CS 240E – Data Structures and Data Management (Enriched)

Module 7: Dictionaries via Hashing

Therese Biedl

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

Winter 2025

version 2025–02–16 16:49

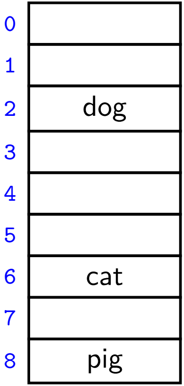
Outline

Outline

Direct Addressing

Special situation: For a known $M \in \mathbb{N}$, every key k is an integer with $0 \leq k < M$.

We can then implement a dictionary easily: Use an array A of size M that stores (k, v) via $A[k] \leftarrow v$.

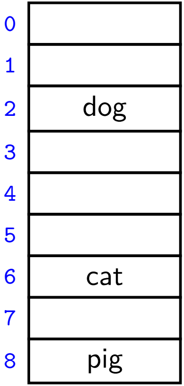


- *search*(k): Check whether $A[k]$ is `NULL`
- *insert*(k, v): $A[k] \leftarrow v$
- *delete*(k): $A[k] \leftarrow$ `NULL`

Direct Addressing

Special situation: For a known $M \in \mathbb{N}$, every key k is an integer with $0 \leq k < M$.

We can then implement a dictionary easily: Use an array A of size M that stores (k, v) via $A[k] \leftarrow v$.



- *search*(k): Check whether $A[k]$ is `NULL`
- *insert*(k, v): $A[k] \leftarrow v$
- *delete*(k): $A[k] \leftarrow$ `NULL`

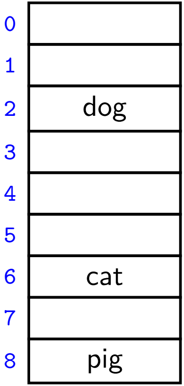
Each operation is $\Theta(1)$.
Total space is $\Theta(M)$.

What sorting algorithm does this remind you of?

Direct Addressing

Special situation: For a known $M \in \mathbb{N}$, every key k is an integer with $0 \leq k < M$.

We can then implement a dictionary easily: Use an array A of size M that stores (k, v) via $A[k] \leftarrow v$.

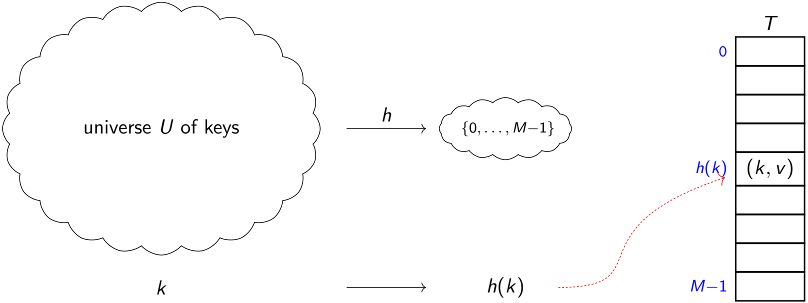


- *search*(k): Check whether $A[k]$ is `NULL`
- *insert*(k, v): $A[k] \leftarrow v$
- *delete*(k): $A[k] \leftarrow$ `NULL`

Each operation is $\Theta(1)$.
Total space is $\Theta(M)$.

What sorting algorithm does this remind you of?
Bucket Sort

Hashing Details



- **Assumption:** We know that all keys come from some **universe** U. (Typically U = non-negative integers, sometimes $|U|$ finite.)
- We pick a **table-size** M.
- We pick a **hash function** $h : U \to \{0, 1, \ldots, M-1\}$. (Commonly used: $h(k) = k \bmod M$. We will see other choices later.)

- Store dictionary in **hash table**, i.e., an array T of size M.
- An item with key k wants to be stored in **slot** $h(k)$, i.e., at $T[h(k)]$.

Hashing example

$U = \mathbb{N}$, $M = 11$, $\qquad h(k) = k \bmod 11$.

The hash table stores keys 7, 13, 43, 45, 49, 92. (Values are not shown).

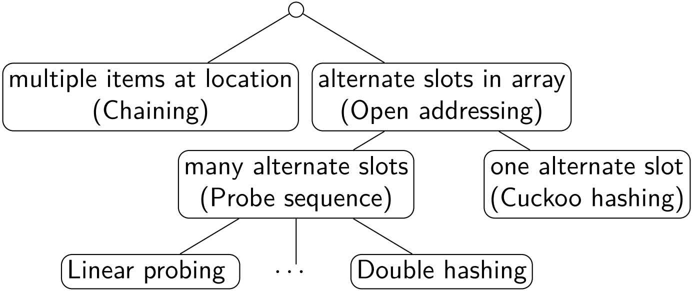
0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Collisions

- Generally hash function h is not injective, so many keys can map to the same integer.
  - For example, $h(46) = 2 = h(13)$ if $h(k) = k \bmod 11$.
- We get **collisions**: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.

Collisions

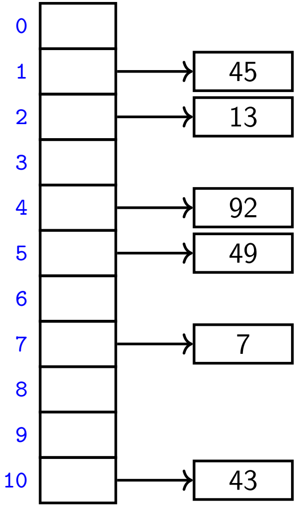
- Generally hash function h is not injective, so many keys can map to the same integer.
  - For example, $h(46) = 2 = h(13)$ if $h(k) = k \bmod 11$.
- We get **collisions**: we want to insert (k, v) into the table, but $T[h(k)]$ is already occupied.
- There are many strategies to resolve collisions:



