CS 240E – Data Structures and Data Management (Enriched)

Module 8: Range-Searching in Dictionaries for Points

Therese Biedl

Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Outline

8. Range-Searching in Dictionaries for Points
   - Range Searches
   - Multi-Dimensional Data
   - Quadtrees
   - kd-Trees
   - Range Trees
   - 3-sided range search

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Range searches

- So far: *search*(k) looks for *one* specific item.
- **range-search**: look for *all* items in a given range.
  - Input: A **range**, i.e., an interval $Q = (x, x')$ (open or closed)
  - Want: Report all KVPs in the dictionary whose key k satisfies $k \in Q$

Example:

5	10	11	17	19	33	45	51	55	59

range-search((18,45]) should return $\{19, 33, 45\}$

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- As usual n denotes the number of input-items.
- Let s be the **output-size**, i.e., the number of items in the range.
- We need $\Omega(s)$ time simply to report the items.
- Note that sometimes $s = 0$ and sometimes $s = n$; we therefore keep it as a separate parameter when analyzing the run-time.

Typical run-time: $O(\log n + s)$.

Range searches in existing dictionary realizations

Unsorted list/array/hash table: Range search requires $\Omega(n)$ time: We have to check for each item explicitly whether it is in the range.

Sorted array: Range search in A can be done in $O(\log n + s)$ time:

range-search((18,45])

5	10	11	17	19	33	45	51	55	59
			$\uparrow i$			$\uparrow i'$			

- Using binary search, find i such that x is at (or would be at) $A[i]$.
- Using binary search, find i' such that x' is at (or would be at) $A[i']$
- Report all items $A[i{+}1...i'{-}1]$
- Report $A[i]$ and $A[i']$ if they are in range

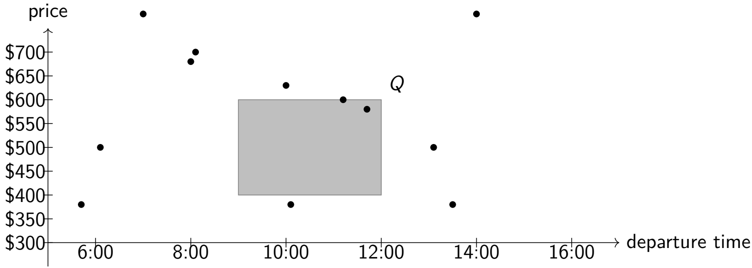
BST: Range searches can similarly be done in time $O(\text{height}{+}s)$ time. We will see this in detail later.

Outline

Multi-Dimensional Data

Range searches are of special interest for **multi-dimensional data**.

Example: flights that leave between 9am and noon, and cost \$400-\$600



- Each item has d **aspects** (coordinates): $(x_0, x_1, \cdots, x_{d-1})$ so corresponds to a point in d-dimensional space
- We concentrate on $d = 2$, i.e., points in Euclidean plane
- (Orthogonal) ***d*-dimensional range search**: Given a **query rectangle** $Q = [x_1, x_1'] \times \cdots \times [x_d, x_d']$, find all points that lie within Q.

Multi-dimensional Range Search

The time for range searches depends on how the points are stored.

- Two naive ideas that do not work well:
  - Could store a 1-dimensional dictionary (where the key is some combination of the aspects.)
    Problem: Range search on one aspect is not straightforward
  - Could use one dictionary for each aspect
    Problem: inefficient, wastes space
- **Better idea**: Design new data structures specifically for points.
  - Quadtrees
  - kd-trees
  - range-trees

Assumption: Points are in **general position**:

- No two points on a horizontal line.
- No two points on a vertical line.

This simplifies presentation; data structures can be generalized.

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   - Multi-Dimensional Data
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   - Range Trees
   - 3-sided range search

Quadtrees

We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane.

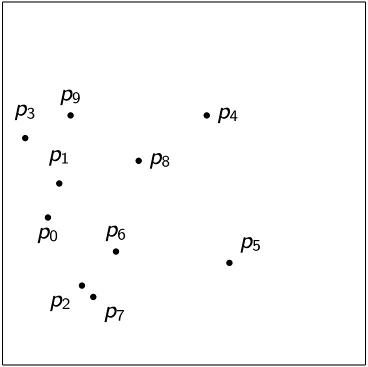
Find a **bounding box** $R = [0, 2^k) \times [0, 2^k)$: a square containing all points.

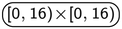
- Assume (after translation) that all coordinates are non-negative.
- Find max-coordinate in P, use the smallest k such that it is $< 2^k$.

Structure (and also how to *build* the quadtree that stores P):

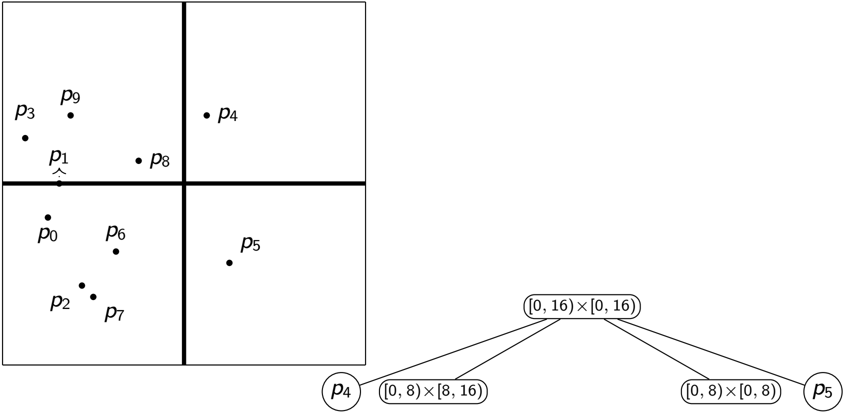
- Root r of the quadtree is associated with region R
- If R contains 0 or 1 points, then root r is a leaf that stores point.
- Else *split*: Partition R into four equal subsquares (**quadrants**) $R_{NE}, R_{NW}, R_{SW}, R_{SE}$
- Partition P into sets $P_{NE}, P_{NW}, P_{SW}, P_{SE}$ of points in these regions.
  - **Convention:** Points on split lines belong to right/top side
- Recursively build tree T_i for points P_i in region R_i and make them children of the root.

Quadtree example





Quadtree example



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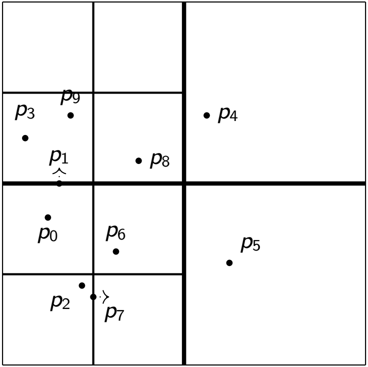
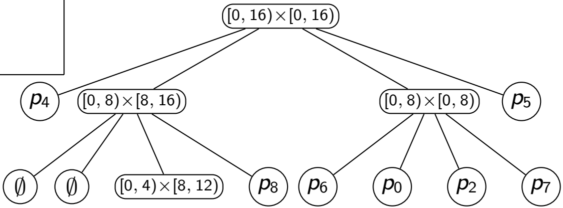
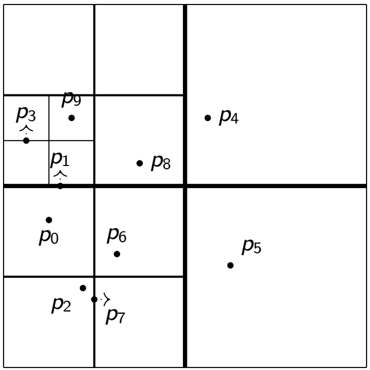
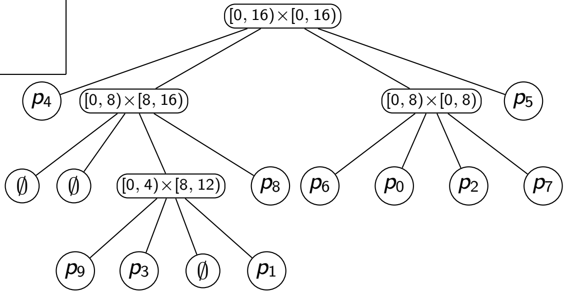




Quadtree example

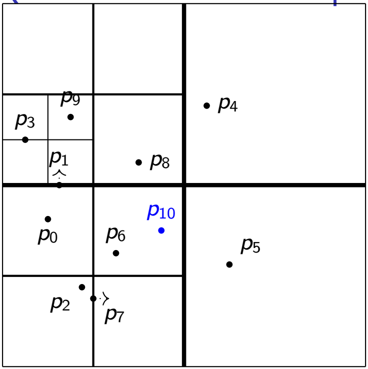




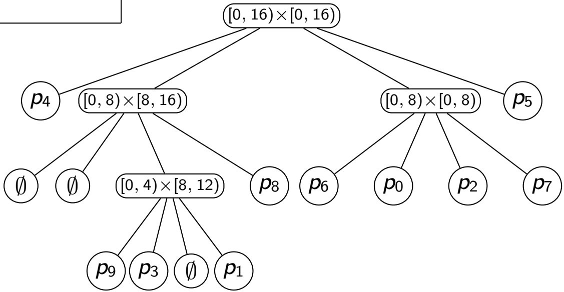
Quadtree Dictionary Operations

- *search*: Analogous to binary search trees and tries
- *insert*:
  - Search for the point
  - Split the leaf while there are two points in one region
- *delete*:
  - Search for the point
  - Remove the point
  - If its parent has only one point left: delete parent
    (and recursively all ancestors that have only one point left)

