Outline

1. Inductive Definitions of Sets
2. Structural Induction
3. Introduction to the Syntax of Propositional Logic
4. Unique Readability of Propositional Formulas
5. Parse Trees
6. Precedence Rules for Propositional Connectives
   1. Generation Sequences for Propositional Formulas
7. Structural Induction - More Examples
Q & A

1. Should we treat an English “or” as inclusive-or (\(\lor\)) in logic, by default?
   
   **A:** Yes. This convention is established in the course slides.

2. When translating from English into propositional logic, should we choose proposition symbols to represent non-negated statements?
   
   **A:** Yes. This rule is established in the course slides. E.g. to translate “I do not like ice cream”, we should define \(p\) to stand for “I like ice cream”, then translate the provided sentence as (\(\neg p\)).
In CS 245, **structural induction** will be a theme of the course.

In this lecture we give the setup which will permit us to carry out structural induction correctly in every situation.

The first example of an inductively defined set, for which we can employ structural induction, is the set of **propositional formulas**, $\text{Form}(\mathcal{L}_P)$.

**Remark:** You may find this approach is too abstract for your taste, especially so early in the course. While I understand this reaction, I assure you that if we invest the time to understand the general setup, then we will reap the rewards throughout the rest of the course.
What techniques do we have for declaring sets?

1. The empty set, \( \emptyset \), is a set.

2. via some property of interest, e.g.
   
   - **Even Natural Numbers** \( \{ n \in \mathbb{N} \mid n \text{ is even} \} \), or
   \( \{ n \in \mathbb{N} \mid 2 \mid n \} \).

3. The **power set of a set** \( S \), denoted \( P(S) \) (i.e. the set of all subsets of \( S \)) is a set.

4. Explicitly list all elements of the set. For example, \( \{3, 6, 7\} \). Drawback: we cannot do this for infinite sets.

5. Inductively, for example, the set of all my blood relatives. Core set = \( \{ \text{me} \} \) Operations = \{ daughter of, son of, brother of, sister of, mother of, father of \}
Inductive Definition of a Set: 3 ingredients:

1. **Universe** of all elements denoted by $X$ (e.g. $X = \mathbb{R}$),

2. **Core set** denoted by $A$ (for “atoms”), with $A \subseteq X$, (e.g. $A = \{0\}$) and

3. a set of **operations (functions)** $X \rightarrow X$, denoted by $F$. The elements of $F$ are **functions**, $f$, each having some **arity**, $k \geq 1$. I.e. $k$ is the number or arguments that $f$ takes.

E.g. $F = \{s(x) = x + 1\}$, i.e. $s$ is the **successor function**. Different functions have different arities - there is not a single arity $k$ which applies to all of the $f$s.
**Definition** Given any subset \( Y \subseteq X \), and any set \( F \) of operations (functions \( f: X^k \rightarrow X \) for any \( k \geq 1 \)), \( Y \) is **closed under** \( F \) if, for every \( f \in F \), (say \( f \) is a \( k \)-ary function) and every \( y_1, \ldots, y_k \in Y \), \( f(y_1, \ldots, y_k) \in Y \).

**Examples:**

1. Letting \( Y = \emptyset \), the above definition is vacuously satisfied.

2. Let \( Y \) be the set of even Natural numbers. Let \( F \) be the set of addition, and multiplication. Then we know \( Y \) is closed under \( F \).

   Y is not closed if we include subtraction in \( F \). (Time permitting, give an example to demonstrate this.)
Definition $Y$ is a minimal set with respect to a property $R$ if

1. $Y$ satisfies $R$, and
2. for every set $Z$ that satisfies $R$, $Y \subseteq Z$.

Now we have the formal definition.

