Propositional Language - Syntax

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Based in part on materials and comments by Borzoo Bonakdarpour, Daniela Maftuleac, Peter Van Beek, Jonathan Buss, Collin Roberts, Richard Trefler, Stephen Watt

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1. **Syntax of propositional language $\mathcal{L}^p$: atoms, formulas**
2. Generating and parsing formulas of $\mathcal{L}^p$
3. Mathematical induction (optional), structural induction
4. Unique Readability Theorem
5. Precedence rules for connectives, scope of a connective

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- If an expression in $\mathcal{L}^p$ can be produced by the formation rules then it is a formula in $\text{Form}(\mathcal{L}^p)$, otherwise it is not a formula in $\text{Form}(\mathcal{L}^p)$.

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- Scanning of expressions proceeds from left to right.

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The compound proposition becomes

$$((p \rightarrow (q \wedge r)) \wedge ((\neg p) \rightarrow s))$$

One can use parse trees to analyze formulas.

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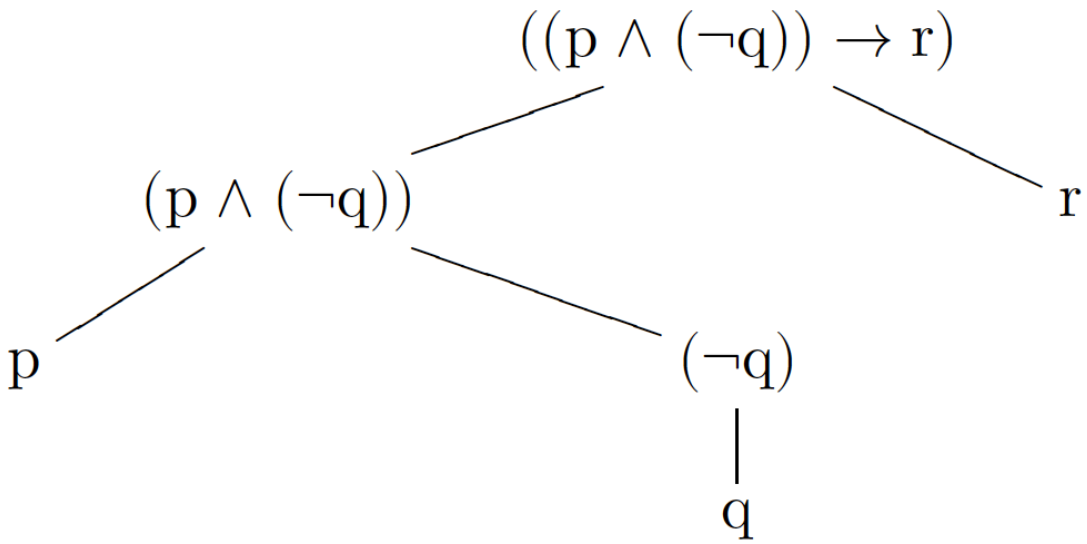

Note: Later in this module we will informally describe an algorithm that determines the main connective of a formula, i.e., the root of its parse tree (as a consequence of the Unique Readability Theorem).

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Or both a negation and a disjunction?

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- (Unique Readability Theorem) Every formula in Form$(\mathcal{L}^p)$ is of exactly one of the six forms: an atom, $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, or $(A \leftrightarrow B)$ and, in each case, it is of that form in exactly one way.

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- To prove such claims, we will use induction.

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Review of mathematical induction

The "natural numbers" are the numbers we use to count things.
Before we start, we count 0; as we find things we count 1, 2, etc.
The natural numbers form an unbounded sequence

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A statement "every natural number has property P" corresponds to a sequence of statements

$$P(0),\ P(1),\ P(2),\ P(3),\ P(4),\ \ldots$$

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Then we may conclude that every natural number has property P.

Mathematical induction example

Example: Show that $\sum_{x=0}^{n} x = \frac{n(n+1)}{2}$ for every natural number n.

Let P be the property; that is, let $P(n)$ be "$\sum_{x=0}^{n} x = \frac{n(n+1)}{2}$".

Proof for the example

Step 1 (Base Case): The property $P(0)$ is $\sum_{x=0}^{0} x = \frac{0(0+1)}{2}$.

