Towards High-Level Languages: Formal Language Theory

High-Level Languages

- Programs are represented using low-level **machine language**, which is hard for humans to read and write.
- Assembly language is more convenient, but still low-level, and can be difficult to write and understand.
- Humans sought to develop higher-level languages that make it easier to express what we want our programs to do.
- Common goals of early high level languages were:
  - To be able to write mathematical formulas more naturally. (FORTRAN)
  - To be able to write programs using natural languages like English. (COBOL)

FORTRAN Example

The FORTRAN Automatic Coding System for the IBM 704 EDPM: Programmer's Reference Manual (1956)

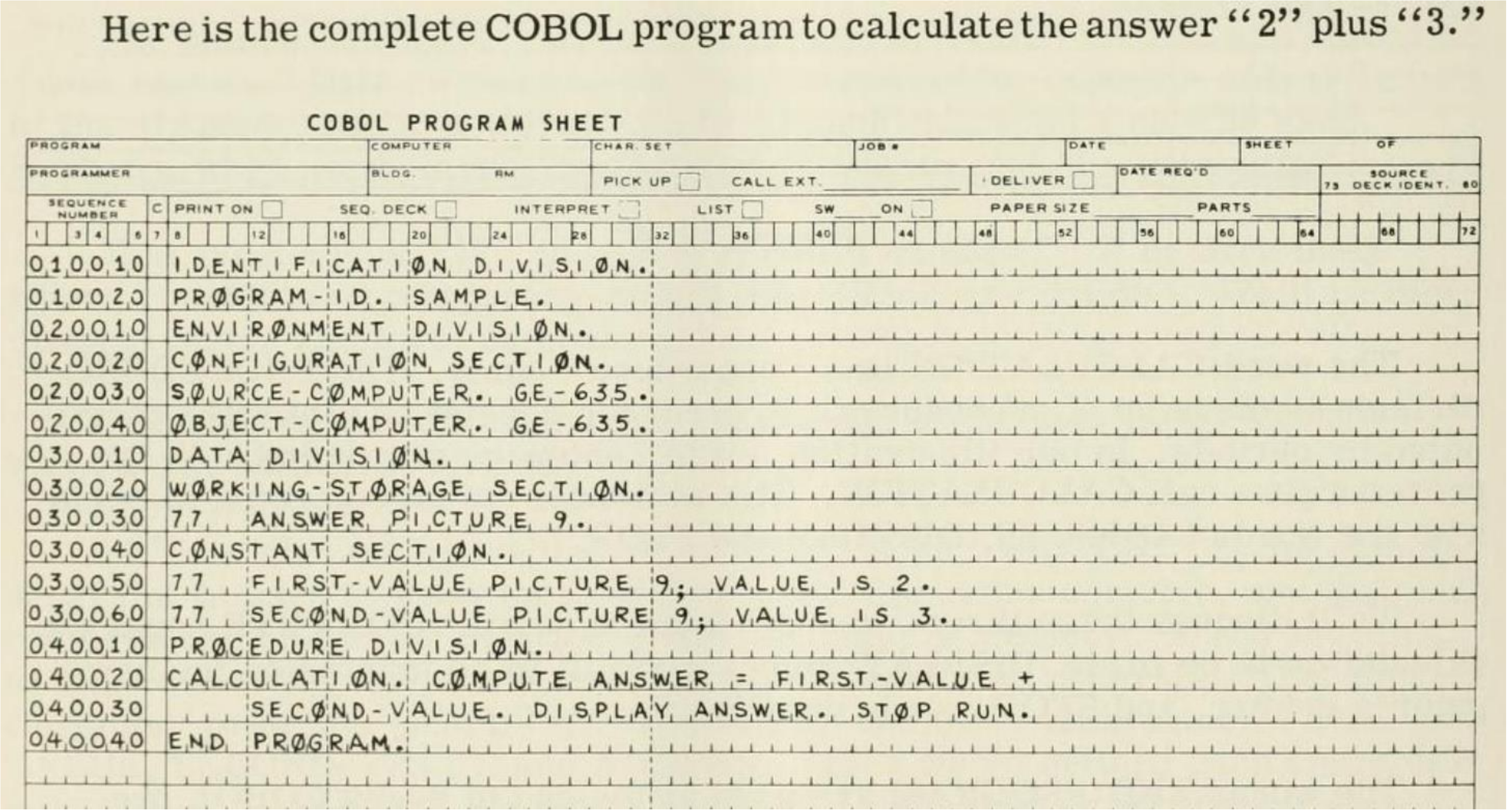
A DO Nest with Exit and Return

Given an N x N square matrix A, to find those off-diagonal elements which are symmetric and to write them on binary tape.

