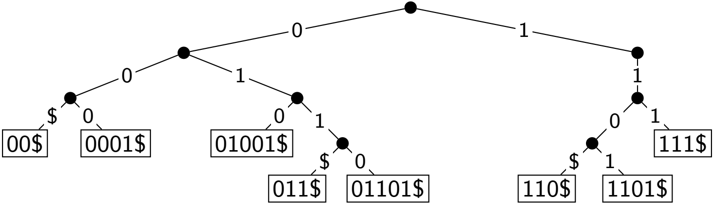


- Worst-case space is $\Theta(n \cdot \text{(maximum length of a word)})$
- What can we do to save space?

Variations of Tries: Pruned Tries

Pruned Trie: Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Saves space if there are only few bitstrings that are long.
- Could even store infinite bitstrings (e.g. real numbers)

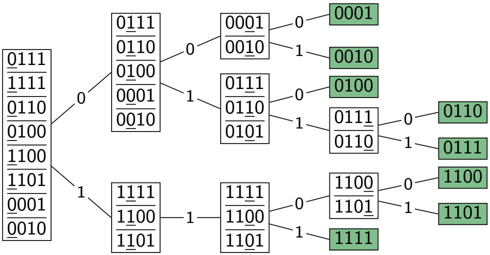


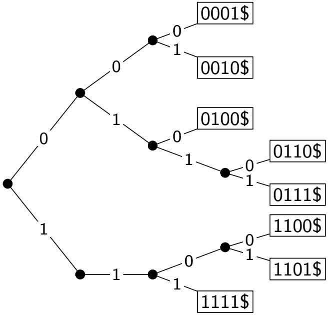
A more efficient version of tries, but the operations get a bit more complicated.

Pruned tries and MSD-radix sort

We have (implicitly) seen pruned tries before:

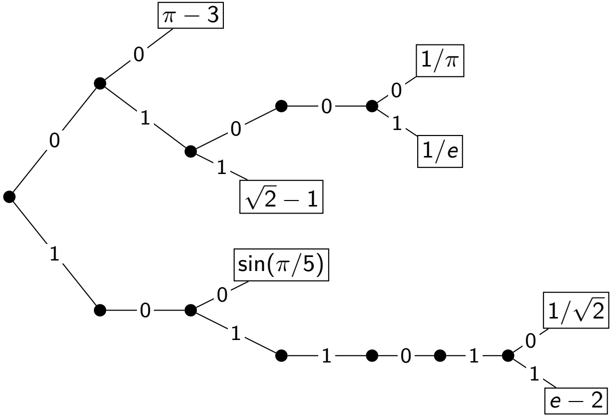
- For equal-length bitstrings:
  Pruned trie equals recursion tree of MSD radix-sort.





Pruned tries can store real numbers

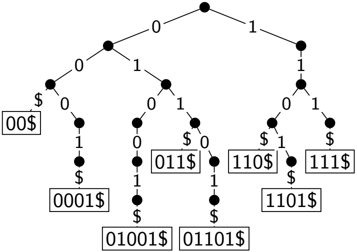
If we have a generator for each bit of a real number, then we can store them in a pruned trie.

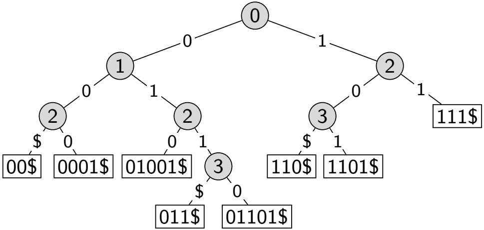


Compressed Tries

Another (important!) variation:

- Compress paths of nodes with only one child.
- Each node stores an *index*, corresponding to the level of the node in the uncompressed trie. (On level d, we searched for link with $w[d]$.)





Also known as **Patricia-Tries**:
Practical Algorithm to Retrieve Information Coded in Alphanumeric

Compressed Tries: Search

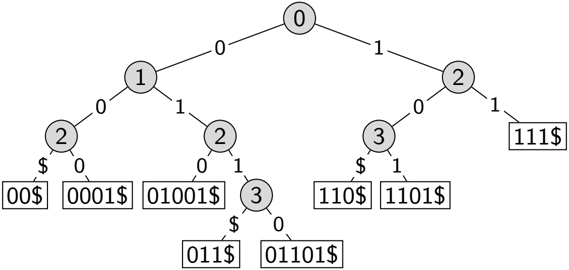
- As for tries, follow links that corresponds to current bits in w
- Main difference: stored indices say which bits to compare.
- Also: must compare w to word found at the leaf (why?)

```
CompressedTrie::get-path-to(w)
$P \leftarrow$ empty stack; $z \leftarrow root$; $P.push(z)$
while $z$ is not a leaf and $(d \leftarrow z.index \leq w.size)$ do
    if ($z$ has a child-link labelled with $w[d]$) then
        $z \leftarrow$ child at this link; $P.push(z)$
    else break
return $P$
```

```
CompressedTrie::search(w)
$P \leftarrow$ get-path-to$(w)$, $z \leftarrow P.top$
if ($z$ is not a leaf or word stored at $z$ is not $w$) then
    return "not found"
return key-value pair at $z$
```