Definition $I(X, A, F) =$ The minimal subset of $X$ that

1. contains $A$, and
2. is closed under the operations in $F$. 
Motivating Example The set of Natural numbers:

\[ \mathbb{N} = I \left( \mathbb{R}, \{0\}, \left\{ \begin{array}{l} s(x) = x + 1 \\ \text{successor function} \end{array} \right\} \right). \]

Exercise: Prove it. One containment is easy. The other containment can be proved using POMI, the way you would have in MATH 135.
Example: \( I(X, A, F) \) is the set of polynomials in variable \( z \) over a field \( K \) (say the field of real numbers), with:

\[
X = \text{the set of all strings that can be written using } \mathbb{R} \cup \{+, \cdot, z\}
\]

e.g. \( + z \cdot 5 + +18 \cdot 0 \cdot 5 \) is garbage, while \( z \cdot z + 1 \) is a polynomial in \( z \) over \( \mathbb{R} \)

\[
A = \{z\} \cup \mathbb{R}
\]

\[
F = \{+, \cdot\}
\]
Motivation: Here we explain our strategy to use structural induction to prove a desirable property $R$ holds for every element of an inductively defined set, $I(X, A, F)$.

1. Prove that $R(a)$ holds for every $a$ in the core set $A$ (the base case).

2. Prove that, for every $k$-ary $f \in F$ (for any $k \geq 1$), and any $y_1, \ldots, y_k \in X$ such that $R(y_1), \ldots, R(y_k)$ all hold, we also have that $R(f(y_1, \ldots, y_k))$ holds (the inductive case).
Remarks:

1. Our first example using this technique will be to prove the unique readability of propositional formulas in Form(ℒ_p).

2. Recalling that

\[ \mathbb{N} = I \left( \mathbb{R}, \{0\}, \left\{ s(x) = x + 1 \right\} \right). \]

we see that structural induction, as above, reduces to the familiar POMI (strong perhaps, but there is no real difference between strong and regular induction anyway).
Using our setup above, we will define the set $\text{Form}(\mathcal{L}^p)$ inductively. We need these ingredients:

1. Let $P$ be a set of **proposition symbols**, e.g. $P = \{p, q, r\}$. These will be our **atoms**.

2. Let $C$ be the set of **propositional connectives**, namely $C = \{\neg, \land, \lor, \rightarrow, \leftrightarrow\}$.

3. Let $X$ be the set of all strings that can be written using $P \cup C \cup \{(, )\}$.
Let $F$ be the set containing the following functions defined on $X$:

1. $\text{neg}(x) = (\neg x)$ (unary)
2. $\text{and}(x, y) = (x \land y)$ (binary)
3. $\text{or}(x, y) = (x \lor y)$ (binary)
4. $\text{impl}(x, y) = (x \rightarrow y)$ (binary)
5. $\text{equiv}(x, y) = (x \leftrightarrow y)$ (binary)
Definition 1

Using the notation above, the set $\text{Form}(\mathcal{L}_p)$ of **propositional formulas over** $F$ is defined inductively, as

$$I(X, P, F)$$

See also Definitions 2.2.1, 2.2.2 and 2.2.3 in the text.
Examples:

1. Each of the following is a in $\text{Form}(\mathcal{L}_p)$ over $P = \{p, q, r\}$.
   1. $p$
   2. $(\neg q)$
   3. $(p \land q)$
   4. $((p \land q) \land p)$
   5. $(p \rightarrow (q \lor r))$

2. This string is not in $\text{Form}(\mathcal{L}_p)$.

   $)p \rightarrow \land \leftrightarrow rq()$

How would you prove this fact? We will answer this soon.
**Problem:** Prove by structural induction that every propositional formula in $\text{Form}(\mathcal{L}_p)$ contains at least one proposition symbol.

**Solution:** Exercise.
Problem: Prove that every propositional formula in Form(ℒ^p), A, has the same number of ‘(’ and ‘)’ symbols.

Solution: Let A be any propositional formula in Form(ℒ^p). The proof is by structural induction on A.

Let R(A) be the property

“A has equally many ‘(’ and ‘)’ symbols.”

Base Case (A is p for some proposition symbol, p): Then A has zero ‘(’ and zero ‘)’ symbols. Therefore R(A) holds in the base case.
Induction Case: Define the notation

- $\ell(A)$ denotes the number of ‘(’ symbols in $A$.
- $r(A)$ denotes the number of ‘)’ symbols in $A$.

We have these subcases.

- $A$ is $(\neg B)$:
  - The inductive hypothesis is $R(B)$, i.e. that $\ell(B) = r(B)$.
  - Then we have

$$
\ell((\neg B)) = 1 + \ell(B) \quad \text{(inspection)}
$$
$$
= 1 + r(B) \quad \text{(induction hypothesis: R(B))}
$$
$$
= r((\neg B)) \quad \text{(inspection)}
$$
A is \( (B \star C) \), for some formulas \( B \) and \( C \) and some binary connective \( \star \):

“Without loss of generality” clearly applies to all the binary connectives.