Quadtree Insert example



insert(p_{10})



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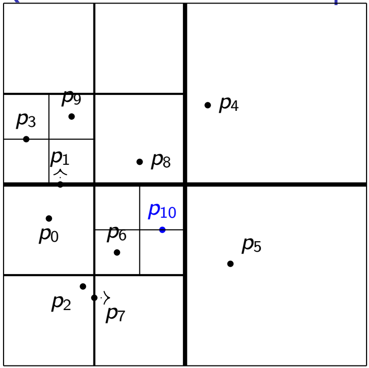

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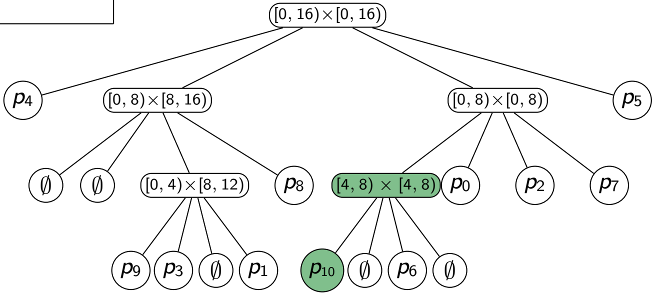


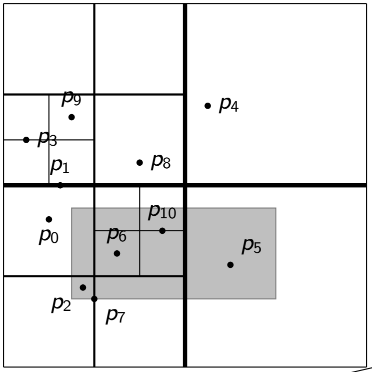
Quadtree Range Search

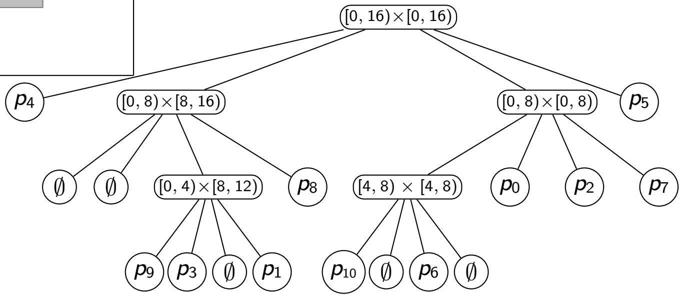
QTree::range-search($r \leftarrow root, Q$)
r: The root of a quadtree, Q: Query-rectangle
1. $R \leftarrow$ region associated with node r
2. **if** ($R \subseteq Q$) **then** // inside node, stop searching
    report all points below r and **return**
3. **else if** ($R \cap Q$ is empty) **then return** // outside node, stop searching

    // boundary node, recurse
4. **if** (r is a leaf) **then**
5. $p \leftarrow$ point stored at r
6. **if** p is not `NULL` and in Q **then** report it and **return**
7. **else return**
8. **for** each child v of r **do** *QTree::range-search*(v, Q)

Note: We assume here that each node of the quadtree stores the associated square. Alternatively, these could be re-computed during the search (space-time tradeoff).

Quadtree range search example





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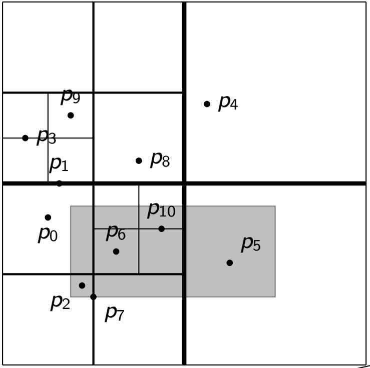
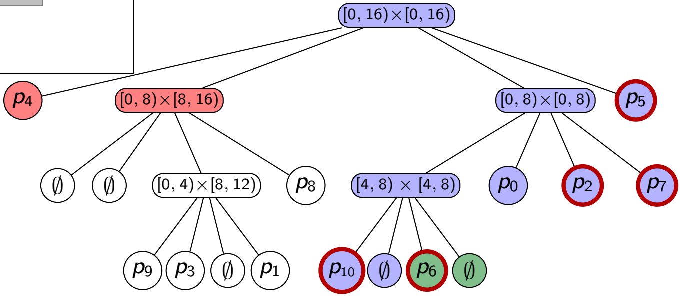


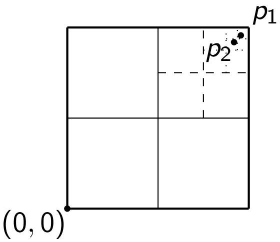
- Green: Search stopped due to $R \subseteq Q$.
- Red: Search stopped due to $R \cap Q = \emptyset$.
- Blue: Must continue search in children / evaluate.



Quadtree Analysis

Complexity of range search:

- In worst-case, we look at nearly all nodes, even if the answer is $\emptyset$.
- The number nodes could be $\Theta(nh)$, where h is the height.
- Can have very large height for bad distributions of points.



(Even with $n = 3$ points, the height can be arbitrarily large.)

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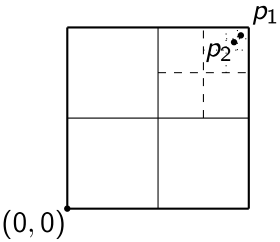

(Even with $n = 3$ points, the height can be arbitrarily large.)

In practice, quad-trees work quite well. Theoretical evidence (no details):

- For n randomly chosen points, the expected height is $O(\log n)$.
- The height depends on the **spread factor**:

$$\frac{\text{sidelength of } R}{\text{minimum distance between points in } P}$$

The height is in $\Theta(\log(\text{spread factor}))$

Quadtrees in other dimensions

- Quad-tree of 1-dimensional points:

"Points:" 0 9 12 14 24 26 28

Quadtrees in other dimensions

- Quad-tree of 1-dimensional points:

“Points:”	0	9	12	14	24	26	28
(in base-2)	00000	01001	01100	01110	11000	11010	11100

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[0,32)

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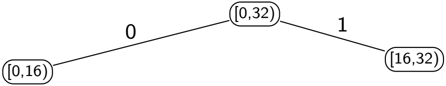

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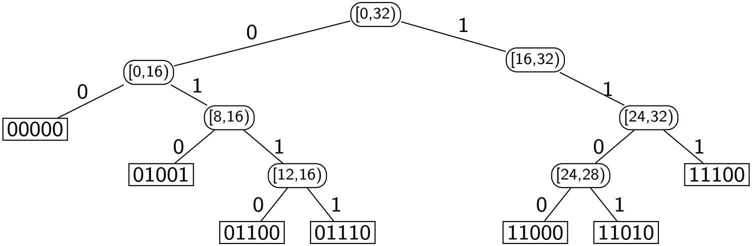

Same as a pruned trie

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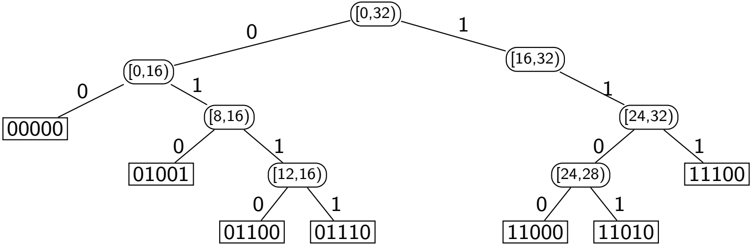
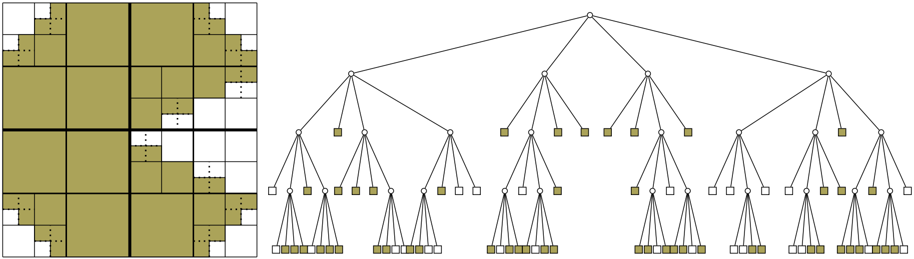


Same as a pruned trie

- Quadtrees also easily generalize to higher dimensions (split into octants $\rightarrow$ octrees, etc.) but are rarely used beyond dimension 3.

Quadtree summary

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (bit-shift!) if the width/height of bounding box R is a power of 2
- Space potentially wasteful, but good if points are well-distributed
- Variation: We could stop splitting earlier and allow up to K points in a leaf (for some fixed bound K).
- Variation: Use quad-tree to store pixelated images.