Outline

Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

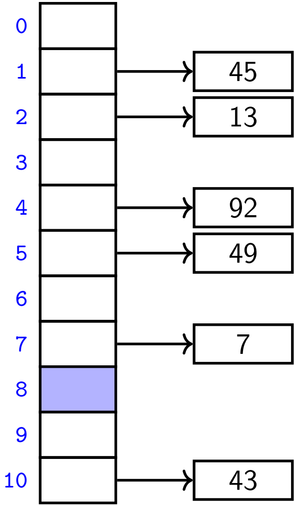


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(41)

$h(41) = 8$

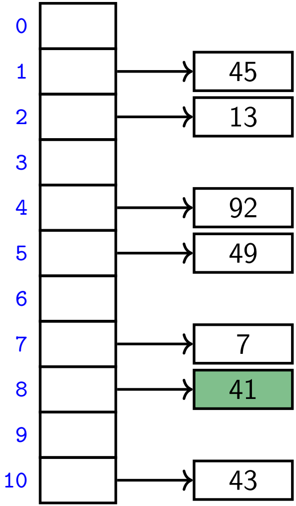


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(41)

$h(41) = 8$

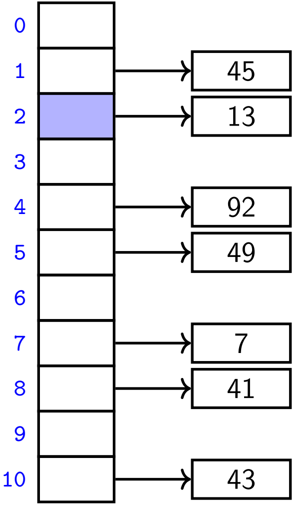


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(46)

$h(46) = 2$

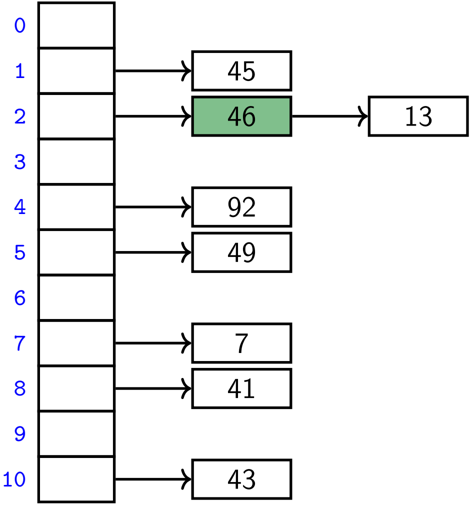


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(46)

$h(46) = 2$

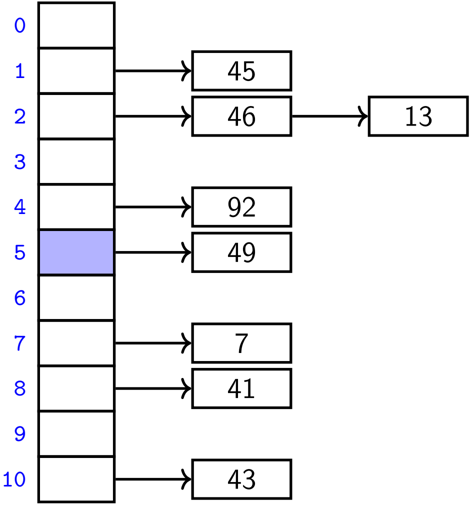


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(16)

$h(16) = 5$

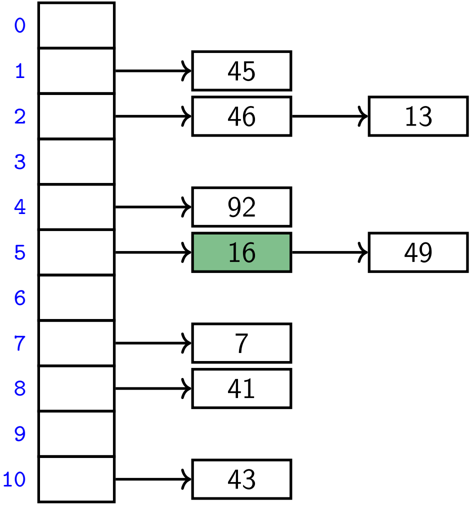


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(16)

$h(16) = 5$

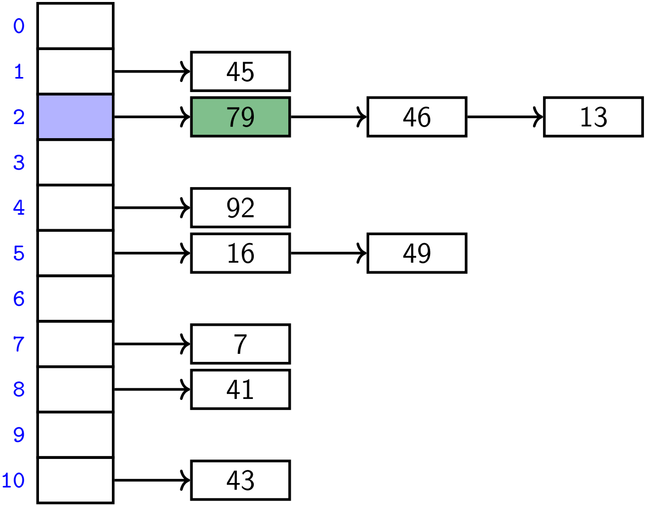


Chaining example

$M = 11, \qquad h(k) = k \bmod 11$

insert(79)

$h(79) = 2$



Complexity of chaining

Run-times: *insert* takes time $\Theta(1)$.

search and *delete* have run-time $\Theta\Big(1 + \text{size of bucket } T[h(k)]\Big)$.

- The *average* bucket-size is $\frac{n}{M} =: \alpha$.
  (α is also called the **load factor**.)

Complexity of chaining

Run-times: *insert* takes time $\Theta(1)$.
search and *delete* have run-time $\Theta\Big(1 + \text{size of bucket } T[h(k)]\Big)$.