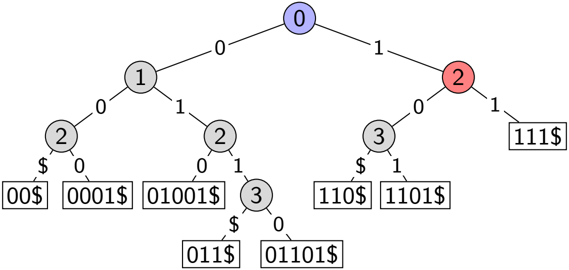
Compressed Tries: Search Example

Example 1: CompressedTrie::search($\overset{0}{\boxed{1}}\overset{1}{\boxed{\$}}$)



Compressed Tries: Search Example

Example 1: CompressedTrie::search($\begin{array}{|c|c|} \multicolumn{1}{c}{0} & \multicolumn{1}{c}{1} \\ \hline 1 & \$ \\ \hline \end{array}$) unsuccessful (*d* too big)



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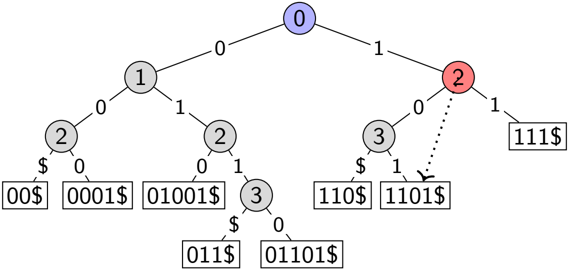
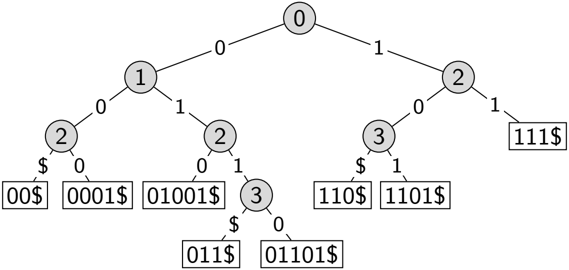


prefix-search(w): Compare w to z.*leaf* at last visited node z.

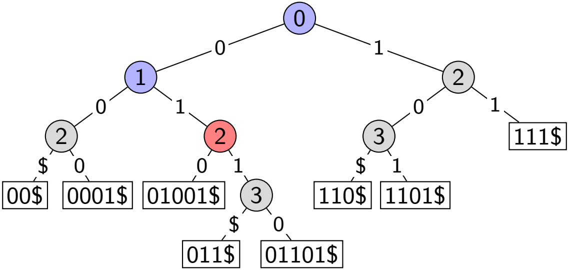
Compressed Tries: Search Example

Example 2: CompressedTrie::search($\begin{array}{ccc} \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 2 \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\$} \\ \hline \end{array}$)



Compressed Tries: Search Example

Example 2: CompressedTrie::search($\begin{array}{ccc} 0 & 1 & 2 \\ \hline \boxed{0} & \boxed{1} & \boxed{\$} \end{array}$) unsuccessful (no \$-child)



Compressed Tries: Search Example

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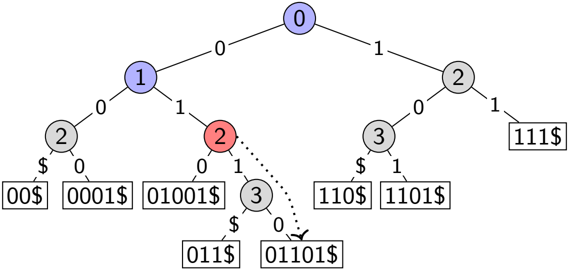

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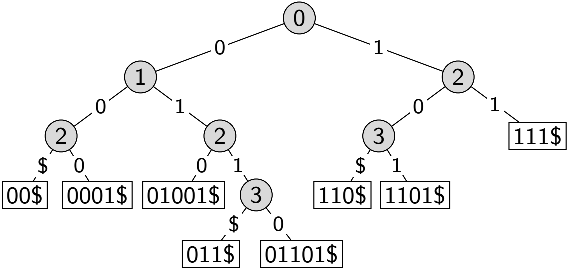
Compressed Tries: Search Example

Example 3: CompressedTrie::search(

0	1	2	3
1	0	1	$

)