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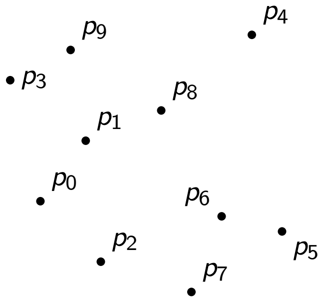
kd-trees

- We have n points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$
- Quadtrees: Split region into quadrants regardless of where points are
- kd-tree idea: Split region based on where points are.
  - We split at upper median of coordinates
  - $\rightsquigarrow$ roughly half of the point are in each subtree
- Each node of the kd-tree keeps track of a **splitting line** in one dimension (2D: either vertical or horizontal)
- **Convention:** Points on split lines belong to right/top side
- Continue splitting, switching between vertical and horizontal lines, until every point is in a separate region

  (There are alternatives, e.g., split by the dimension that has better aspect ratios for the resulting regions.)

kd-tree example



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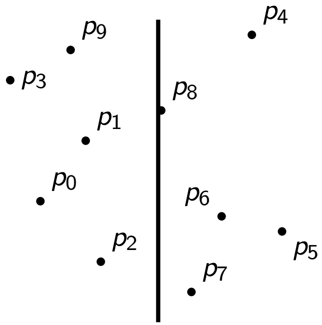

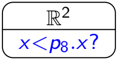

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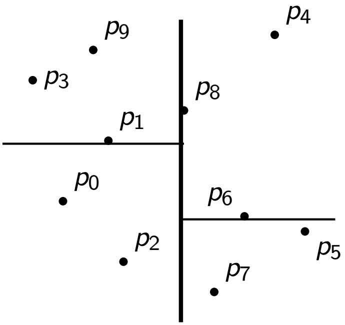
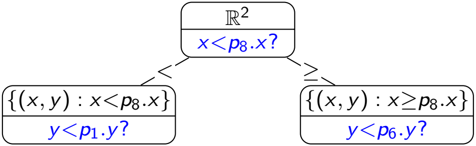
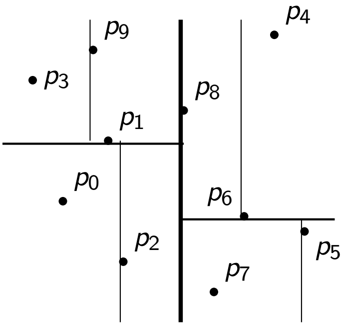
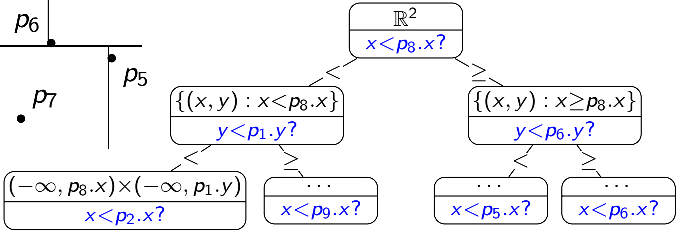




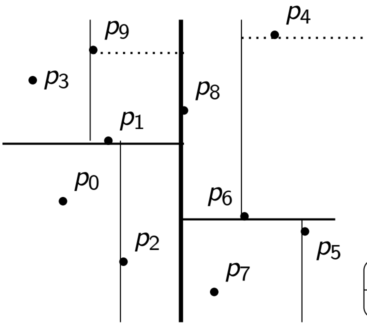
kd-tree example

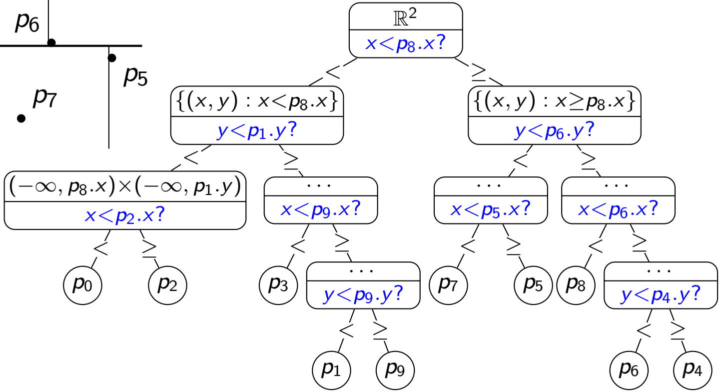




For ease of drawing, we will usually not list the associated regions of nodes.

kd-tree example





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Constructing kd-trees

The algorithm to build a kd-tree is immediate from the definition of a kd-tree:

To build a kd-tree with initial split by x on points P:

- If $|P| \leq 1$ create a leaf and return.
- Else $X :=$ *randomized-quick-select*$(P, \lfloor \frac{n}{2} \rfloor)$ (select by x-coordinate)
- Partition P by x-coordinate into $P_{x<X}$ and $P_{x \geq X}$
  - $\lfloor \frac{n}{2} \rfloor$ points on one side and $\lceil \frac{n}{2} \rceil$ points on the other. (Recall: Points in general position.)
- Create left subtree recursively (splitting by y) for points $P_{x<X}$.
- Create right subtree recursively (splitting by y) for points $P_{x \geq X}$.

Building with initial y-split symmetric.

Constructing kd-trees

Run-time:

- Find X and partition P takes $\Theta(n)$ expected time.
- Both subtrees have $\approx n/2$ points.

$$T^{\exp}(n) = 2T^{\exp}(n/2) + O(n) \quad \text{(sloppy recurrence)}$$

  This resolves to $\Theta(n \log n)$ expected time.

- This can be reduced to $\Theta(n \log n)$ *worst-case* time by pre-sorting.

Height: $h(1) = 0$, $h(n) \le h(\lceil n/2 \rceil) + 1$.

- This resolves to $O(\log n)$ (specifically $\lceil \log n \rceil$).
- This is tight (binary tree with n leaves)

Space: All interior nodes have exactly two children.

- Therefore have $n - 1$ interior nodes.
- Space is $\Theta(n)$.

kd-tree Dictionary Operations

- *search* (for single point): as in binary search tree using indicated coordinate
- *insert*: search, insert as new leaf.
- *delete*: search, remove leaf.

Problem: After insert or delete, the split might no longer be at exact median and the height is no longer guaranteed to be $\lceil \log_2 n \rceil$.

We can maintain $O(\log n)$ height by occasionally re-building entire subtrees. (No details.) But *range-search* will be slower.

kd-trees do not handle insertion/deletion well.

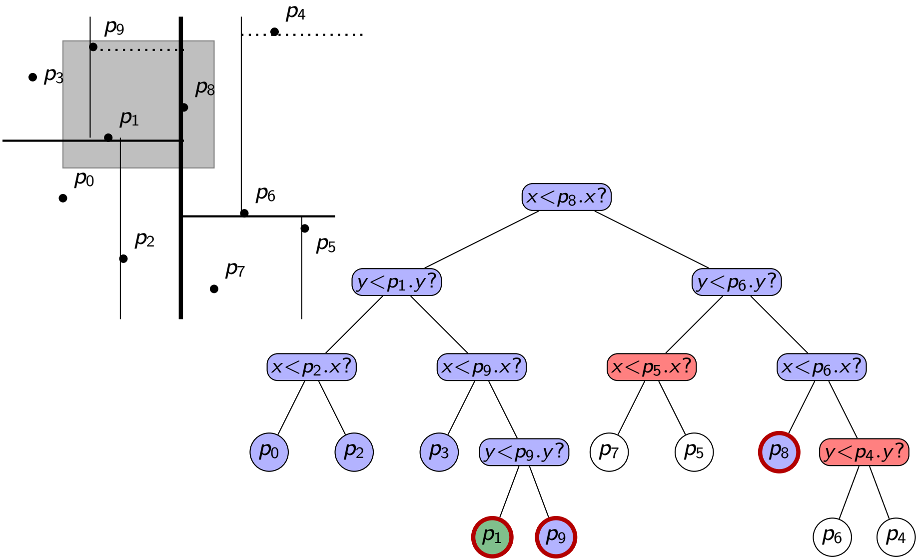
kd-tree Range Search

- Range search is *exactly* as for quad-trees, except that there are only two children and leaves always store points.

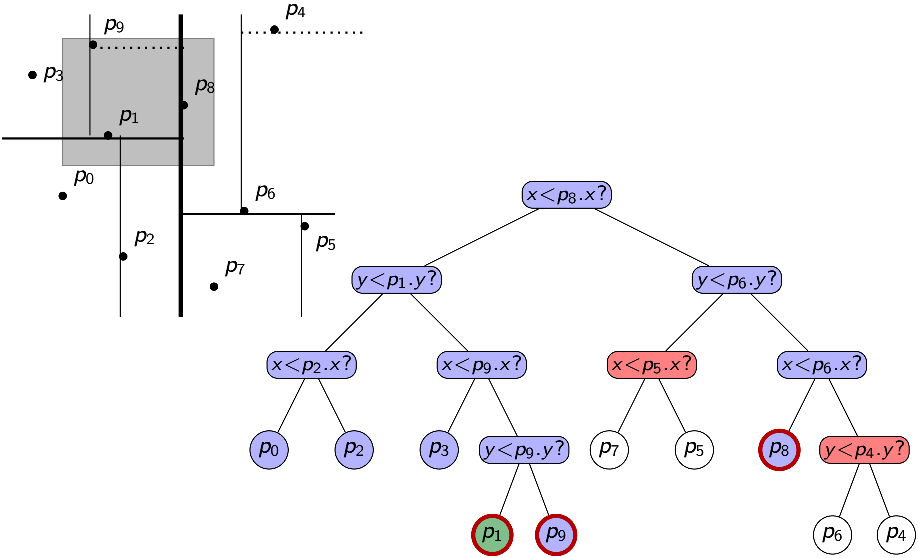
```
kdTree::range-search(r ← root, Q)
r: The root of a kd-tree, Q: Query-rectangle
    R ← region associated with node r
    if (R ⊆ Q) then report all points below r; return
    if (R ∩ Q is empty) then return
    if (r is a leaf) then
        p ← point stored at r
        if p is in Q return p
        else return
    for each child v of r do kdTree::range-search(v, Q)
```

- We assume again that each node stores its associated region.
- To save space, we could instead pass the region as a parameter and compute the region for each child using the splitting line.

kd-tree: Range Search Example



kd-tree: Range Search Example



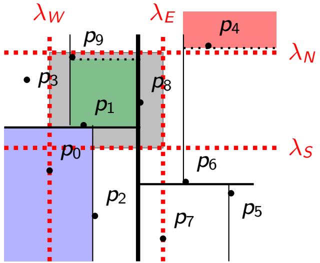
Red: Search stopped due to $R \cap Q = \emptyset$. Green: Search stopped due to $R \subseteq Q$.