- The *average* bucket-size is $\frac{n}{M} =: \alpha$.
  (α is also called the **load factor**.)
- However, this does not imply that the *average-case* cost of *search* and *delete* is $\Theta(1+\alpha)$.
  - Consider the case where all keys hash to the same slot
  - The average bucket-size is still α
  - But the operations take $\Theta(n)$ time on average
- To get meaningful average-case bounds, we need some assumptions on the hash-functions and the keys!

Complexity of chaining

- To analyze what happens 'on average', switch to *randomized* hashing.
- How can we randomize?

Complexity of chaining

- To analyze what happens 'on average', switch to *randomized* hashing.
- How can we randomize?
  Assume that the *hash-function* is chosen randomly.
  - We will later see examples how to do this.
- To be able to analyze, we assume the following:

Uniform Hashing Assumption*: Any possible hash-function is equally likely to be chosen as hash-function.*

(This is not at all realistic, but the assumption makes analysis possible.)

Complexity of chaining

UHA implies that the distribution of keys is unimportant.

- **Claim 1:** Hash-values are uniform.
  Formally: $P(h(k) = i) = \frac{1}{M}$ for any key k and slot i.

- **Claim 2:** Hash-values of any two keys are independent of each other.

Complexity of chaining

UHA implies that the distribution of keys is unimportant.

- **Claim 1:** Hash-values are uniform.
  Formally: $P(h(k) = i) = \frac{1}{M}$ for any key k and slot i.

- **Claim 2:** Hash-values of any two keys are independent of each other.

Back to complexity of chaining:

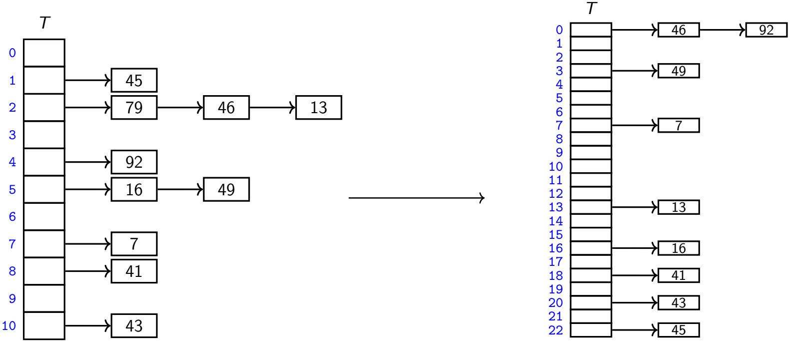
- Each bucket has expected length $\frac{n}{M} \leq \alpha$
  - n other keys are in this slot with probability $\frac{1}{M}$
- Each key in dictionary is expected to collide with $\frac{n-1}{M}$ other keys
  - $n-1$ other keys are in same slot with probability $\frac{1}{M}$
- Expected cost of *search* and *delete* is hence $\Theta(1 + \alpha)$

Load factor and re-hashing

- For hashing with chaining (and also other collision resolution strategies), the run-time bound depends on α

  (Recall: *load factor* $\alpha = n/M$.)

- We keep the load factor small by **rehashing** when needed:



  - Keep track of n and M throughout operations
  - If α gets too large, create new (roughly twice as big) hash-table, new hash-function(s) and re-insert all items in the new table.

Hashing with Chaining summary

- For Hashing with Chaining: Rehash so that $\alpha \in \Theta(1)$ throughout
- Rehashing costs $\Theta(M + n)$ time (plus the time to find a new hash function).
- Rehashing happens rarely enough that we can ignore this term when amortizing over all operations.
- We should also re-hash when α gets too small, so that $M \in \Theta(n)$ throughout, and the space is always $\Theta(n)$.

Summary: The amortized expected cost for hashing with chaining is $O(1)$ and the space is $\Theta(n)$

(assuming uniform hashing and $\alpha \in \Theta(1)$ throughout)

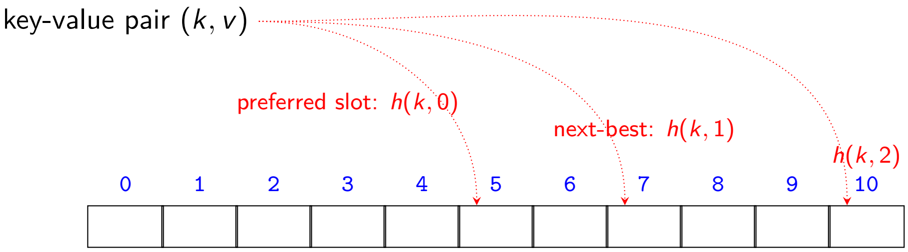
Theoretically perfect, but too slow in practice.

Outline

Open addressing

Main idea: Avoid the links needed for chaining by permitting only one item per slot, but allowing a key k to be in multiple slots.

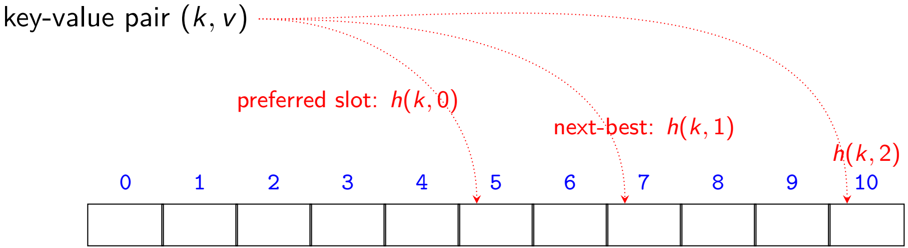
search and *insert* follow a **probe sequence** of possible locations for key k: $\langle h(k,0), h(k,1), h(k,2), \ldots h(k, M-1)\rangle$ until an empty spot is found.