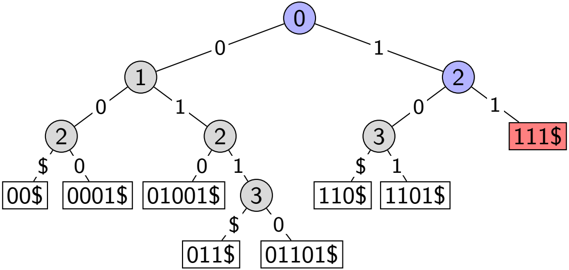
Compressed Tries: Search Example

Example 3: CompressedTrie::search(

0	1	2	3
1	0	1	$

) unsuccessful

(wrong word at leaf)



Compressed Tries: Search Example

Example 3: CompressedTrie::search($\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\$} \\ \hline \end{array}$) unsuccessful

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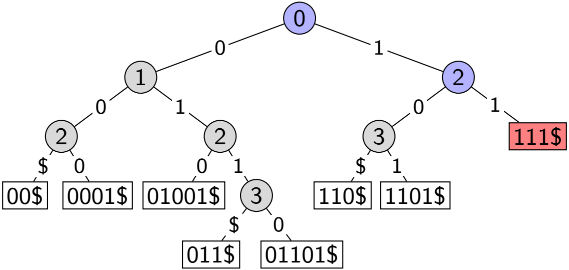


prefix-search(w): Compare w to word at reached leaf.

Compressed Tries: Summary

- *search*(w) and *prefix-search*(w) are easy.
- *insert*(w) and *delete*(w) are conceptually simple:
  - Search for path P to word w (say we reach node z)
  - Uncompress this path (using characters of z.*leaf*)
  - Insert/Delete w as in an uncompressed trie.
  - Compress path from root to where change happened

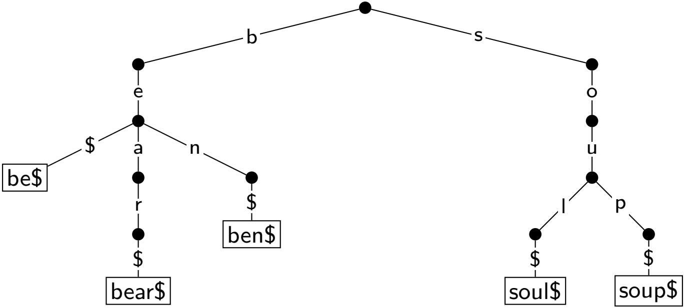
(Pseudocode gets more complicated and is omitted.)

- All operations take $O(|w|)$ time for a word w.
- Compressed tries use $O(n)$ *space*
  - We have n leaves.
  - Every internal node has two or more children.
  - **Can show:** Therefore more leaves than internal nodes.

Overall, code is more complicated, but space-savings are worth it if words are unevenly distributed.

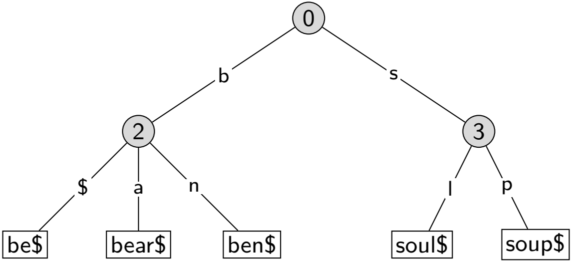
Multiway Tries: Larger Alphabet

- To represent *strings* over any *fixed alphabet* Σ
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings $\{$bear\$, ben\$, be\$, soul\$, soup\$$\}$



Compressed Multiway Tries

- **Variation**: Compressed multi-way tries: compress paths as before
- Example: A compressed trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



Multiway Tries: Summary

- Operations *search*(w), *prefix-search*(w), *insert*(w) and *delete*(w) are exactly as for tries for bitstrings.
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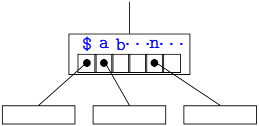
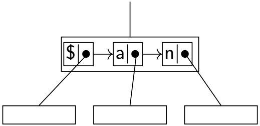
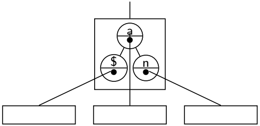


Array?



List?



Dictionary?

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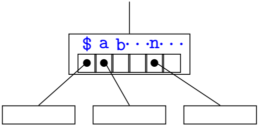

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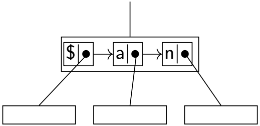

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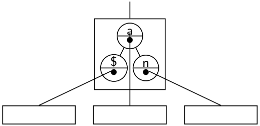

Dictionary?

- Time/space tradeoff: arrays are fast, lists are space-efficient.
- Dictionary best in theory, not worth it in practice unless $|\Sigma|$ is huge.

- In practice, use *hashing* (→ module 07).