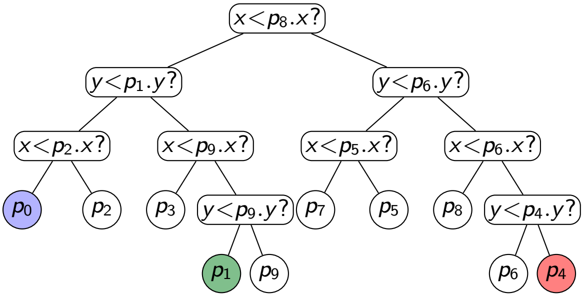
kd-tree: Range Search Complexity

- We spend $O(1)$ time at each visited node, except in line 2.
- All calls to line 2 together take $O(s)$ time (recall: s is the output-size)
- **Observe**: $\#$ visited nodes is $O(\beta(n))$
  where $\beta(n)$ is the number of "boundary" nodes (blue):
  - *kdTree::range-search* was called.
  - Neither $R \subseteq Q$ nor $R \cap Q = \emptyset$
- **We will show:** $\beta(n) \in O(\sqrt{n})$
- Therefore, the complexity of range search in kd-trees is $O(s + \sqrt{n})$

Boundary nodes in kd-trees

Goal: The number of boundary-nodes satisfies $\beta(n) \in O(\sqrt{n})$.

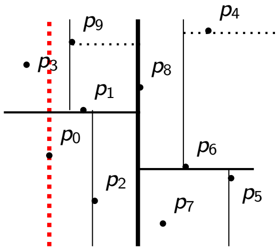




Observation: If z is a boundary-node, then its associated region intersects one of the lines $\lambda_W, \lambda_N, \lambda_E, \lambda_S$ that support the query-rectangle.

Boundary nodes in kd-trees

$$\beta(n,\lambda) := \max_{\text{kd-trees with } n \text{ points}} \left\{ \begin{array}{l} \text{number of associated regions} \\ \text{that intersect a given line } \lambda \end{array} \right\}$$

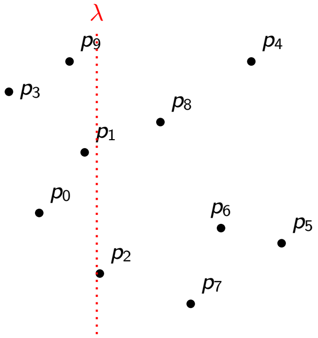


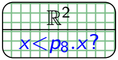
$$\beta_{ver}(n) := \max_{\text{vertical lines } \lambda} \beta_{ver}(n,\lambda) \qquad \beta_{hor}(n) := \max_{\text{horizontal lines } \lambda} \beta_{hor}(n,\lambda)$$

$$\begin{aligned} \beta(n) &\leq \beta(n,\lambda_W) + \beta(n,\lambda_N) + \beta(n,\lambda_E) + \beta(n,\lambda_S) \\ &\leq 2\beta_{ver}(n) + 2\beta_{hor}(n) \end{aligned}$$

Boundary nodes in kd-trees

Goal: Recursive formula for $\beta_{ver}(n)$.





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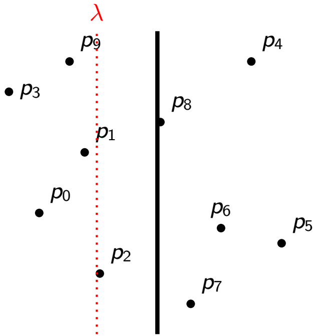

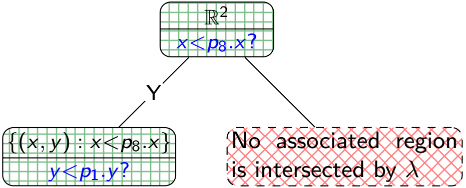

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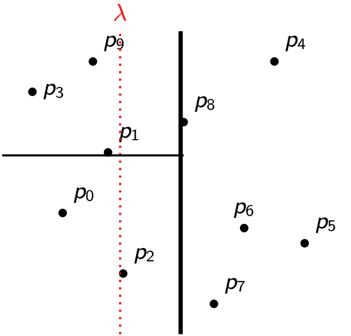

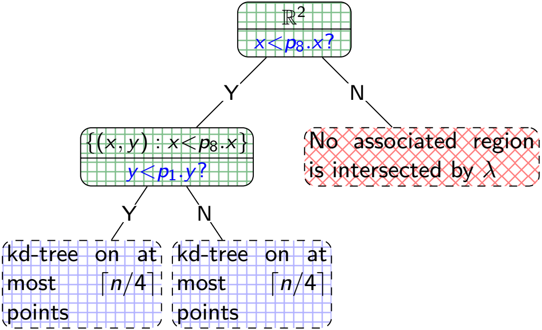

Boundary nodes in kd-trees

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Theorem: In a range-query in a kd-tree (of points in general position) there are $O(\sqrt{n})$ boundary-nodes.

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Theorem: In a range-query in a kd-tree (of points in general position) there are $O(\sqrt{n})$ boundary-nodes.

- So range-search takes $O(\sqrt{n} + s)$ time.
- Note: It is *crucial* that we have $\approx n/4$ points in each grand-child of the root.

kd-tree: Higher Dimensions

- kd-trees for d-dimensional space:
  - At the root the point set is partitioned based on the first coordinate
  - At the subtrees of the root the partition is based on the second coordinate
  - At depth $d-1$ the partition is based on the last coordinate
  - At depth d we start all over again, partitioning on first coordinate
- **Storage**: $O(n)$
- **Height**: $O(\log n)$
- **Construction time**: $O(n \log n)$
- **Range search time**: $O(s + n^{1-1/d})$

This assumes that d is a constant.

Outline

Towards Range Trees

- Both Quadtrees and kd-trees are intuitive and simple.
- But: both may be very slow for range searches.
- Quadtrees are also potentially wasteful in space.

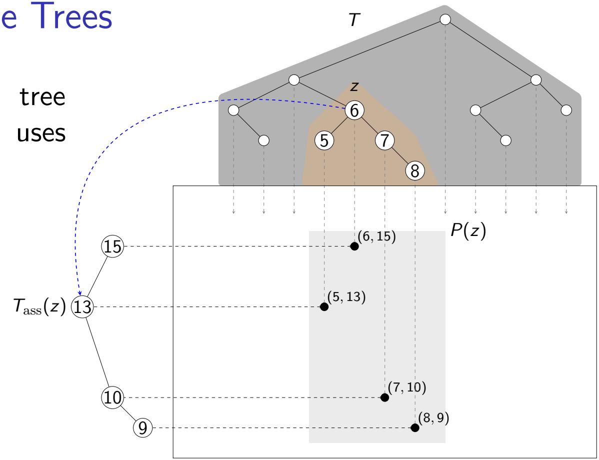
New idea: **Range trees**

- **Tree of trees** (a *multi-level* data structure)
  - So far, nodes in our trees stored a key-value pair and references to children and (maybe) the parent
  - But we can store much more in a node!
  - Here: Each node stores in another binary search tree (!)
- They are wasteful in space, but permit much faster range search.

2-dimensional Range Trees

Primary structure:

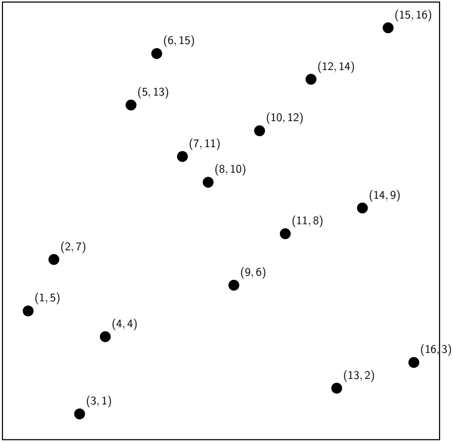
Balanced binary search tree T that stores P and uses *x-coordinates* as keys.