C ← FOR COMMENT STATEMENT NUMBER 1 5	CONTINUATION 6	FORTRAN STATEMENT 7 72	IDENTI-FICATION 73 80
		REWIND 3	
		DO 3 I = 1,N	
		DO 3 J = 1,N	
		IF(A(I,J)-A(J,I)) 3,20,3	
3		CONTINUE	
		END FILE 3	
		MORE PROGRAM	
20		IF(I-J) 21,3,21	
21		WRITE TAPE 3,I,J, A(I,J)	
		GO TO 3	

COBOL Example

Cobol Simplified (1968) by Mario V. Farina



Here is the complete COBOL program to calculate the answer "2" plus "3."

COBOL PROGRAM SHEET

```
010010 IDENTIFICATIØN DIVISIØN.
010020 PRØGRAM-ID. SAMPLE.
020010 ENVIRØNMENT DIVISIØN.
020020 CØNFIGURATIØN SECTIØN.
020030 SØURCE-CØMPUTER. GE-635.
020040 ØBJECT-CØMPUTER. GE-635.
030010 DATA DIVISIØN.
030020 WØRKING-STØRAGE SECTIØN.
030030 77  ANSWER PICTURE 9.
030040 CØNSTANT SECTIØN.
030050 77  FIRST-VALUE PICTURE 9; VALUE IS 2.
030060 77  SECØND-VALUE PICTURE 9; VALUE IS 3.
040010 PRØCEDURE DIVISIØN.
040020 CALCULATIØN. CØMPUTE ANSWER = FIRST-VALUE +
040030     SECØND-VALUE. DISPLAY ANSWER. STØP RUN.
040040 END PRØGRAM.
```

The Grammar of MIPS Assembly

- The "grammar" of MIPS assembly is simple to process line-by-line.
  - Each line starts with zero or more label definitions.
    → Read until a non-label token is found.
  - Then there is optionally an instruction.
    → Match against one of six possible syntax patterns for instructions.
  - Then there is optionally a comment.
    → Check for a semicolon, if found, skip until the end of the line.
  - The only difficult part is matching label uses with their definitions.
    → Solved by doing two passes and building a symbol table.
- Math and natural language are both significantly more complex in structure than assembly, and thus, so are high-level languages.

The Grammar of High-Level Languages

```
if (say_hello) {
     printf("Hello world!\n");
} else {
  if (a+b*(c+d) == 0) {
     printf("Yay!\n");
  } else {
     printf("I'm sad!!! The number was %d\n", a+b*(c+d));
  }
}
```

Need to match braces and parentheses (possibly across different lines!), deal with nesting, order of arithmetic operations, distinguish between "if(condition)" and "procedure(arg1, arg2, …)", and more!

Formal Language Theory

- It is a mathematical approach to describing and studying languages.
- A lot of early development was by linguists who were looking to formalize the structure of *natural languages*.
- Computer scientists found the same ideas useful for formalizing *programming languages*.
- From the early days, there was an interest in finding connections between *models of grammar* and *models of computation*.
  - A formal grammar specifies rules that can be applied to "generate" sentences.
  - The idea was to find models of computation that have the same "generative power" as a certain kind of formal grammar.

"Thus, remarkably, the same important ideas emerged independently for the automatic translation of both natural and artificial languages:

- Separating syntax and semantics.
- Using a generative grammar to specify the set of all and only legal sentences (programs).
- Analyzing the syntax of the sentence (program) and then using the analysis to drive the translation (compilation)."

Formal Languages: Origins and Directions (S. A. Greibach, 1981)

Basic Definitions: Alphabets

- An **alphabet** is a finite set. Its elements are called **symbols**.
- $\Sigma = \{a, b, c\}$ is an alphabet containing 3 symbols.
  - Up until now, the symbols in our strings have been single characters…
- $\Sigma = \{cat, dog, mouse, iguana\}$ is an alphabet containing *4 symbols.*
  - Individual letters like "c" and "a" are not symbols in this alphabet!
- $\Sigma = \{ (x, y) : 0 \leq x, y \leq 9, x \in \mathbb{Z} \}$ is an alphabet containing 100 symbols. Each symbol is an ordered pair of integers with values from 0 to 9.
- An alphabet typically cannot be infinite. For example, the set of all integers is not an alphabet (in this course).

Basic Definitions: Strings (Words)

- A *string* over an alphabet Σ is a sequence of symbols from Σ.
  - Strings are frequently called "words" in formal language theory, but we already use this term for machine-architecture words.
- The length of a string x is denoted $|x|$ and is the total number of symbols in the string (including repeats).
- Examples:
  - *x = cabba* is a string over Σ = {a,b,c}. We have $|x| = 5$.
  - *x = cat dog cat dog iguana mouse* is a string over Σ = {cat, dog, iguana, mouse}. We have $|x| = 6$.
- A sequence of *zero* symbols is allowed. This is called the *empty string* and it is denoted by the Greek letter ε (epsilon). We have $|\varepsilon| = 0$.