The inductive hypothesis is \( R(B) \) and \( R(C) \), i.e. that \( \ell(B) = r(B) \) and \( \ell(C) = r(C) \).

Then we have

\[
\ell \left( (B \star C) \right) = 1 + \ell(B) + \ell(C) \quad \text{(inspection)} \\
= 1 + r(B) + r(C) \quad \text{(induction hypothesis)} \\
= r \left( (B \star C) \right) \quad \text{(inspection)}
\]
Remarks:

1. **Q:** How could we use this result to demonstrate that our non-examples of formulas from Lecture 02 were correct?
Theorem (Unique Readability of Propositional Formulas) 2

Every propositional formula in Form(\mathcal{L}^p), A, is exactly one of an atom, (\neg B), (B \land C), (B \lor C), (B \rightarrow C), or (B \leftrightarrow C); and in each case A is of that form in exactly one way.

Property R(A): A formula A has property R(A) iff it satisfies all three of the following.

1: The first symbol of A is either ‘(’ or a variable.

2: A has an equal number of ‘(’ and ‘)’, and each proper initial segment of A has more ‘(’ than ‘)’.

3: A has a unique construction as a formula.
Remarks:

1. (A proper initial segment of $A$ is a non-empty expression $X$ such that $A$ is $XY$ for some non-empty expression $Y$.)

2. We prove property $R(A)$ for all formulas $A$, by Structural Induction on $A$.

3. We only need Property 3 in the end. Including Properties 1 and 2 gives our inductive hypothesis more strength when needed.
Base (A is p, for some proposition symbol, p):

1: trivial.

2:
   first part: trivial.
   second part: vacuous (since A has no proper initial segments in this case).

3: trivial.
The induction step has two sub-cases.

**First Sub-case:**

A is \((\neg B)\), for some propositional formula B in Form\((\mathcal{L}_p)\):

The inductive hypothesis is that the formula B has property R.

- **1:** By construction, \((\neg B)\) has Property 1, since it begins with ‘(’.

- **2:** Since B has an equal number of left and right parentheses, therefore so does \((\neg B)\).
For the second part of Property 2, we check these subcases for every possible proper initial segment, $x$, of $A$.

1. $x$ is "("): Then $x$ has 1 "(" symbol and 0 ")" symbols.

2. $x$ is "(¬"): Then $x$ has 1 "(" symbol and 0 ")" symbols.

3. $x$ is "(¬"$z$, for some proper initial segment, $z$, of $B$: Since $z$ has more "(" than ")" symbols, therefore so does $x$.

4. $x$ is "(¬"$B$: Since $B$ equally many "(" and ")" symbols, therefore $x$ has more "(" than ")" symbols.

In every case, $x$ has more "(" than ")" symbols. Hence $(¬B)$ has Property 2.
3: Because B has Property 3, therefore by construction so does (¬B).

This shows that A has Property R.
Second Sub-case:
A is \((B \star C)\), for some propositional formulas \(B, C\) in \(\text{Form}(\mathcal{L}_p)\) and some binary connective \(\star\):

The inductive hypothesis is that each formula \(B\) and \(C\) has property \(R\).

1: Clearly, \(A\) has property 1.

2: Since \(B\) and \(C\) have equal numbers of left and right parentheses, therefore so does \((B \star C)\).
For the second part of Property 2, we check these subcases for every possible proper initial segment, \( x \), of \( A \).

1. \( x \) is "("": Then \( x \) has 1 "(‘ ’ symbol and 0 “)”) symbols.

2. \( x \) is "("\( z \), for some proper initial segment, \( z \), of \( B \): Since \( z \) has more "(" than "") symbols, therefore so does \( x \).

3. \( x \) is "("\( B \): Since \( B \) equally many "(" and "")" symbols, therefore \( x \) has more "(" than "")" symbols.