Every node z of T stores an **associate structure** $T_{\mathrm{ass}}(z)$:

- Let $P(z)$ be all points in subtree of z in T (including point at z)
- $T_{\mathrm{ass}}(z)$ stores $P(z)$ in a balanced binary search tree, using the *y-coordinates* as key
- Note: Point of z is not necessarily the root of $T_{\mathrm{ass}}(z)$

Bigger example



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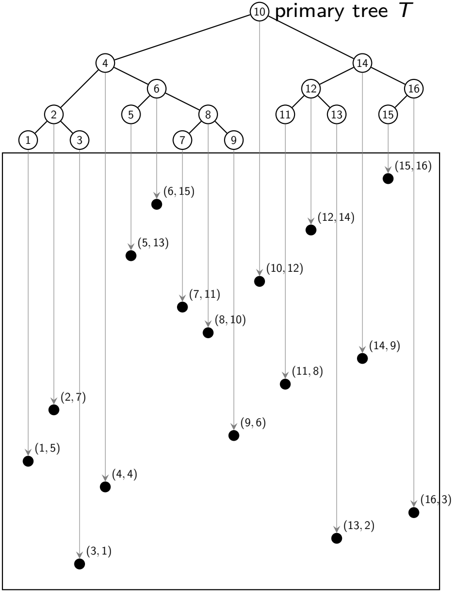

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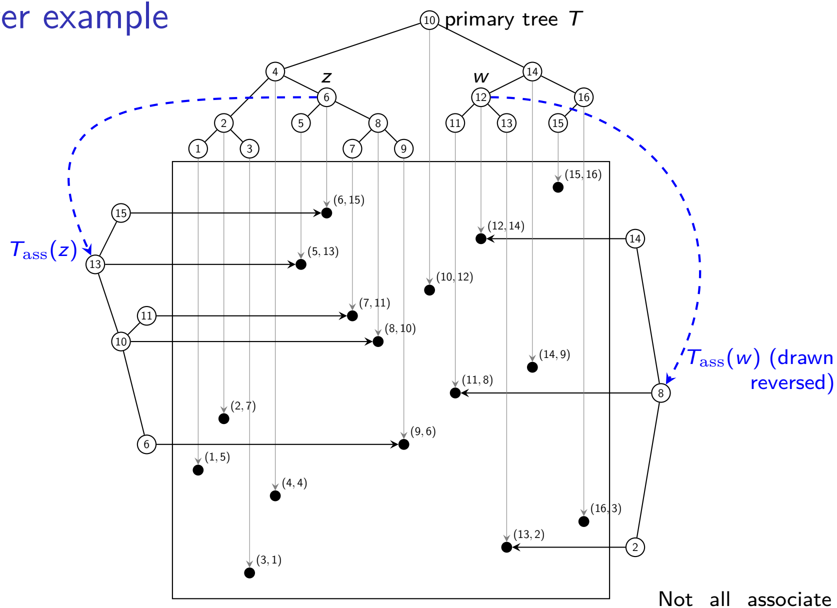
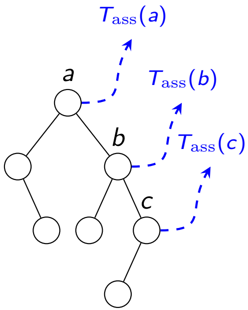


Not all associate trees are shown.

Range Tree Space Analysis

- Primary tree T uses $O(n)$ space.
- How many nodes do all associate trees together have?



  - point of a is only in associate tree $T_{\text{ass}}(a)$
  - point of b is in associate trees $T_{\text{ass}}(a)$, $T_{\text{ass}}(b)$
  - point of c is in associate trees $T_{\text{ass}}(a)$, $T_{\text{ass}}(b)$, $T_{\text{ass}}(c)$
  - **Key insight**: point of z is in associate tree $T_{\text{ass}}(u)$ if and only if u is an ancestor of z in T
  - So every point belongs to $O(\log n)$ associate trees.
  - So all associate trees together use $O(n \log n)$ space.
- A range-tree with n points uses $O(n \log n)$ space.

This is tight for some primary trees.

Range Trees Operations

- *search*: search by x-coordinate in T
- *insert*/*delete*: First, insert/delete point by x-coordinate into T. Then, walk back up to the root and insert/delete the point by y-coordinate in *all* associate trees $T_{\text{ass}}(z)$ of nodes z on path.

  Problem: We want the binary search trees to be balanced.
  - This makes *insert*/*delete* very slow if we use AVL-trees. (A rotation at z changes $P(z)$, so requires a re-build of $T_{\text{ass}}(z)$.)
  - **Solution**: Use Scapegoat trees! (No rotations.)
  - Run-time for *insert*/*delete* becomes $O(\log^2 n)$ amortized.

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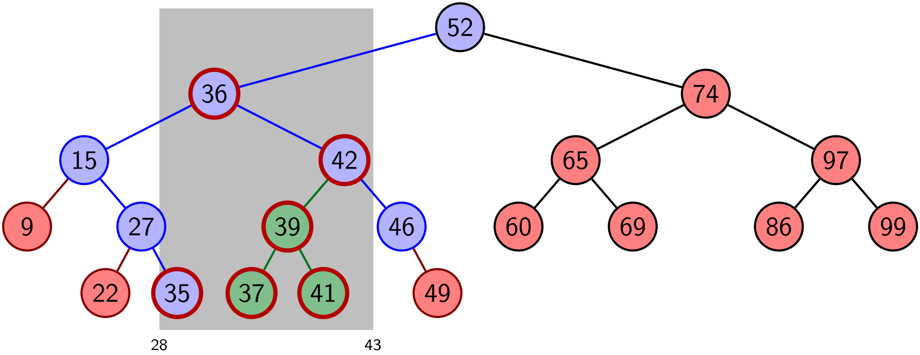
  - **Solution**: Use Scapegoat trees! (No rotations.)
  - Run-time for *insert*/*delete* becomes $O(\log^2 n)$ amortized.

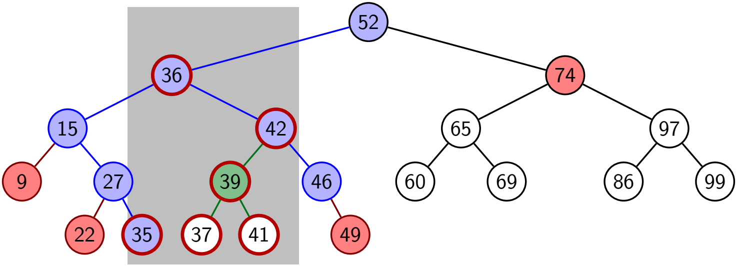
- *range-search*: search by x-range in T. Among found points, search by y-range in some associated trees.
- Must understand first: How to do (1-dimensional) range search in binary search tree?

BST Range Search



- Search for left boundary x_1: this gives path P_1
- Search for right boundary x_2: this gives path P_2
- Three types of nodes: outside, on, or between the paths.
- This classification will be crucial later!

BST Range Search re-phrased

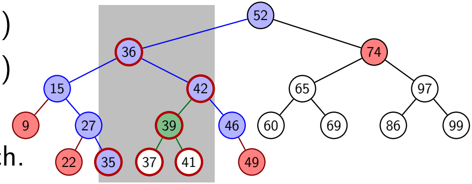


- boundary nodes: nodes in P_1 or P_2
  - For each boundary node, test whether it is in the range.
- outside nodes: nodes that are left of P_1 or right of P_2
  - These are *not* in the range, we do not search in them.
- inside nodes: nodes that are right of P_1 and left of P_2
  - We keep a list of the topmost inside nodes.
  - *All* descendants of such a node are in the range.
    For a 1d range search, report all of them.

BST Range Search analysis

Assume that the binary search tree is balanced:

- Search for path P_1: $O(\log n)$
- Search for path P_2: $O(\log n)$
- $O(\log n)$ boundary nodes
- We spend $O(1)$ time on each.



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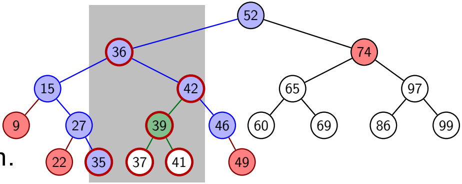


- We spend $O(1)$ time per topmost inside node v.
  - They are children of boundary nodes, so this takes $O(\log n)$ time.
- For 1d range search, also report the descendants of v.
  - We have $\sum_{z \text{ topmost inside}} \#\{\text{descendants of } z\} \leq s$ since subtrees of topmost inside nodes are disjoint. So this takes time $O(s)$ overall.

Run-time for 1d range search: $O(\log n + s)$.

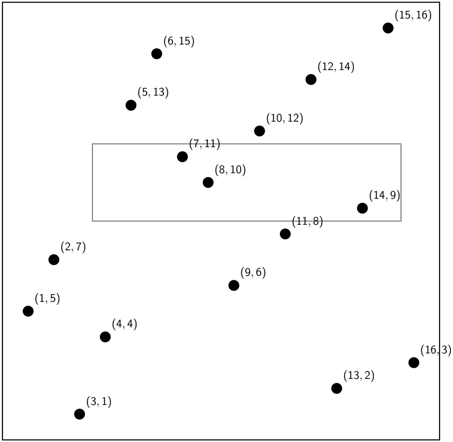
The ability to report the topmost inside nodes will be important for 2d range search.