Open addressing

Main idea: Avoid the links needed for chaining by permitting only one item per slot, but allowing a key k to be in multiple slots.

search and *insert* follow a **probe sequence** of possible locations for key k: $\langle h(k,0), h(k,1), h(k,2), \ldots h(k, M-1)\rangle$ until an empty spot is found.



Simplest method for open addressing: *linear probing*
$h(k, j) = (h(k) + j) \bmod M$, for some hash function h.

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

insert(41)

$h(41, 0) = 8$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k, j) = (h(k) + j) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

insert(84)

$h(84, 0) = 7$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

insert(84)

$h(84, 1) = 8$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

insert(84)

$h(84, 2) = 9$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k) + j) \bmod 11.$

insert(20)

$h(20, 0) = 9$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	**84**
10	43

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k, j) = (h(k) + j) \bmod 11.$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

insert(20)

$h(20, 1) = 10$

Linear probing example

$M = 11, \qquad h(k) = k \bmod 11, \qquad h(k,j) = (h(k)+j) \bmod 11.$

0	20
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	84
10	43

insert(20)

$h(20,2) = 0$

Probe sequence operations

Use *lazy deletion* (cannot handle updates after *delete* efficiently).

```
probe-sequence::insert(T, (k, v))
for (j = 0; j < M; j++)
    if T[h(k, j)] is NULL or "deleted"
        T[h(k, j)] = (k, v)
        return "success"
return "failure to insert"      // need to re-hash
```

```
probe-sequence-search(T, k)
for (j = 0; j < M; j++)
    if T[h(k, j)] is NULL return "item not found"
    if T[h(k, j)] has key k return T[h(k, j)]
    // key is incorrect or "deleted"
    // try next probe, i.e., continue for-loop
return "item not found"
```

Independent hash functions

- Some hashing methods require *two* hash functions h_0, h_1.
- These hash functions should be *independent* in the sense that the random variables $P(h_0(k) = i)$ and $P(h_1(k) = j)$ are independent.
- Using two modular hash-functions often leads to dependencies.

Independent hash functions

- Some hashing methods require *two* hash functions h_0, h_1.
- These hash functions should be *independent* in the sense that the random variables $P(h_0(k) = i)$ and $P(h_1(k) = j)$ are independent.
- Using two modular hash-functions often leads to dependencies.
- Better idea: Use *multiplication method* for second hash function:
  - Fix some floating-point number A with $0 < A < 1$

$$h(k) = \Big\lfloor \underbrace{M \cdot \big(\underbrace{\underbrace{A \cdot k}_{\text{multiply}} - \underbrace{\lfloor A \cdot k \rfloor}_{\text{integral part}}}_{\text{fractional part, in } [0,1)} \big)}_{\text{integer in } [0,M)} \Big\rfloor .$$

  - Our examples use $\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618033988749....$ as A.

Double Hashing

- Assume we have two hash independent functions h_0, h_1.
- Assume further that $h_1(k) \neq 0$ and that $h_1(k)$ is relative prime with the table-size M for all keys k.
  - Choose M prime.
  - Modify standard hash-functions to ensure $h_1(k) \neq 0$
    E.g. modified multiplication method: $h(k) = 1 + \lfloor (M-1)(kA - \lfloor kA \rfloor) \rfloor$
- **Double hashing**: open addressing with probe sequence

$$h(k, j) = \big(h_0(k) + j \cdot h_1(k)\big) \bmod M$$

- *search*, *insert*, *delete* work just like for linear probing, but with this different probe sequence.

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(41)

$h_0(41) = 8$

$h(41, 0) = 8$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

$h_1(194) = 9$

$h(194, 1) = 5$

0	
1	45
2	13
3	
4	92
5	49
6	
7	7
8	41
9	
10	43

Double hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 10(\varphi k - \lfloor \varphi k \rfloor) \rfloor + 1$

insert(194)

$h_0(194) = 7$

$h(194, 0) = 7$

$h_1(194) = 9$

$h(194, 1) = 5$

$h(194, 2) = 3$

0	
1	45
2	13
3	194
4	92
5	49
6	
7	7
8	41
9	
10	43

Analysis of uniform probing

- Analyzing linear probing and double hashing is difficult (no details).
- Instead, analyze an idealized setup: **uniform probing**

$$P(\text{slot } i \text{ is occupied}) = \frac{1}{M}$$

- As before, $\alpha = \frac{n}{M}$ is the load factor.

Analysis of uniform probing

- Analyzing linear probing and double hashing is difficult (no details).
- Instead, analyze an idealized setup: **uniform probing**

$$P(\text{slot } i \text{ is occupied}) = \frac{1}{M}$$

- As before, $\alpha = \frac{n}{M}$ is the load factor.

Claim 1: The expected run-time of *search* is $O(\frac{1}{1-\alpha})$.

Analysis of uniform probing

- Analyzing linear probing and double hashing is difficult (no details).
- Instead, analyze an idealized setup: **uniform probing**

$$P(\text{slot } i \text{ is occupied}) = \frac{1}{M}$$

- As before, $\alpha = \frac{n}{M}$ is the load factor.

Claim 1: The expected run-time of *search* is $O(\frac{1}{1-\alpha})$.

Claim 2: The expected-luck average-instance run-time of a successful *search* is $O(\frac{1}{\alpha}) \ln\left(\frac{1}{1-\alpha}\right)$.

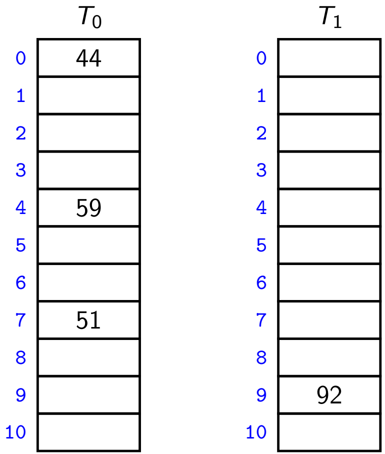
Outline

Cuckoo hashing

We use two independent hash functions h_0, h_1 and two tables T_0, T_1.