The left side of the equation is just 0. Also the right side evaluates to 0.
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Step 2 (Inductive Step): Hypothesize that an arbitrary natural number, say k, has property P (this is called the Inductive Hypothesis, I.H.), that is,

$$\sum_{x=0}^{k} x = \frac{k(k+1)}{2}$$

Note: We chose the letter k to differentiate this arbitrary but fixed natural number, from n which denotes any natural number.

Step 2 (Inductive Step), continued:

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We calculate:

$$\begin{aligned}
\sum_{x=0}^{k+1} x &= \left(\sum_{x=0}^{k} x\right) + (k+1) && \text{definition of } \textstyle\sum \\
&= \tfrac{k(k+1)}{2} + (k+1) && \text{by the Inductive Hypothesis} \\
&= \left(\tfrac{k}{2} + 1\right)(k+1) && \text{"algebra"} \\
&= \tfrac{(k+1)(k+2)}{2} && \text{DONE!}
\end{aligned}$$

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The induction principle does not say how to actually do either step. We must invent those parts ourselves, and every problem is different.

"Simple" induction vs. "Strong" induction

	Simple Induction	**Strong Induction** *(course-of-values induction)*
Base Case:	Show $P(0)$	Show $P(0)$
Ind. Hyp.:	$P(k)$ holds	$P(m)$ holds, ***for every*** $m \leq k$
Ind. Step:	Show $P(k+1)$ holds	Show $P(k+1)$ holds
Conclusion:	$P(k)$ holds for every k	$P(k)$ holds for every k

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Induction example: Tiling checkerboards

Tiling Checkerboards

Example: Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes.

A right triomino is an L-shaped tile which covers three squares at a time.

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A right triomino is an L-shaped tile which covers three squares at a time.

Solution: Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes. Use mathematical induction to prove that $P(n)$ is true for all positive integers n.

- BASIS STEP: P(1) is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino.

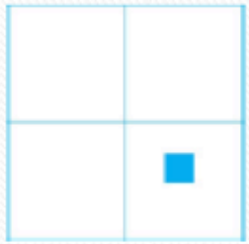
- INDUCTIVE STEP: Assume that $P(k)$ is true for every $2^k \times 2^k$ checkerboard, for some positive integer k.

Induction example: Tiling checkerboards

Tiling Checkerboards

Inductive Hypothesis: Every $2^k \times 2^k$ checkerboard, for some positive integer k, with one square removed can be tiled using right triominoes.

- Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions.



- Remove a square from one of the four $2^k \times 2^k$ checkerboards. By the inductive hypothesis, this board can be tiled. Also by the inductive hypothesis, the other three boards can be tiled with the square from the corner of the center of the original board removed. We can then cover the three adjacent squares with a triominoe.
- Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes.

◀

All horses are the same colour

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  The proof of the Inductive Step
  does not work for the Base Case, $k = 1$.

How to prove properties of formulas

We want to prove: "Every formula in Form$(\mathcal{L}^p)$ has property P".

A formula is not a natural number, but it suffices to prove any one of the following.

> *For every natural number n, every formula with n or fewer symbols has property P.*

OR

> *For every natural number n, every formula with n or fewer connectives has property P.*

OR

> *For every natural number n, every formula whose parse tree has height less than or equal to n has property P.*

OR

> *For every natural number n, every formula producible with n or fewer uses of the formation rules has property P.*

Recursively defined sets

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III. RESTRICTION: A statement that no objects belong to the set other than those coming from I, and II.

Examples

Example: The set of natural numbers $\mathbb{N}$ is a recursively defined set with one formation rule ("add 1").

I. BASE: 0 is a natural number in $\mathbb{N}$.

II. RECURSION: If k is a natural number in $\mathbb{N}$, then $k+1$ is a natural number in $\mathbb{N}$.

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Example: The set Form$(\mathcal{L}^p)$ is a recursively defined set, where the core set in BASE consist of all atomic formulas (proposition symbols), and the RECURSION has 5 formation rules (one for each connective).

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Structural Induction for Recursively Defined Sets

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Because, due to RESTRICTION, no objects other than those obtained through the BASE and RECURSION conditions are contained in S, it must then be the case than every object in S satisfies the property.