Range Trees: Range Search

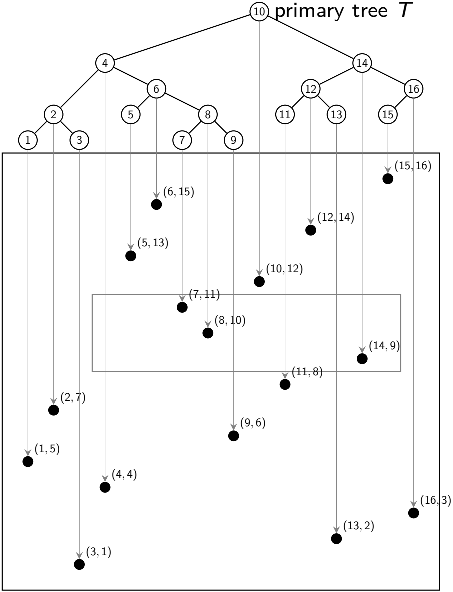
Range search for $Q = [x_1, x_2] \times [y_1, y_2]$ is a two stage process:

- Perform a range search (on the x-coordinates) for the interval $[x_1, x_2]$ in primary tree T (*BST::range-search*(T, x_1, x_2))
- Get boundary and topmost inside nodes as before.
- For every boundary node, test to see if the corresponding point is within the region Q.
- For every topmost inside node v:
  - Let $P(z)$ be the points in the subtree of z in T.
  - We know that all x-coordinates of points in $P(z)$ are within range.
  - Recall: $P(z)$ is stored in $T_{\mathrm{ass}}(z)$.
  - To find points in $P(z)$ where the y-cordinates are within range as well, perform a range search in $T_{\mathrm{ass}}(z)$: *BST::range-search*$(T_{\mathrm{ass}}(z), y_1, y_2)$

Range tree range search example



Range tree range search example



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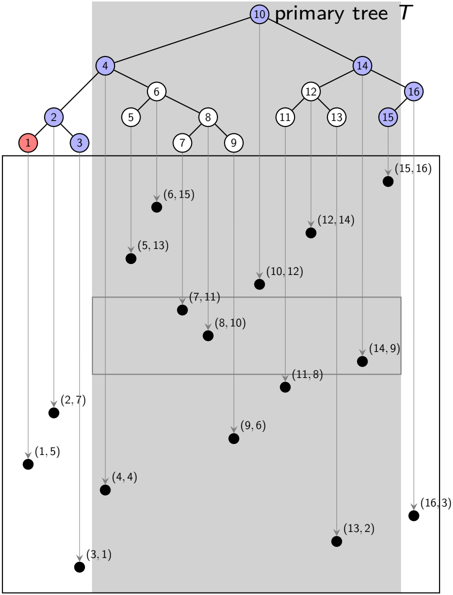

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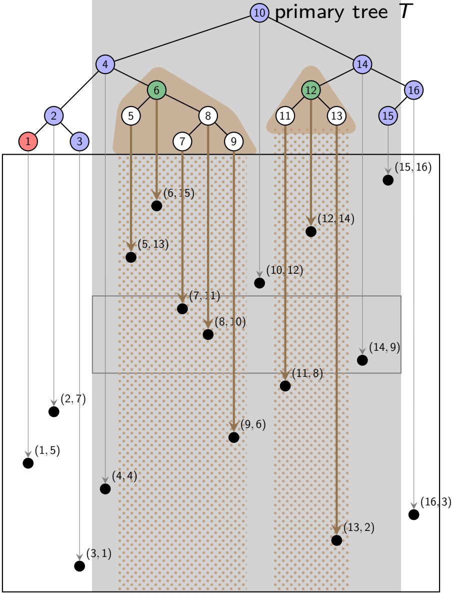

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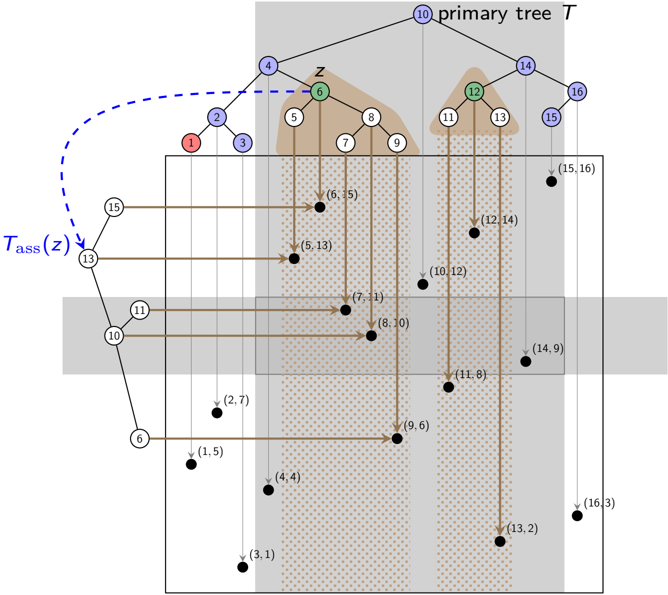

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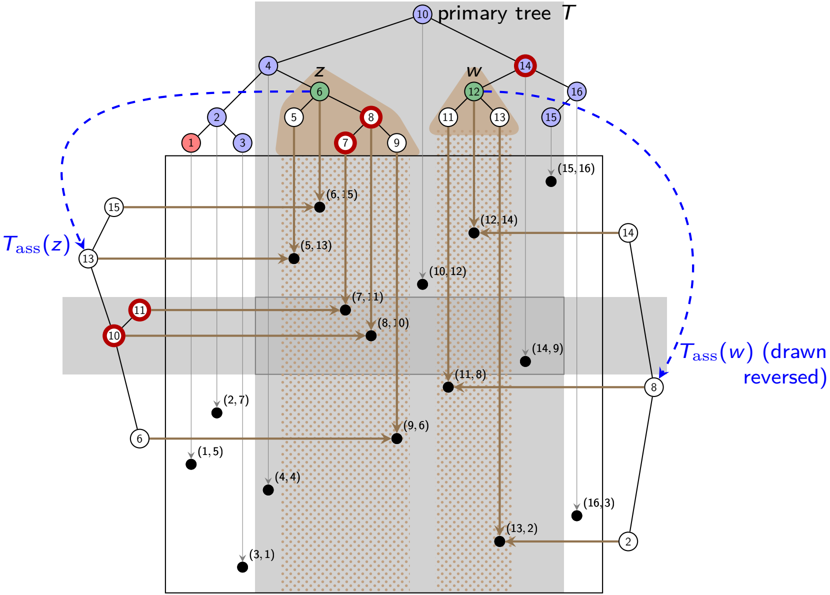


Range Trees: Range Search Run-time

- $O(\log n)$ time to find boundary and topmost inside nodes in primary tree.
- There are $O(\log n)$ such nodes.
- $O(\log n + s_z)$ time for each topmost inside node z, where s_z is the number of points in $T_{\text{ass}}(z)$ that are reported
- Two topmost inside nodes have no common point in their trees
  $\Rightarrow$ every point is reported in at most one associate structure
  $\Rightarrow \sum_{z \text{ topmost inside}} s_z \leq s$

Time for range search in range-tree is proportional to

$$\sum_{z \text{ topmost inside}} (\log n + s_z) \in O(\log^2 n + s)$$

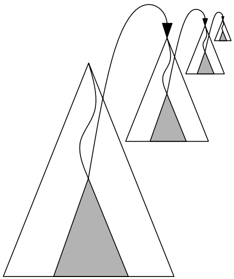
(There are ways to make this even faster. No details.)

Range Trees: Higher Dimensions

- Range trees can be generalized to d-dimensional space.

Space	$O(n\,(\log n)^{d-1})$
Construction time	$O(n\,(\log n)^{d})$
Range search time	$O(s + (\log n)^{d})$

(Note: d is considered to be a constant.)



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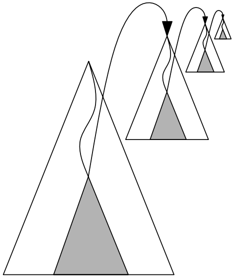
- Range trees can be generalized to d-dimensional space.

Space	$O(n\,(\log n)^{d-1})$	kd-trees: $O(n)$
Construction time	$O(n\,(\log n)^{d})$	kd-trees: $O(n \log n)$
Range search time	$O(s + (\log n)^{d})$	kd-trees: $O(s + n^{1-1/d})$

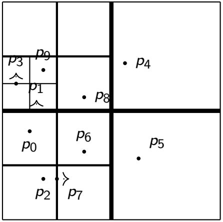
(Note: d is considered to be a constant.)

- Space/time trade-off compared to kd-trees.

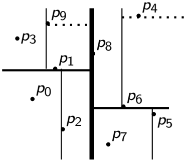


Range search data structures summary

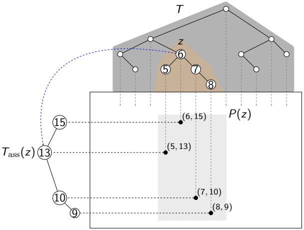
- Quadtrees
  - simple (also for dynamic set of points)
  - work well only if points evenly distributed
  - wastes space for higher dimensions



- kd-trees
  - linear space
  - range search time $O(\sqrt{n} + s)$
  - inserts/deletes destroy balance and range search time (no simple fix)



- range-trees
  - range search time $O(\log^2 n + s)$
  - wastes some space
  - inserts/deletes destroy balance (can fix this with occasional rebuild)



Convention: Points on split lines belong to right/top side.

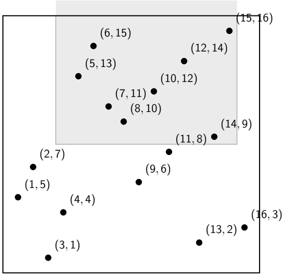
Outline

8 Range-Searching in Dictionaries for Points
- Range Searches
- Multi-Dimensional Data
- Quadtrees
- kd-Trees
- Range Trees
- 3-sided range search

3-sided range search

Consider a special kind of range-search:

3sidedRangeSearch(x_1, x_2, y'): *return* (x, y) *with* $x_1 \leq x \leq x_2$ *and* $y \geq y'$.