Aside about String Notation

- There is no single universal notation for "sequences" in mathematics because the cleanest notation varies wildly depending on context.
- For strings, we often just write them out with no spacing.
  - *cat* is a string over the alphabet {a, c, t}. It has length 3.
  - *11110001* is a string over the alphabet {0,1}. It has length 8.
- But if the symbols consist of multiple characters, we might put spaces between each symbol of a string for readability.
  - *ID REG COMMA REG* is a string over {ID, REG, COMMA}. It has length 4.
- Pay attention to what the alphabet is.
  - *jr $31* is a string over the ASCII alphabet. It has length 6.

Basic Definitions: Languages

- A *language* over an alphabet Σ is a set of strings over Σ.
- Equivalently, a language over Σ is a subset of Σ*.
  - The Kleene star of an alphabet is the set of all strings over the alphabet!
- Strings in a formal language don't necessarily have to be "meaningful" the way strings in a natural language are.
  - "The set of grammatically correct English sentences" is a language, assuming you can agree on what "grammatically correct" means.
  - The set {sfsfdsdf, fghghfgh, eivlyS} is also a language.
- The "interesting" classes of formal languages are restricted classes that have more structure than just "a set".

Extending Regular Languages with Recursion

- What if we could use "recursion" in regular expressions?
  - For example: Let L be described by the "recursive regular expression" aLb.
- We can think of this as an equation L = aLb. Is there a language L that makes this equation true?
- Yes, the language $\{ a^n b^n : n \geq 0 \}$ (where n is an integer).
  - The notation a^n means aa…a where there are n occurrences of a.
  - So this language contains words with n a's followed by n b's.
- Can we recognize this language with a DFA? Or describe it with a (non-recursive) regular expression?

The Need for Non-Regular Languages

- It is *impossible* to construct a DFA for $\{ a^n b^n : n \geq 0 \}$.
- Suppose you had such a DFA, and it had *m* states total.
- Run the strings $a, a^2, a^3, \ldots a^{m+1}$ through the DFA and write down the states you get. Call them $q_1, q_2, \ldots q_{m+1}$.
- There are only m states, so two of these are equal. Say $q_i = q_j$ but $i \neq j$.
- Since $a^i b^i$ is accepted, following the sequence of transitions on b^i from q_i must lead to an accepting state. So this must be true for q_j too.
- But $a^j b^i$ is not accepted since $i \neq j$! This is a contradiction, so there is no such DFA!

The Need for Non-Regular Languages

- Does it matter that we can't handle this weird $\{ a^n b^n : n \geq 0 \}$ language? What about something more practical?
- Practical languages are *harder*. Consider the language of *sequences of balanced parentheses* (left brackets matching with right brackets).
- We can make the same argument we just did to show there is no DFA for this language using strings like (((((()))))).
- But this language also contains more complex strings like (()())()((())).
- Both $\{ a^n b^n : n \geq 0 \}$ and the language of sequences of balanced parentheses are examples of **context-free languages**.

Context-Free Languages and Grammars

- Context-free languages are conceptually like "regular languages with recursion", but they are usually defined in terms of *formal grammars*.
- A **formal grammar** has four elements:
  - An alphabet called the *terminal alphabet*, which is the alphabet of the language the grammar is describing.
  - A disjoint alphabet called the *nonterminal alphabet*, which can be thought of as a set of "meta-symbols" that appear in the grammar but not the language.
  - A *start symbol* which is one of the nonterminals (meta-symbols).
  - A set of *production rules*, which are "string rewriting rules", i.e., they tell you that it is valid to replace certain strings of symbols with certain other strings.
- Idea: A string is in the language if you can start with the start symbol and repeatedly apply production rules to eventually obtain the string.

Production Rules

- A production rule, in its most general form, looks like x → y where x is a non-empty string and y is a string. This says x can be rewritten as y.
- In an "unrestricted" formal grammar, all productions are permitted. The only restriction is that the left hand side is not an empty string.
- In a **context-free grammar**, the left hand side of each production rule must be a *single nonterminal symbol*.
  - We can only rewrite "meta-symbols", not actual symbols from the language.
  - "Terminal" refers to the fact that terminal symbols can't be rewritten.
  - Also, we can only rewrite one of these symbols at a time. No surrounding **context** is allowed in context-free production rules.