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Note: In general, the number of subcases of the composite inductive step in a proof by structural induction equals the number of formation rules in the RECURSION part of the definition of that set (e.g., 1 rule/subcase for $\mathbb{N}$, and 5 rules/subcases for Form$(\mathcal{L}^p)$).

Example: Parentheses in formulas

To illustrate structural induction, we shall prove the following.

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for every formula A in Form$(\mathcal{L}^p)$.

Base Case: A is an atom. Then, A has zero left and right parentheses, as it is only a proposition symbol. Thus $R(A)$ holds. This completes the proof of the Base Case.

Proof cont'd: Inductive Step, subcase $\neg$

Inductive Step, the subcase of $\neg$

Notation: For any formula A, let $\ell(A)$ denote the number of '(' in A, and let $r(A)$ denote the number of ')' in A.

Assume A is $(\neg B)$.

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Then we have

$$\begin{aligned} \ell\left((\neg B)\right) &= 1 + \ell(B) \text{ (inspection)} \\ &= 1 + r(B) \text{ (induction hypothesis: } R(B)) \\ &= r\left((\neg B)\right) \text{ (inspection)} \end{aligned}$$

Proof cont'd: Inductive Step, subcases $\wedge, \vee, \rightarrow, \leftrightarrow$

Inductive Hypothesis: formulas B and C both have property R.
To prove: Each of the formulas $(B \wedge C)$, $(B \vee C)$, $(B \rightarrow C)$ and $(B \leftrightarrow C)$ has property R.

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This concludes the proof of the (composite) inductive step, the inductive proof, and thus the Example.

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This concludes the proof of the (composite) inductive step, the inductive proof, and thus the Example.

1. Syntax of propositional language $\mathcal{L}^p$: atoms, formulas
2. Generating and parsing formulas of $\mathcal{L}^p$
3. Mathematical induction (optional), structural induction
4. **Unique Readability Theorem**
5. Precedence rules for connectives, scope of a connective

The Unique Readability Theorem

Theorem (Unique Readability). *Every formula in* Form$(\mathcal{L}^p)$ *is of exactly one of the six forms: an atom,* $(\neg B)$, $(B \wedge C)$, $(B \vee C)$, $(B \rightarrow C)$ *or* $(B \leftrightarrow C)$ *and, in each case, it is of that form in exactly one way.*

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Proof. Use the following Claims:

(a) *The first symbol of a formula* A *is either '(' or a proposition symbol* (by the recursive definition of formulas)

(b) *Every formula* A *has an equal number of '(' and ')'* (*Lemma 1*)

(c) *If* A' *is a non-empty proper initial segment of a formula* A*, then* A' *has more '(' than ')'* (can be proved by structural induction)

Proof of the Unique Readability Theorem

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Assume that a formula F can be of two different forms, that is:

1. $F = p = A$, where A is not an atom.
2. $F = (\neg A) = (B \star C)$, where $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.
3. $F = (B \star C) = (B' \star' C')$ where $\star, \star' \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

where all of F, A, B, C, B', C' are formulas in Form$(\mathcal{L}^p)$.

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where all of F, A, B, C, B', C' are formulas in Form$(\mathcal{L}^p)$.

Note: In, e.g., case (3), $(B \star C)$ and $(B' \star' C')$ are two decompositions of the same formula F, into its constituent (sub)formulas.

Recall that $(B \star C) = (B' \star' C')$ means that the formulas $(B \star C)$ and $(B' \star' C')$ are equal (identical), i.e., they have same length, and the same sequence of symbols, in the same order.

Proof of Unique Readability, cont'd

1. Assume $p = A$. Due to Claim (a), a non-atomic formula A starts with $'('$. Since the first two symbols on each side are the same, this implies $p \neq ($ which is a contradiction.
2. Assume $(\neg A) = (B \star C)$. Delete the first symbol on each side, obtaining $\neg A) = B \star C)$. This implies either $\neg = p$ (if $B = p$ is an atom), or $\neg = ($ (if B is a non-atomic formula), due to Claim (a), a contradiction.
3. Assume $(B \star C) = (B' \star' C')$. Delete the first symbol on each side, obtaining $B \star C) = B' \star' C')$.