Main idea: An item with key k can *only* be at $T_0[h_0(k)]$ or $T_1[h_1(k)]$.

search and *delete* then *always* take constant time.



Cuckoo Hashing Insertion

insert *always* initially puts the new item into $T_0[h_0(k)]$

- Evict item that may have been there already.
- If so, evicted item inserted at alternate position
- This may lead to a loop of evictions.
  - **Can show:** If insertion is possible, then there are at most $2n$ evictions.
  - So abort after too many attempts.

cuckoo::insert(k, v)

$(k_{insert}, v_{insert}) \leftarrow$ new key-value pair with (k, v)
$i \leftarrow 0$
do at most $2n$ times:
    $(k_{evict}, v_{evict}) \leftarrow T_i[h_i(k_{insert})]$ `// save old KVP`
    $T_i[h_i(k_{insert})] \leftarrow (k_{insert}, v_{insert})$ `// put in new KVP`
    if (k_{evict}, v_{evict}) is `NULL` **return** "success"
    else `// repeat in other table`
        $(k_{insert}, v_{insert}) \leftarrow (k_{evict}, v_{evict})$; $i \leftarrow 1 - i$
return "failure to insert" `// need to re-hash`

Cuckoo hashing example

$M = 11, \qquad h_0(k) = k \bmod 11, \qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

T_0

0	44
1	
2	
3	
4	59
5	
6	
7	
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	92
10	

Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(51)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$

	T_0
0	44
1	
2	
3	
4	59
5	
6	
7	
8	
9	
10	

	T_1
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	92
10	

Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(51)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$

	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		
6		
7	51	
8		
9		92
10		

Cuckoo hashing example

$M = 11$, $\quad h_0(k) = k \bmod 11$, $\quad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 0$
$k = 95$

$h_0(k) = 7$
$h_1(k) = 7$

	T_0	T_1
0	44	
1		
2		
3		
4	59	
5		
6		
7	51	
8		
9		92
10		

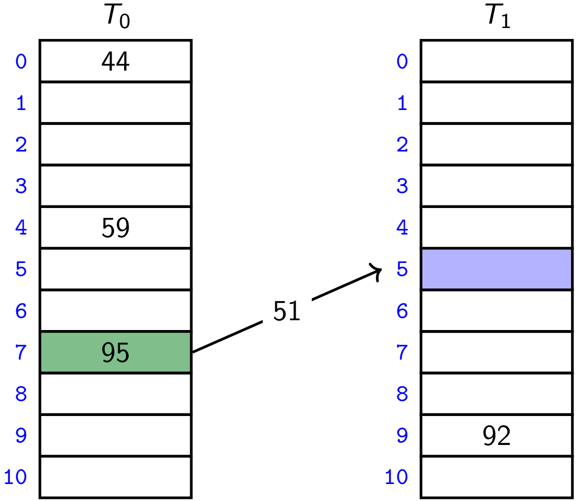
Cuckoo hashing example

$M = 11,$ $h_0(k) = k \bmod 11,$ $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 1$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$



Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(95)

$i = 1$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$

	T_0
0	44
1	
2	
3	
4	59
5	
6	
7	95
8	
9	
10	

	T_1
0	
1	
2	
3	
4	
5	51
6	
7	
8	
9	92
10	

Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 0$
$k = 26$

$h_0(k) = 4$
$h_1(k) = 0$

T_0

0	44
1	
2	
3	
4	59
5	
6	
7	95
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	51
6	
7	
8	
9	92
10	

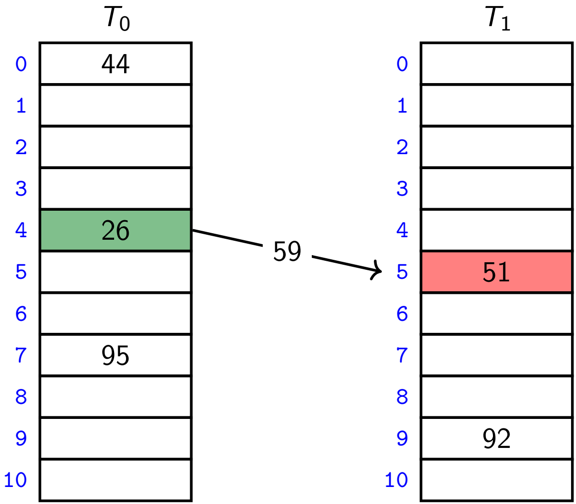
Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 1$
$k = 59$

$h_0(k) = 4$
$h_1(k) = 5$



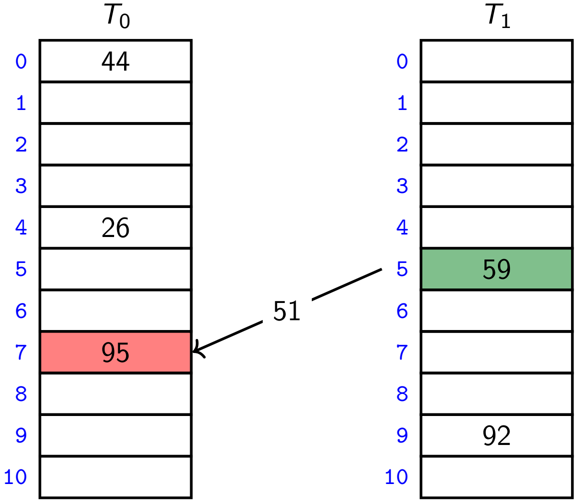
Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 0$
$k = 51$