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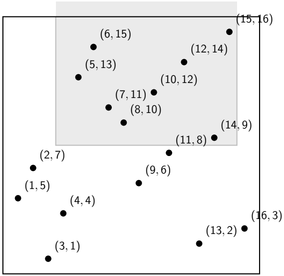
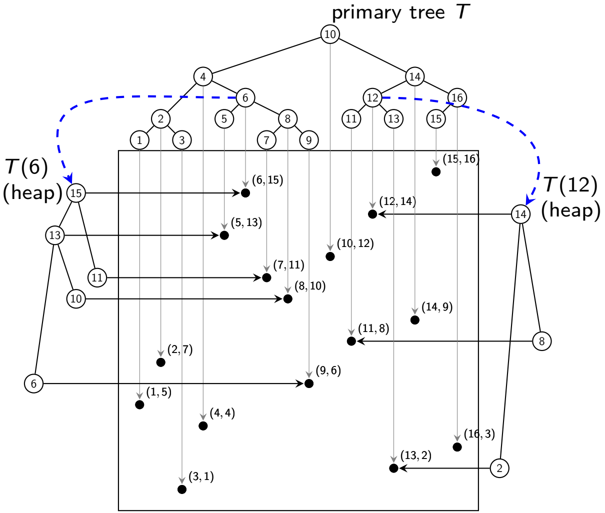


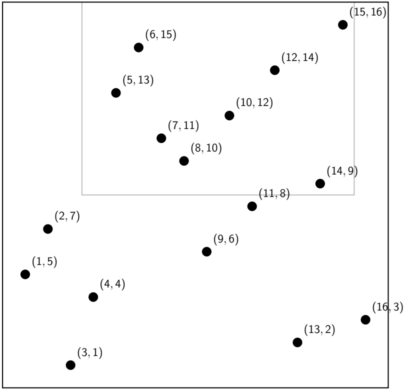
- We can do this with a range tree in $O(\log^2 n + s)$ with $\Theta(n \log n)$ space.
- Can we do this faster or using less space by adapting previous ideas to the special situation?

Idea 1: Associated heaps



- Primary tree: balanced binary search tree.
- Associated tree: binary heap.
- Space: $\Theta(n \log n)$.
- Range-search time?

Idea 1: Associated heaps - 3-sided range search



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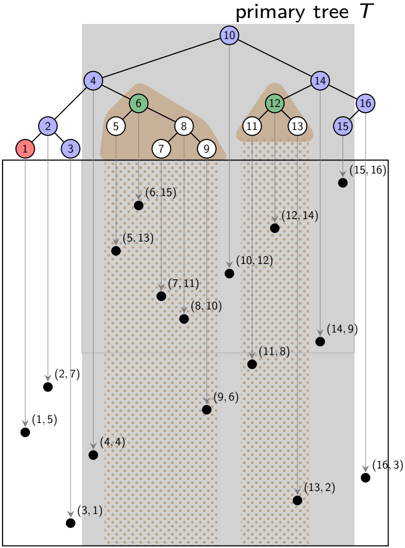

- Search in primary as before.

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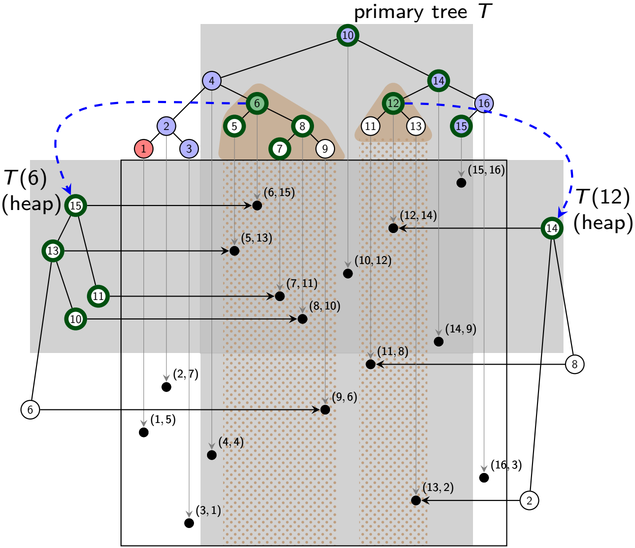

- Search in primary as before.
- In associated heap: Search by y-coordinate in $O(1 + s)$ time. (Exercise.)

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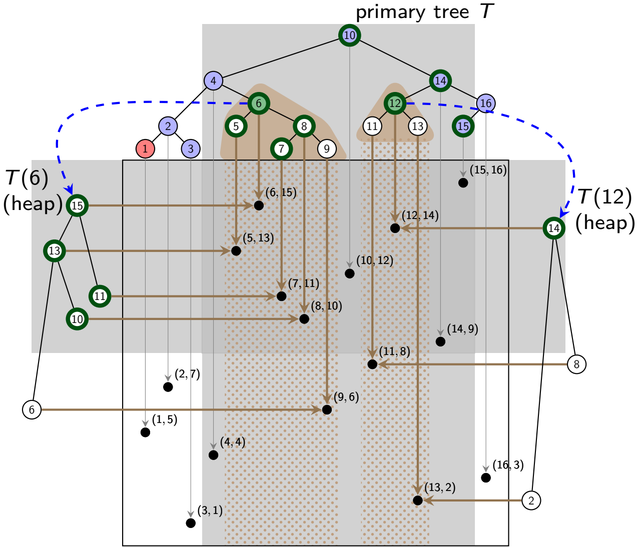
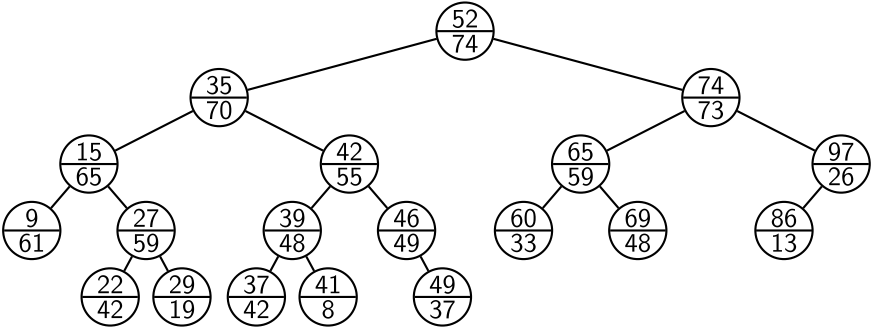


- Search in primary as before.
- In associated heap: Search by y-coordinate in $O(1 + s)$ time. (Exercise.)
- Total time: $O(\log n + s)$
- But space is $\omega(n)$

Idea 2: Cartesian Trees

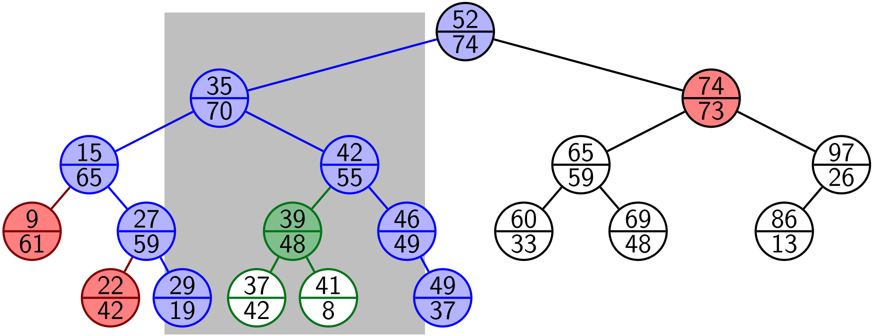
Recall: Treap = binary search tree (with respect to keys)
+ heap (with respect to priorities)



Cartesian tree: Use x-coordinate as key, y-coordinate as priority.
Space: $\Theta(n)$.

Idea 2: Cartesian Tree - 3-sided range search

CartesianTree::3-sided-range-search$(T, 28, 47, 36)$:



- BST::range-search(x_1, x_2) to get boundary and topmost inside nodes.

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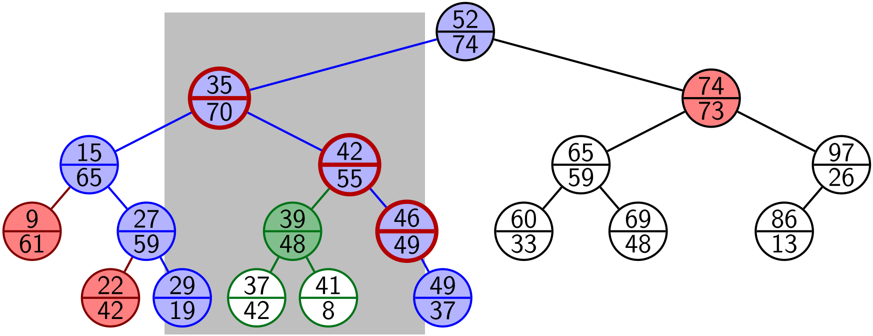

- BST::range-search(x_1, x_2) to get boundary and topmost inside nodes.
- Boundary-nodes: Explicitly test whether in x-range and y-range.

