CS 245 - Lecture 04 - Tautological Consequence

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Outline

1. Tautological Consequence
    1. Satisfaction of a Set of Formulas
    2. Definition of Tautological Consequence
    3. Subtleties About Tautological Consequence
    4. Argument Validity

Definition 1

We say that a truth valuation, t, ***satisfies a set*** Σ***, of propositional formulas in*** $\mathrm{Form}(\mathcal{L}^{\mathrm{p}})$*, (Notation: $\Sigma^t = 1$) if, for every $\mathrm{C} \in \Sigma$, we have $\mathrm{C}^t = 1$. If there exists $\mathrm{C} \in \Sigma$ such that $\mathrm{C}^t = 0$, then we say that t* ***does not satisfy*** Σ *(Notation: $\Sigma^t = 0$).*

Remarks

1. Analogously to a formula, we say that Σ is **satisfiable** if there exists a truth valuation t such that $\Sigma^t = 1$. Otherwise, we say that Σ is **not satisfiable**.
2. When we write $\Sigma^t = 0$, we are **not** asserting that every $C \in \Sigma$ has $C^t = 0$; we just assert that there is **at least one** such C.

Problems

1. Verify that $\Sigma = \{((p \to q) \vee r), ((p \vee q) \vee s)\}$ is satisfiable.

Solution 1: Let t be any truth valuation such that $q^t = 1$. p^t, r^t, s^t can be 0 or 1 - it will not matter in this example. Then

- $(p \to q)^t = 1$ (by $\to$-rule), so that $((p \to q) \vee r)^t = 1$ (by $\vee$-rule), and
- $(p \vee q)^t = 1$ (by $\vee$-rule), so that $((p \vee q) \vee s)^t = 1$ (by $\vee$-rule).

This shows that t satisfies Σ. N.B. This defines a family of truth valuations that work. There are $2^3 = 8$ such truth valuations.

Problems

Solution 2: Construct a joint truth table for $((p \to q) \vee r)$ and $((p \vee q) \vee s)$ and verify that at least one row has 1 in both of its final columns.

Problems

2. Is the set $\Sigma = \{(p \to (p \wedge q)), ((\neg p) \vee (\neg q)), ((\neg p) \to p)\}$ satisfiable? Prove your answer.

Solution 1: I claim that the set is not satisfiable. For a contradiction, suppose that there exists a truth valuation, t, such that $\Sigma^t = 1$. Then

- $(p \to (p \wedge q))^t = 1$, so we must have one of
  - $(p \wedge q)^t = 1$, which requires $p^t = q^t = 1$. However this implies that $((\neg p) \vee (\neg q))^t = 0$, which contradicts $\Sigma^t = 1$, completing this case.
  - $p^t = 0$, which implies $((\neg p) \to p)^t = 0$, which contradicts $\Sigma^t = 1$, completing this case.
- All possibilities lead to contradiction. This completes the proof.

Problems

Solution 2: Construct a joint truth table for $(p \rightarrow (p \wedge q))$, $((\neg p) \vee (\neg q))$ and $((\neg p) \rightarrow p)$ and verify that no row has 1 in all three of its final columns.

Definition 2

We say that a set Σ *of propositional formulas in* $\mathrm{Form}(\mathcal{L}^{\mathrm{p}})$ ***tautologically implies*** *a propositional formula* C *in* $\mathrm{Form}(\mathcal{L}^{\mathrm{p}})$ *(Notation:* $\Sigma \models \mathrm{C}$*), if, whenever* $\Sigma^t = 1$ *for some truth valuation* t*, we also have* $\mathrm{C}^t = 1$*. In this situtation, we also say that* C *is a* ***tautological consequence*** *of* Σ*.*

Remarks

1. The notion of **tautological consequence** will be a key ingredient in defining the **soundness** and **completeness** of our proof system(s), later on.
2. It is crucial to understand that Definition **??** asserts **nothing** in the situation where $\Sigma^t = 0$.
3. To prove that $\Sigma \models C$, apply Defintion **??**.

Remarks

4. To prove that $\Sigma \not\models C$, for some Σ and C, we must exhibit a choice of a truth valuation, t, such that $\Sigma^t = 1$ and $C^t = 0$.
5. If Σ is not satisfiable, then $\Sigma \models C$, for **any** C.
6. Let $\Sigma = \emptyset$. Let t be any truth valuation. Then $\emptyset^t = 1$. This is counter-intuitive. But we must accept it, given Definition **??**. There is no $C \in \emptyset$ such that $C^t = 0$.

Problems

1. Q: Is it true that $\{(p \wedge q)\} \models p$?

   A: Yes. Any truth valuation, t such that $\{(p \wedge q)\}^t = 1$ has $(p \wedge q)^t = 1$, and by properties of $\wedge$, this requires $p^t = 1 = q^t$. Hence t satisfies p.

Problems

2. Q: Is it true that $\{(p \wedge q)\} \models r$?

   A: No. Consider the truth valuation

$$
\begin{array}{rcccc}
t & : & \{p, q, r\} & \to & \{0, 1\} \\
 & & p^t & = & 1 \\
 & & q^t & = & 1 \\
 & & r^t & = & 0
\end{array}
$$

Then we have that $\{(p \wedge q)\}^t = 1$, and $r^t = 0$.