$h_0(k) = 7$
$h_1(k) = 5$



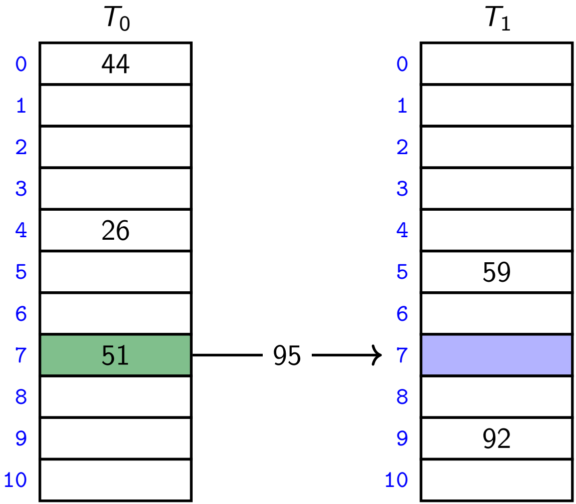
Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 1$
$k = 95$

$h_0(k) = 4$
$h_1(k) = 7$



Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

insert(26)

$i = 1$
$k = 95$

$h_0(k) = 4$
$h_1(k) = 7$

	T_0	T_1
0	44	
1		
2		
3		
4	26	
5		59
6		
7	51	95
8		
9		92
10		

Cuckoo hashing example

$M = 11$, $h_0(k) = k \bmod 11$, $h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

search(59)

$h_0(59) = 4$
$h_1(59) = 5$

T_0

0	44
1	
2	
3	
7	26
5	
6	
7	51
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	59
6	
7	95
8	
9	92
10	

Cuckoo hashing example

$M = 11,\qquad h_0(k) = k \bmod 11,\qquad h_1(k) = \lfloor 11(\varphi k - \lfloor \varphi k \rfloor) \rfloor$

delete(59)

$h_0(59) = 4$
$h_1(59) = 5$

T_0

0	44
1	
2	
3	
7	26
5	
6	
7	51
8	
9	
10	

T_1

0	
1	
2	
3	
4	
5	
6	
7	95
8	
9	92
10	

Cuckoo hashing discussions

- **Can show**: expected number of evictions during *insert* is $O(1)$.
  - So in practice, stop evictions much earlier than $2n$ rounds.
- This crucially requires load factor $\alpha < \frac{1}{2}$.
  - Here $\alpha = n/(\text{size of } T_0 + \text{size of } T_1)$
- So cuckoo hashing is wasteful on space.
- In fact, space is $\omega(n)$ if *insert* forces lots of re-hashing.
- **Can show**: expected space is $O(n)$.

Cuckoo hashing discussions

- **Can show**: expected number of evictions during *insert* is $O(1)$.
  - So in practice, stop evictions much earlier than $2n$ rounds.
- This crucially requires load factor $\alpha < \frac{1}{2}$.
  - Here $\alpha = n/(\text{size of } T_0 + \text{size of } T_1)$
- So cuckoo hashing is wasteful on space.
- In fact, space is $\omega(n)$ if *insert* forces lots of re-hashing.
- **Can show**: expected space is $O(n)$.

There are many possible variations:

- The two hash-tables could be combined into one.
- Be more flexible when inserting: Always consider both possible positions.
- Use $k > 2$ allowed locations (i.e., k hash-functions).

Complexity of open addressing strategies

For any open addressing scheme, we *must* have $\alpha \leq 1$ (why?).
For the analysis, we require $0 < \alpha < 1$ (not arbitrarily close).
Cuckoo hashing requires $0 < \alpha < 1/2$ (not arbitrarily close).

Under these restrictions (and the universal hashing assumption):

- All strategies have $O(1)$ expected time for *search*, *insert*, *delete*.
- Cuckoo Hashing has $O(1)$ worst-case time for *search*, *delete*.
- Probe sequences use $O(n)$ worst-case space,
  Cuckoo Hashing uses $O(n)$ expected space.

But for any hash-function the worst-case run-time is $\Theta(n)$ for *insert*.

Complexity of open addressing strategies

For any open addressing scheme, we *must* have $\alpha \leq 1$ (why?).
For the analysis, we require $0 < \alpha < 1$ (not arbitrarily close).
Cuckoo hashing requires $0 < \alpha < 1/2$ (not arbitrarily close).

Under these restrictions (and the universal hashing assumption):

- All strategies have $O(1)$ expected time for *search*, *insert*, *delete*.
- Cuckoo Hashing has $O(1)$ worst-case time for *search*, *delete*.
- Probe sequences use $O(n)$ worst-case space,
  Cuckoo Hashing uses $O(n)$ expected space.

But for any hash-function the worst-case run-time is $\Theta(n)$ for *insert*.

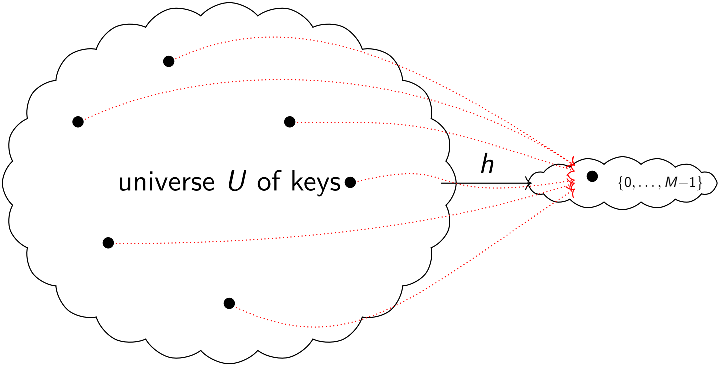
In practice, double hashing seems the most popular, or cuckoo hashing if there are many more searches than insertions.

Outline

Hash functions

Every hash function *must* do badly for some inputs:

- If the universe is big enough ($|U| \geq M(n-1)+1$), then there are n keys that all hash to the same value.



- If we insert this set of keys, then we have $\Theta(n)$ run-time.

Choosing a good hash function

- Analysis works only under **uniform hashing assumption**: Hash function is randomly chosen among all possible hash-functions.
- Satisfying this is impossible: There are too many hash functions; we would not know how to look up $h(k)$.