   We now have three possibilities:

   (i) B has the same length as B'
   (ii) B is strictly shorter than B'
   (iii) B is strictly longer than B'

Proof of Unique Readability, cont'd

3(i) If B has the same length as B', then $B = B'$, $\star = \star'$, and $C = C'$.

3(ii) If B is strictly shorter than B', this implies that B is a non-empty initial segment of the formula B'. By Claim (c), this implies that B has more $'('$ than $')'$.
On the other hand, since B is itself a formula, by Claim (b), B has the same number of $'('$ and $')'$.
This is a contradiction.

3(iii) If B is strictly longer than B', reasoning similar to 3(ii) leads to a contradiction.

Thus, the only case that did not lead to a contradiction, 3(i), implies that the decomposition of a formula F into its constituent (sub)formulas is unique. □

Consequences of Unique Readability

There are six kinds of formulas:

- A proposition symbol p is called an atom.
- $(\neg A)$ is called a negation (formula). It is the negation of A.
- $(A \wedge B)$ is called a conjunction (formula). It is the conjunction of A and B. A and B are called the conjuncts of $(A \wedge B)$.
- $(A \vee B)$ is called a disjunction (formula). It is the (inclusive) disjunction of A and B. A and B are called the disjuncts of $(A \vee B)$.
- $(A \rightarrow B)$ is called an implication (formula). It is the implication of A and B. A and B are called the antecedent and consequent of $(A \rightarrow B)$.
- $(A \leftrightarrow B)$ is called an equivalence (formula). It is the equivalence of A and B.

Moreover, each formula is of exactly one kind.

Algorithms for parsing formulas

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Algorithm: Given a formula in Form($\mathcal{L}^p$), determine its subformulas:

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(	$\underbrace{(\ \langle\texttt{rest of subformula}\rangle\ )}$			$\star$	$\langle\texttt{subformula 2}\rangle$			**)**

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After scanning the first left parenthesis, start a counter equal to the difference between the encountered number of left parentheses and the number of encountered right parentheses.
When the count returns zero, the first subformula (which starts at symbol 2 and ends at symbol m) has been determined.

1. Syntax of propositional language $\mathcal{L}^p$: atoms, formulas
2. Generating and parsing formulas of $\mathcal{L}^p$
3. Mathematical induction (optional), structural induction
4. Unique Readability Theorem
5. **Precedence rules for connectives, scope of a connective**

Precedence rules (for human logicians)

$\neg$ has precedence over $\wedge$

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- $p \rightarrow q \vee r$ is to be understood as $(p \rightarrow (q \vee r))$
- $p \leftrightarrow p \rightarrow q$ must be understood as $(p \leftrightarrow (p \rightarrow q))$
- The proposition in the example of parsing formulas can now be written without all the parentheses, as $(p \rightarrow q \wedge r) \wedge (\neg p \rightarrow s)$.

Scope

- If $(\neg A)$ is a segment of a formula C, then A is called the scope of this negation in the formula C.
- If $(A \wedge B)$ is a segment of a formula C, then A is called the left scope of the conjunction and B is called the right scope of this conjunction in the formula C.
- Similar definitions hold for left/right scopes of disjunction, implication and equivalence.

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Suppose $A = (\neg((p \wedge q) \vee ((\neg p) \rightarrow r)))$.

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The left and right scopes of $\vee$ are $(p \wedge q)$ and $((\neg p) \rightarrow r)$.

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The left and right scopes of $\wedge$ are p and q.
The left and right scopes of $\vee$ are $(p \wedge q)$ and $((\neg p) \rightarrow r)$.
The left and right scopes of $\rightarrow$ are $(\neg p)$ and r.

Learning Objectives

- Know what is the propositional language $\mathcal{L}^p$ and the definitions of: expressions of $\mathcal{L}^p$, atoms (atomic formulas) of $\mathcal{L}^p$ (Atom($\mathcal{L}^p$)), and the set of formulas of $\mathcal{L}^p$ (Form($\mathcal{L}^p$))
- Know how to generate and parse formulas of $\mathcal{L}^p$ (parse tree)
- Review the mathematical induction proof technique
- Know the definition of a recursive set
- Know how to prove statements by structural induction
- Know the statement and proof of Unique Readability Theorem, and its consequences
- Know the precedence rules for logical connectives
- Know the definition of the scope of a connective