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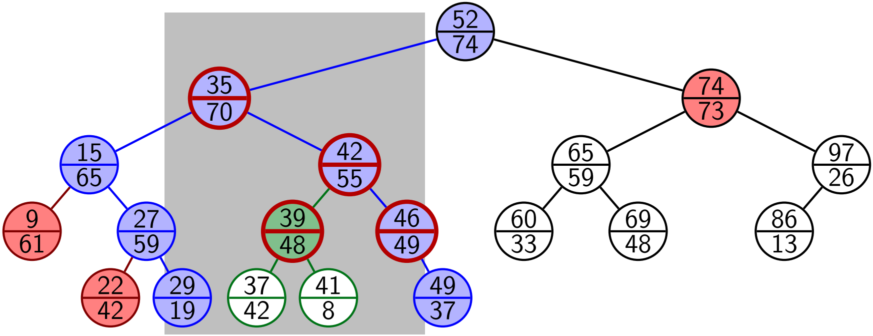
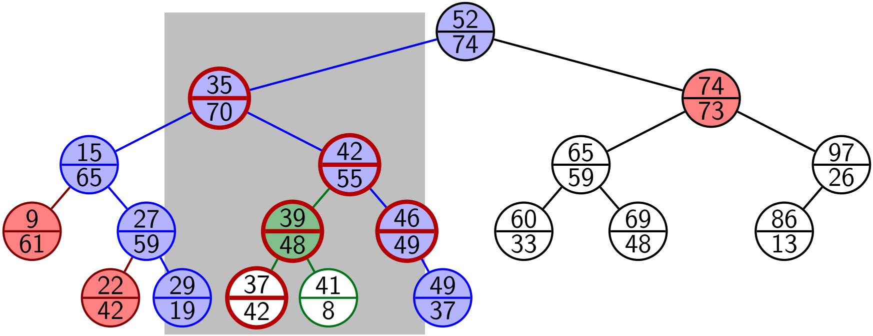


- BST::range-search(x_1, x_2) to get boundary and topmost inside nodes.
- Boundary-nodes: Explicitly test whether in x-range and y-range.
- Topmost inside-nodes: If $y \geq y_1$, report and recurse in children.

Idea 2: Cartesian Tree - 3-sided range search

CartesianTree::3-sided-range-search$(T, 28, 47, 36)$:



- BST::range-search(x_1, x_2) to get boundary and topmost inside nodes.
- Boundary-nodes: Explicitly test whether in x-range and y-range.
- Topmost inside-nodes: If $y \geq y_1$, report and recurse in children.

Idea 2: Cartesian Tree - 3-sided range search

Run-time for 3-sided range search:

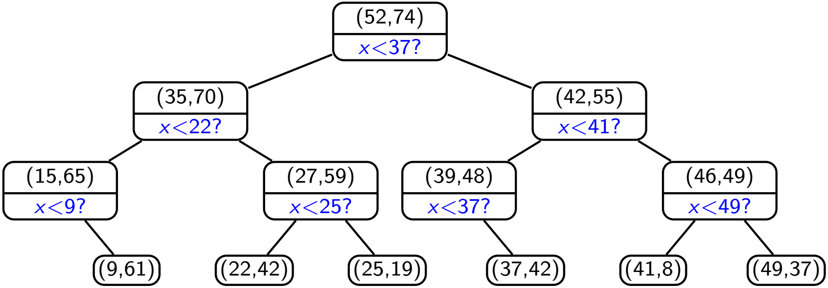
- BST::range-search(x_1, x_2) — $O(\textit{height})$ since we do not report points.
- Testing boundary-nodes: $O(\textit{height})$
- Testing heap: $O(1 + s_z)$ per topmost inside-node z

$\Rightarrow O(\textit{height} + s)$ run-time, $O(n)$ space

But: No guarantees on the height (not even in expectation) since we cannot choose priorities.

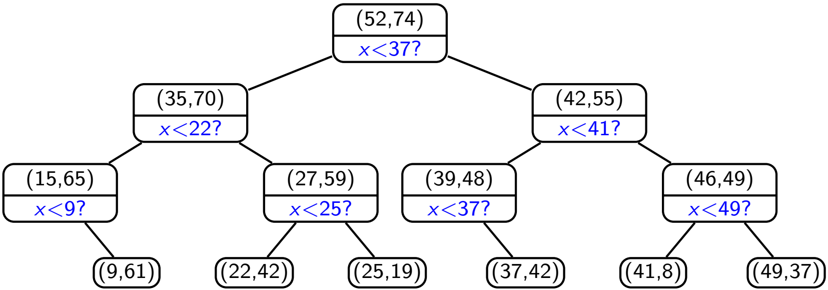
Idea 3: Priority search trees

- Design a new data structure
- Keep good aspects of Cartesian trees (store y-coordinates in heap-order)
- Keep good aspects of kd-tree (split in half by x-coordinate)



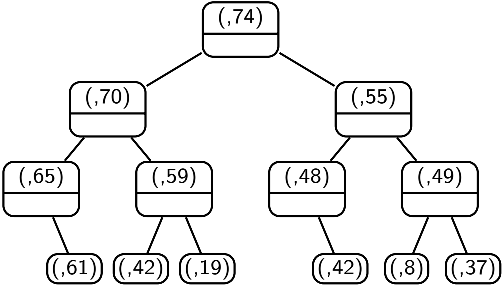
Key idea: The x-coordinate stored for splitting can be *different* from the x-coordinate of the stored point.

Idea 3: Priority search trees



- Every node z stores a point $p_z = (x_z, y_z)$,
  - y_z is the maximum y-coordinate in subtree
- Every non-leaf z stores an x-coordinate x'_z (split-line)
  - Every point p in left subtree has $p.x < x'_z$
  - Every point p in right subtree has $p.x \geq x'_z$
- x'_z is chosen so that tree is balanced $\Rightarrow$ height $O(\log n)$.

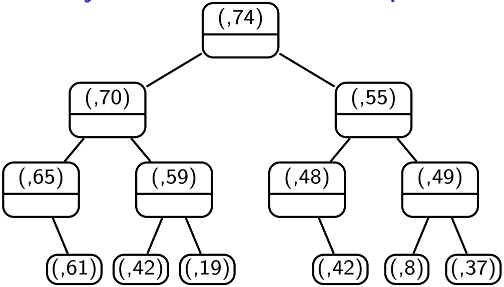
Priority search tree closeup



Looking only at *y*-coordinates:

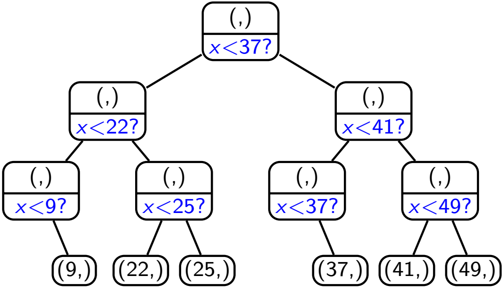
- Heap-order property
- But not heap-structure

Priority search tree closeup



Looking only at y-coordinates:

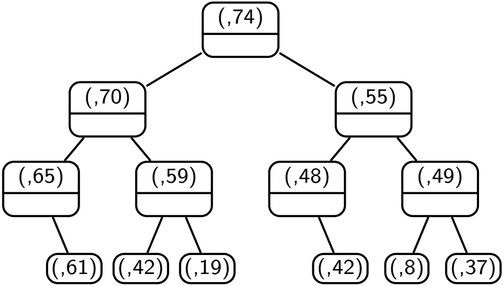
- Heap-order property
- But not heap-structure



Looking only at x-coordinates:

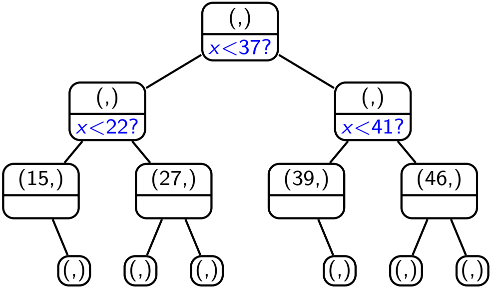
- Points at leaves ≈ kd-tree (1-dimensional)

Priority search tree closeup



Looking only at y-coordinates:

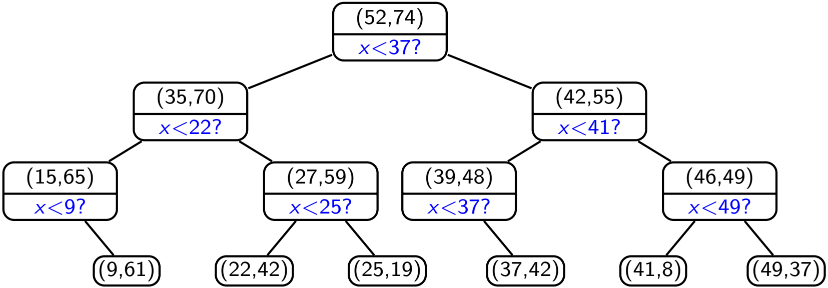
- Heap-order property
- But not heap-structure



Looking only at x-coordinates:

- Points at leaves $\approx$ kd-tree (1-dimensional)
- Points at level $\ell \approx$ kd-tree if we ignore points below.

Idea 3: Priority search trees



- Construction: $O(n \log n)$ time (exercise)
- *search*: $O(\log n)$ time
  - Get search-path by following split-lines, check all nodes on path
- *insert*, *delete*: Re-balancing is difficult, but can be done (no details).
- 3-sided range search: As for Cartesian trees, but height now $O(\log n)$.

  - Run-time $O(\log n + s)$

3-sided range search summary

- Idea 1: Scapegoat tree + associated heaps
  $O(\log n + s)$ time for range search, but $\omega(n)$ space.
- Idea 2: Cartesian Tree
  $O(n)$ space, but range search takes $O(\mathit{height} + s)$, could be slow
- Idea 3: Priority search tree
  $O(n)$ space, $O(\log n + s)$ time for range search.

Sometimes it pays to design purpose-built data structures.