Choosing a good hash function

- Analysis works only under **uniform hashing assumption**: Hash function is randomly chosen among all possible hash-functions.
- Satisfying this is impossible: There are too many hash functions; we would not know how to look up $h(k)$.

Two ways to compromise:

1. Deterministic: hope for good performance by choosing a hash-function that is
   - unrelated to any possible patterns in the data, and
   - depends on all parts of the key.
2. Randomized: Choose randomly among a limited set of functions.
   - But aim for $P(\text{two keys collide}) = \frac{1}{M}$ w.r.t. key-distribution.
   - This is enough to prove the expected run-time bounds for chaining

Deterministic hash functions

We saw two basic methods for integer keys:

- **Modular method**: $h(k) = k \bmod M$.
  - We should choose M to be a prime.
  - This means finding a suitable prime quickly when re-hashing.
  - This can be done in $O(M \log\log n)$ time (no details).

Deterministic hash functions

We saw two basic methods for integer keys:

- **Modular method**: $h(k) = k \bmod M$.
  - We should choose M to be a prime.
  - This means finding a suitable prime quickly when re-hashing.
  - This can be done in $O(M \log\log n)$ time (no details).
- **Multiplication method**: $h(k) = \lfloor M(kA - \lfloor kA \rfloor)\rfloor$,
  for some floating-point number A with $0 < A < 1$.
  - Multiplying with A is used to scramble the keys.
    So A should be irrational to avoid patterns in the keys.
  - Experiments show that good scrambling is achieved when A is the golden ratio $\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618033988749....$

Deterministic hash functions

We saw two basic methods for integer keys:

- **Modular method**: $h(k) = k \bmod M$.
  - We should choose M to be a prime.
  - This means finding a suitable prime quickly when re-hashing.
  - This can be done in $O(M \log\log n)$ time (no details).
- **Multiplication method**: $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$, for some floating-point number A with $0 < A < 1$.
  - Multiplying with A is used to scramble the keys. So A should be irrational to avoid patterns in the keys.
  - Experiments show that good scrambling is achieved when A is the golden ratio $\varphi = \frac{\sqrt{5}-1}{2} \approx 0.618033988749.....$
  - How many bits should we use?
  - Won't the computation be terribly slow?

Multiplication method

- $h(k) = \lfloor M(kA - \lfloor kA \rfloor) \rfloor$
- Consider what happens at bit-level:

$A = 0.a_1 a_2 a_3 \ldots$
$k = b_1 b_2 \ldots b_6$
(both in base 2)

	(leading bits)	(bits of fractional part)	
$A \cdot k =$	$0\ 0\ 0\ ..\ 0\ 0$		
$+a_1\cdot$	$0\ b_1 b_2 b_3\ ..\ b_5$	b_6	
$+a_2\cdot$	$0\ 0\ b_1 b_2 b_3\ ..$	$b_5\ b_6$	
$+a_3\cdot$	$0\ 0\ 0\ b_1 b_2 b_3$	$..\ b_5\ b_6$	
$\vdots$	$\ddots\ \ddots$	$\ddots\ \ddots\ \ddots$	$\ddots$
$+a_5\cdot$	$0\ 0\ 0\ 0\ 0\ b_1$	$b_2\ b_3\ ..$	$b_5\ b_6$
$+a_6\cdot$	$0\ 0\ 0\ 0\ 0\ 0$	$b_1\ b_2\ b_3$	$..\ b_5\ b_6$
$+a_7\cdot$	$0\ 0\ 0\ 0\ 0\ 0$	$0\ b_1\ b_2$	$b_3\ ..\ b_5\ b_6$
$+a_8\cdot$	$0\ 0\ 0\ 0\ 0\ 0$	$0\ 0\ b_1$	$b_2\ b_3\ ..\ b_5\ b_6$
$\vdots$		$\leftarrow h(k) \rightarrow$	$\ddots\ \ddots\ \ddots\ \ddots\ \ddots\ \ddots$

- Use $M = 2^\ell$. Then $h(k)$ = first ℓ bits of fractional part.
- Only $\log |U| + \ell$ bits of A influence $h(k)$.
- Computing $h(k)$ = multiplication plus bit-shift.
  This may actually be faster than taking "modulo a prime number".

Outline

Randomly chosen hash-functions

- Ideally we would choose randomly among all hash functions. But this is impossible.
- **Idea:** Fix a family $\mathcal{H}$ of hash-functions that are easy to compute. Then choose uniformly among them.
- Example:
  - $U = \mathbb{Z}_5,\ M = 2$
  - $h_b(k) = ((k + b) \bmod 5) \bmod 2$
  - $\mathcal{H} = \{h_b : b \in \mathbb{Z}_5\}$
  - Choose $b \in \mathbb{Z}_5$ randomly to get hash-function

$\mathcal{H}$	keys				
	0	1	2	3	4
h_0	0	1	0	1	0
h_1	1	0	1	0	0
h_2	0	1	0	0	1
h_3	1	0	0	1	0
h_4	0	0	1	0	1

- But how do we measure whether these are "good"?

Universal hash-functions

- For analysis, we needed *uniform hash-values*:,

$$P(h(k) = i) = \frac{1}{M}$$

- But this is *not* good enough.

$\mathcal{H}$	keys				
	0	1	2	3	4
h_0	0	0	0	0	0
h_1	1	1	1	1	1

- $P(h(k) = i) = \frac{1}{2}$ for $i = 0, 1$ and any k
- But these hash-functions are terrible!
- Problem: hash-values not independent

- Also want: Small probability of collisions ($\mathcal{H}$ is **universal**):

$$P\Big(h(k) = h(k')\Big) \leq \frac{1}{M} \quad \text{for any two keys } k \neq k'$$

- This is enough for analyzing hashing with chaining as before.