Problems

3. Q: Is it true that $\{(p \vee q)\} \models p$?

   A: No. Consider the truth valuation

$$\begin{array}{ccccc} t & : & \{p, q\} & \to & \{0, 1\} \\ & & p^t & = & 0 \\ & & q^t & = & 1 \end{array}$$

Then we have that $\{(p \vee q)\}^t = 1$, and $p^t = 0$.

Problems

4. Prove or disprove the tautological consequence $\{(p \to r), (q \to (\neg r))\} \models (p \to (\neg q))$.

Solution: The truth table below shows the valuations of all of the formulas involved. The lines marked with $\lhd$ are the ones for which all the assumptions $\{(p \to r), (q \to (\neg r))\}$ are all true. In all such cases, the conclusion $(p \to (\neg q))$ is also true.

p	q	r	$(p \to r)$	$(q \to (\neg r))$	$(p \to (\neg q))$	
0	0	0	1	1	1	$\lhd$
0	0	1	1	1	1	$\lhd$
0	1	0	1	1	1	$\lhd$
0	1	1	1	0	1	
1	0	0	0	1	1	
1	0	1	1	1	1	$\lhd$
1	1	0	0	1	0	
1	1	1	1	0	0	

Thus the tautological consequence holds.

1. Any statement like "For all $x \in \emptyset$, $x < 6$" is true. The empty set, $\emptyset$, provides no counterexample $x \in \emptyset$, such that $x < 6$ fails to hold.
2. By the same token, "For all $C \in \emptyset$, $C^t = 1$" is true, for any truth valuation t.
    1. It is important to understand that the negation of the statement in the previous bullet is "There exists $C \in \emptyset$, such that $C^t = 0$" (which is clearly not true), and **not** "For all $C \in \emptyset$, $C^t = 0$" (which is true, as above).
3. As pointed out earlier, **the empty set is satisfied under any truth valuation,** t.

4. This shows by Definition **??** that if $\emptyset \vDash C$, then C is a tautology.
   1. Some authors will write "fresh air" instead of $\emptyset$. I will always write $\emptyset$ explicitly.
5. For the reverse implication, note that if C is a tautology, $\Sigma \vDash C$, for **any** Σ.
6. As pointed out earlier, in the case where Σ is not satisfiable, we see that $\Sigma \vDash C$, for **any** C.
7. An equivalent alternative for $\models\!\!\!\!\!\dashv$, using Definition **??** is that

$$A \models\!\!\!\!\!\dashv B \text{ if and only if } \left(\{A\} \vDash B \text{ and } \{B\} \vDash A\right).$$

8. Writing $\Sigma \models \varnothing$ makes no sense. $\varnothing$ is not a Propsitional formula.

9. In class we developed this summary table about the subtleties of tautological consequence. Make sure that you understand the definition of tautological consequence, and you do not try to rely on this table alone!

Σ	C	$\Sigma \models C$?
not satisfiable	contradiction	yes
not satisfiable	satisfiable, not a tautology	yes
not satisfiable	tautology	yes
satisfiable	contradiction	no
satisfiable	satisfiable, not a tautology	maybe
satisfiable	tautology	yes

As explained in the slides, the argument "Σ (semantically) implies A" is **valid** if and only if $\Sigma \models A$.

Example

Given the premises:

1. If I study before my mid-term, and I get 8 hours' sleep before my mid-term, then I will pass my mid-term.
2. I studied before my mid-term.
3. I did not pass my mid-term.

We may conclude:

1. Therefore I must not have gotten 8 hours' sleep before my mid-term.

Example

First, we translate from English into propositional logic; second we prove the resulting tautological consequence.

Define these atomic propositions:

1. s: I studied before my mid-term.
2. h: I got 8 hours' sleep before my mid-term.
3. p: I passed my mid-term.

We then encode the above argument as follows. We will explain all the notation below, soon.

$$\{((s \wedge h) \to p), s, (\neg p)\} \quad \underbrace{\models}_{\text{"tautologically implies"}} \quad (\neg h).$$

Example

We will see soon that the premises on the left of $\models$ are sufficient to tautologically imply $(\neg \mathrm{h})$, the formula on the right of $\models$. This is what I meant by "convincing", above.

Solution 1:

- Let $\Sigma = \{((\mathrm{s} \wedge \mathrm{h}) \to \mathrm{p}), \mathrm{s}, (\neg \mathrm{p})\}$.
- Towards a contradiction, suppose that there exists a truth valuation, t, such that $\Sigma^t = 1$ and $(\neg \mathrm{h})^t = 0$, i.e. $\mathrm{h}^t = 1$.
- Then we have $\mathrm{s}^t = 1$, and $(\neg \mathrm{p})^t = 1$, so that $\mathrm{p}^t = 0$.
- By $\wedge$-properties, it follows that $(\mathrm{s} \wedge \mathrm{h})^t = 1$.
- By $\to$-properties, it follows that $\mathrm{p}^t = 1$.
- This contradiction completes the proof.

Example

Solution 2: Construct a truth table displaying all of the formulas involved. This is left as an exercise.