4. \( x \) is "("\( B\star \): Since \( B \) equally many "(" and "")" symbols, therefore \( x \) has more "(" than "")" symbols.
x is “("B ★ z, for some proper initial segment, z, of C: Since B equally many “(" and ")” symbols, and z has more “(" than ")” symbols, therefore x has more “(" than ")” symbols.

x is “("B ★ C: Since B and C have equally many “(" and ")” symbols, therefore x has more “(" than ")” symbols. Hence (¬B) has Property 2.

In every case, x has more “(" than ")” symbols. Hence (¬B) has Property 2.
3: We must show

If $A$ is $(B' \star' C')$ for formulas $B'$ and $C'$, then $B = B'$, $\star = \star'$ and $C = C'$.

If $\|B'\| = \|B\|$, then $B' = B$ (both start at the second symbol of $A$). Thus also $\star' = \star$ and $C' = C$, as required. So we are finished if we can prove that $\|B'\| = \|B\|$.
Proof that $\|B\| = \|B'\|$.

- Towards a contradiction, assume that either $B'$ is a proper initial segment of $B$ or $B$ is a proper initial segment of $B'$.
- The inductive hypothesis applies to $B$ and $B'$. In particular, each has property 2.
- Therefore $B$ and $B'$ have a balanced number of "(" and ")" characters, by property 2.
- But if $B$ is a proper initial segment of $B'$, then $B$ has more "(" than ")" characters, also by property 2. This is a contradiction.
- We reach a similar contradiction if we assume that $B'$ is a proper initial segment of $B$. Thus neither $B$ nor $B'$ can be a proper initial segment of the other.
Therefore $A$ has a unique derivation; it has Property 3, as required.
By the principle of structural induction, every Propositional formula has Properties 1, 2 and 3.

This shows that Unique Readability (Property 3) holds for every Propositional Formula.

This is what we set out to prove.
Explanation of the Connection Between $B$ and $B'$:

- In the past, some students have been confused about why it holds that either $B = B'$, $B$ is a proper initial segment of $B'$ or vice versa.

- The key fact to remember here is that both $B$ and $B'$ arose from a choice of how to decompose the given formula $A$. In detail,

$$\ (B \star C) = A = (B' \star' C')\ .$$

- Because we actually mean equality of formulas (i.e. symbol-by-symbol equality of the expressions constituting the formulas) here, we now see that the above fact about $B$ and $B'$ must hold.
Why We Care About Unique Readability:

1. For the rules of propositional logic semantics to be well-defined, it is crucial that every propositional formula can be parsed in only one way.
A **parse tree** for a formula represents the formation sequence as a tree with its root at the top, and each internal node corresponding with an application of one of the formation rules.

For example, this is a parse tree for the formula $A$ which is $((p \land (\neg q)) \to r)$:

```
((p \land (\neg q)) \to r)
```

```
(p \land (\neg q))
```

```
p
```

```
(\neg q)
```

```
q
```

This parse tree has height 3.
See also p24 of the text.

Another typical question would be to provide a parse tree, and to ask for the formula that the tree represents.
Remarks:

1. If we follow our construction rules to the letter, then $(p \land q \land r)$ is not in $\text{Form}({\mathcal L_p})$.

2. To put this formula into $\text{Form}({\mathcal L_p})$, we would have to write
   
   1. $((p \land q) \land r)$ or
   
   2. $(p \land (q \land r))$.

3. These are non equal as (syntactic) formulas.

4. These formulas are tautologically equivalent, i.e. they behave the same way in every semantic context (equivalently, they have the same truth tables).
As on p 33 of the text, we may omit some parentheses once we agree on an order of precedence for the connectives. The order is

1. \( \neg \)
2. \( \land \)
3. \( \lor \)
4. \( \rightarrow \)
5. \( \leftrightarrow \)
**Examples:** On each row of the following table, we give a formula with some (or all) parentheses omitted, followed by the formula in $\text{Form}(\mathcal{L}^p)$ that results from adding parentheses according to the above precedence rules.