Example: $\{ a^n b^n : n \geq 0 \}$

- Terminal symbols: {a, b}
- Nonterminal symbols: {S} (only one is needed)
- Start symbol: S
- Production Rules:
    1. S → aSb
    2. S → ε
- As an example, we can produce the string aaabbb by applying rule 1 three times, then rule 2.
- S ⇒ aSb ⇒ aaSbb ⇒ aaaSbbb ⇒ aaabbb

Example: Balanced Sequences of Parentheses

- That is, strings over the alphabet { (,) } where every left parenthesis "(" has a matching right parenthesis ")".
- These rules are not sufficient:
  1. S → (S)
  2. S → ε
- This doesn't include things like ()().
- Adding this third rule is enough (but it's not too easy to prove):
  3. S → SS
- These two rules also work: S → (S)S and S → ε.

Example: a(a|b)*a(a|b)*

- This is a simple example of a language that *requires* two distinct nonterminal symbols to describe.
  - Source: "Nonterminal Complexity of Some Operations on Context-Free Languages", Dassow & Stiebe, 2008.
- Terminal symbols: {a, b}
- Nonterminal symbols: {S, B}
- Start symbol: S
- Production rules: $S \rightarrow aBaB$, $B \rightarrow aB$, $B \rightarrow bB$, $B \rightarrow \varepsilon$

Notation & Conventions

- Formally, a context-free grammar is a 4-tuple (N, Σ, S, P), where N is the nonterminal alphabet, Σ is the terminal alphabet, S is the start symbol, and P is the set of production rules.
- In practice, we rarely write out all these things separately and instead we use the following convention:
  - A grammar is simply written as a list of production rules.
  - The symbol on the left-hand-side of the *first* rule in the list is assumed to be the start symbol.
  - If a symbol never appears on the left-hand-side of a rule, it's a terminal symbol. All other symbols are nonterminal.

Notation & Conventions

- To shorten the description of grammars, we use notation that *looks similar* to regular expressions but *means something different*.
- We will often use the | symbol to combine multiple production rules with the same left hand side.
- Example: Instead of S → aBaB, B → aB, B → bB, B → ε
  We could write S → aBaB, B → aB | bB | ε
- It is **not valid** to put a regular expression on the right hand side of a production rule. You cannot use star, brackets for grouping, etc.
- This notation is just shorthand for "there are multiple production rules that all have the same right hand side".

Notation & Conventions

- To make things easier to read at a glance, we also have conventions for which kinds of letters correspond to which kinds of objects.
- These aren't strict rules and may be broken sometimes.
- Lowercase letters from the start of the English alphabet (a, b, c, …) are usually *terminals*, elements of Σ.
- Lowercase letters from the end of the English alphabet (…, w, x, y, z) are usually *strings of terminals*, elements of Σ*.
- Uppercase English letters (A, B, C, …, X, Y, Z) are usually *nonterminals*, elements of N.
- Greek letters (α, β, γ) are usually *strings that may mix terminals and nonterminals,* that is, elements of (N ∪ Σ)*.

Notation & Conventions: Examples

- "A context-free production rule has the general form A → α."
  - It is implicit that A is a nonterminal, and α is a sequence of symbols that could possibly mix terminals and nonterminals.
- Consider this grammar:

  X → aXa | bXb | Y

  Y → a | b | ε
- It is implicit that the start symbol is X.
- Since a and b don't appear on the left hand side of any rule, they are terminals. X and Y are nonterminals.
- There are **six** production rules in this grammar, not two.

Derivations

- A **derivation** is a sequence of production rules that can be applied to the start symbol of a grammar to produce a string.
- We write derivations by starting with the start symbol, and showing how string evolves with each rule application.
- Example: Find a derivation of (()()) in $S \rightarrow (S)S \mid \varepsilon$.

$$S \Rightarrow (S)S \Rightarrow ((S)S)S \Rightarrow (()S)S \Rightarrow (()(S)S)S \Rightarrow (()()S)S \Rightarrow (()())S \Rightarrow (()())$$

- In the above example, we always rewrote the *leftmost* nonterminal symbol. Here is another valid derivation:

$$S \Rightarrow (S)S \Rightarrow (S) \Rightarrow ((S)S) \Rightarrow ((S)(S)S) \Rightarrow ((S)(S)) \Rightarrow (()(S)) \Rightarrow (()())$$

The Language of a Grammar

- A derivation in a context-free grammar ends once all nonterminals are replaced with terminals.
  - Terminals cannot be replaced or changed, since only nonterminals can appear on the left hand side of a production rule.
- The **language generated by a context-free grammar**, or just the **language of the grammar** for short, is the set of all strings of *terminals* that can be produced as the final result of a derivation.
- A formal language is a **context-free language** if it is the language of some context-free grammar.