Carter-Wegman hash-function

$$\mathcal{H}_{CW} = \left\{ \quad h_{a,b}(k) = \Big(\underbrace{(a \cdot k + b) \bmod p}_{f_{a,b}(k)} \Big) \bmod M \quad : a, b \in \mathbb{Z}_p, a \neq 0 \right\}$$

(where p prime, universe of keys is $\{0, \ldots, p-1\} =: \mathbb{Z}_p$, $M < p$)

Example: ($p = 5, M = 2$):

	keys				
	0	1	2	3	4
$f_{1,0}$	0	1	2	3	4
$f_{2,0}$	0	2	4	1	3
$f_{1,2}$	2	3	4	0	1
$f_{2,1}$	1	3	0	2	4
⋮	⋮	⋮	⋮	⋮	⋮

Carter-Wegman hash-function

$$\mathcal{H}_{CW} = \left\{ \quad h_{a,b}(k) = \Big(\underbrace{(a \cdot k + b) \bmod p}_{f_{a,b}(k)} \Big) \bmod M \quad : a, b \in \mathbb{Z}_p, a \neq 0 \right\}$$

(where p prime, universe of keys is $\{0, \ldots, p-1\} =: \mathbb{Z}_p$, $M < p$)

Example: ($p = 5, M = 2$):

	keys				
	0	1	2	3	4
$f_{1,0}$	0	1	2	3	4
$f_{2,0}$	0	2	4	1	3
$f_{1,2}$	2	3	4	0	1
$f_{2,1}$	1	3	0	2	4
⋮	⋮	⋮	⋮	⋮	⋮

	keys				
	0	1	2	3	4
$h_{1,0}$	0	1	0	1	0
$h_{2,0}$	0	0	0	1	1
$h_{1,2}$	0	1	0	0	1
$h_{2,1}$	1	1	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮

Carter-Wegman hash-function

$$\mathcal{H}_{CW} = \left\{ \quad h_{a,b}(k) = \Big(\underbrace{(a \cdot k + b) \bmod p}_{f_{a,b}(k)} \Big) \bmod M \quad : a, b \in \mathbb{Z}_p, a \neq 0 \right\}$$

(where p prime, universe of keys is $\{0, \ldots, p-1\} =: \mathbb{Z}_p$, $M < p$)

Example: $(p = 5, M = 2)$:

	keys				
	0	1	2	3	4
$f_{1,0}$	0	1	2	3	4
$f_{2,0}$	0	2	4	1	3
$f_{1,2}$	2	3	4	0	1
$f_{2,1}$	1	3	0	2	4
⋮	⋮	⋮	⋮	⋮	⋮

	keys				
	0	1	2	3	4
$h_{1,0}$	0	1	0	1	0
$h_{2,0}$	0	0	0	1	1
$h_{1,2}$	0	1	0	0	1
$h_{2,1}$	1	1	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮

Observe: $f_{a,b}$ is a permutation of $\mathbb{Z}_p$.

Carter-Wegman's universal hashing

- Requires: all keys are in $\{0, \ldots, p-1\}$ for some (big) prime p.
- At initialization, and whenever we re-hash:
  - Choose $M < p$ arbitrarily, power of 2 is ok.
  - Choose (and store) two *random* numbers a, b
    - $b = \mathit{random}(p)$
    - $a = 1 + \mathit{random}(p-1)$ (so $a \neq 0$)
  - Use as hash-function $\boxed{h_{a,b}(k) = \big((ak+b) \bmod p\big) \bmod M}$
- $h(k)$ can be computed quickly.

Carter-Wegman's universal hashing

- Requires: all keys are in $\{0, \ldots, p-1\}$ for some (big) prime p.
- At initialization, and whenever we re-hash:
  - Choose $M < p$ arbitrarily, power of 2 is ok.
  - Choose (and store) two *random* numbers a, b
    - $b = \textit{random}(p)$
    - $a = 1 + \textit{random}(p-1)$ (so $a \neq 0$)
  - Use as hash-function $\boxed{h_{a,b}(k) = \big((ak+b) \bmod p\big) \bmod M}$
- $h(k)$ can be computed quickly.

Theorem: $\mathcal{H}_{CW}$ is universal: $P(h(k) = h(k')) \leq \frac{1}{M}$.

So hashing with chaining and randomly chosen hash function in $\mathcal{H}_{CW}$ has expected run-time $O(1)$.

Multi-dimensional Data

What if the keys are multi-dimensional, such as strings?

Standard approach is to *flatten* string w to integer $f(w) \in \mathbb{N}$, e.g.

$$\begin{aligned} A \cdot P \cdot P \cdot L \cdot E \quad &\rightarrow \quad (65, 80, 80, 76, 69) \quad \text{(ASCII)} \\ &\rightarrow \quad 65R^4 + 80R^3 + 80R^2 + 76R^1 + 69R^0 \\ &\qquad \text{(for some radix } R\text{, e.g. } R = 255) \end{aligned}$$

We combine this with a modular hash function: $\boxed{h(w) = f(w) \bmod M}$

To compute this in $O(|w|)$ time without overflow, use Horner's rule and apply mod early. For exampe, $h(APPLE)$ is

$$\Bigg(\bigg(\Big(\Big(\big(\big((65R+80) \bmod M\big)R+80\big) \bmod M\Big)R+76\Big) \bmod M\bigg)R+69\Bigg) \bmod M$$

Hashing vs. Balanced Search Trees

Advantages of Balanced Search Trees

- $O(\log n)$ worst-case operation cost
- Does not require any assumptions, special functions, or known properties of input distribution
- Predictable space usage (exactly n nodes)
- Never need to rebuild the entire structure
- Supports ordered dictionary operations (successor, select, rank etc.)

Advantages of Hash Tables

- $O(1)$ operation cost (if hash-function random and load factor small)
- We can choose space-time tradeoff via load factor
- Cuckoo hashing achieves $O(1)$ worst-case for search & delete