<table>
<thead>
<tr>
<th>some (or all) parentheses omitted</th>
<th>in $\text{Form}(\mathcal{L}^p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \lor q \land r$</td>
<td>$(p \lor (q \land r))$</td>
</tr>
<tr>
<td>$\neg p \lor q$</td>
<td>$((\neg p) \lor q)$</td>
</tr>
<tr>
<td>$p \rightarrow q \land r$</td>
<td>$(p \rightarrow (q \land r))$</td>
</tr>
<tr>
<td>$p \rightarrow q \leftrightarrow r$</td>
<td>$((p \rightarrow q) \leftrightarrow r)$</td>
</tr>
<tr>
<td>$p \land q \rightarrow \neg r$</td>
<td>$(p \land q) \rightarrow (\neg r))$</td>
</tr>
</tbody>
</table>
Remarks:

1. We will say “propositional formula in Form($\mathcal{L}^P$)” to refer to a formula which is syntactically correct according to the earlier definition.

2. We will say “propositional formula”, to refer to a formula which may be in Form($\mathcal{L}^P$), or may have some parentheses omitted, where the correct formula in Form($\mathcal{L}^P$) could be recovered according to the precedence rules.
Examples

Give a generation sequence for each of the following propositional formulas in $\text{Form}(\mathcal{L}_P)$ over $P = \{p, q, r\}$.

1. $p$
   
   **Solution:**
   
   1. $p$ is in the core set.

2. $q$
   
   **Solution:**
   
   1. $q$ is in the core set.
Examples

3  \((p \land q)\)

Solution:

1  \(p\) is in the core set.

2  \(q\) is in the core set.

3  Applying and to lines 1 and 2 yields \((p \land q)\).
Examples

\( p \rightarrow (q \lor r) \)

Solution:

1. \( p \) is in the core set.
2. \( q \) is in the core set.
3. \( r \) is in the core set.
4. Applying or to lines 2 and 3 yields \( (q \lor r) \).
5. Applying impl to lines 1 and 4 yields \( (p \rightarrow (q \lor r)) \).
Setup:

- Let A be the set \{(0, 1, 0)\}.
- Suppose that we can operate on A by flipping any two elements from 0 to 1 or from 1 to 0.

**Problem:** Is it possible that any sequence of such flips applied to A yields the triple (0, 0, 0)?
Solution: Let $X = \{ \text{all triples of binary digits} \}$. Then consider $I(X, A, F)$, where $A = \{(0, 1, 0)\}$, $F = \{ \text{flip 1 and 2, flip 1 and 3, flip 2 and 3} \}$. The problem is then equivalent to asking is $(0, 0, 0) \in I(X, A, F)$?

I claim that the answer is “No”. I need to prove my answer, by structural induction on $I(X, A, F)$.

For any binary triple $(x, y, z)$, define $R(x, y, z)$ to be $(x, y, z) \text{ has an even number of 0 digits.}$

We will prove by structural induction that every triple in $I(X, A, F)$ has property $R$. 
Base: \(R(0, 1, 0)\) is clear.

Induction: Let \((x, y, z)\) be any binary triple having property \(R\). Then we check that each operation in \(F\) preserves property \(R\), via the following table:

<table>
<thead>
<tr>
<th>input triple</th>
<th>flip 1 and 2</th>
<th>flip 1 and 3</th>
<th>flip 2 and 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1, 1)</td>
<td>(0, 0, 1)</td>
<td>(0, 1, 0)</td>
<td>(1, 0, 0)</td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>(1, 1, 1)</td>
<td>(1, 0, 0)</td>
<td>(0, 1, 0)</td>
</tr>
<tr>
<td>(0, 1, 0)</td>
<td>(1, 0, 0)</td>
<td>(1, 1, 1)</td>
<td>(0, 0, 1)</td>
</tr>
<tr>
<td>(1, 0, 0)</td>
<td>(0, 1, 0)</td>
<td>(0, 0, 1)</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>

All the output triples have property \(R\). This completes the inductive step, and the proof.

Now since \((0, 0, 0)\) does not have property \(R\), therefore \((0, 0, 0) \notin I(X, A, F)\).
Remark: This example provides a strategy for proving that an element of the universe is not a member of an inductively defined set:

1. Prove that all elements of the set have some property $R$.
2. Prove that the element of interest does not have property $R$.

E.g. since all propositional formulas in $\text{Form}(\mathcal{L}_p)$ have equaly many "(" and ")" symbols, therefore $\)p \to \land \leftrightarrow \text{rq()}$ is not in $\text{Form}(\mathcal{L}_p)$. 