Logic and Computation - Introduction

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1. **What is Logic?**
2. Logic and computer science
3. Propositional logic
4. Translations from English into Logic

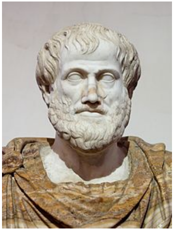
What is logic?

Logic is the **Science of Reasoning**

- Etymology: *Logykos* (Greek) - pertaining to reasoning
- Dictionary definitions:
  - Logic = The science of reasoning, proof, thinking, or inference
  - Logic = The fundamental science of thoughts and its categories
  - Logic = The science or art of reasoning as applied to a department of knowledge
  - Logic = The analysis and appraisal of arguments
- We all do logic when we try to clarify reasoning and separate good from bad reasoning

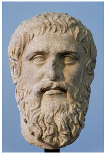
History of logic - the beginnings

- Aristotle (Greek philosopher, 384-322 B.C.) is the first person to offer the outlines of a comprehensive system for codifying and evaluating a wide range of arguments and reasoning.



Aristotle - "Father of Logic"

- Aristotle: Student of Plato ("Father of Western Philosophy")



- Aristotle: Tutor of Alexander the Great (356 - 323 B.C.)

- For Aristotle, logic is the instrument by means of which we come to know anything
- Aristotle wrote the earliest known formal study of logic

Aristotelian logic

- Formalizes the basic principles of good reasoning, and provides a way to evaluate specific cases of reasoning
- A syllogism is a kind of logical argument in which one proposition (the conclusion) is inferred from two or more others (the premises) of a specific form.
- The following is an example of an Aristotelian syllogism

  All humans are mortal.

  Socrates is human.

  Socrates is mortal.
- The horizontal line separates the premises from the conclusion
- This syllogism is an example of good reasoning - constitutes a good argument - because it is truth-preserving.
- That is, if the first two sentences (premises) are true, then the third sentence (conclusion) must also be true.

Aristotelian logic

Correctness of an argument depends on form (structure), not content

The argument

All humans are mortal.	has the form	All x are y.
Socrates is human.		B is an x.
Socrates is mortal.		B is a y.

The argument

All Accords are Hondas.	has the form	All x are y.
All Hondas are Japanese.		All y are z.
All Accords are Japanese.		All x are z.

The second type of argument is called hypothetical syllogism.

Another example

Is the following a correct argument?

All x are y.
Some y are z.

Some x are z.

No. This can be seen by the following example of using the argument

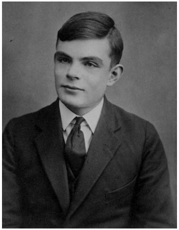
All bunnies are mammals.
Some mammals are fierce predators.

Some bunnies are fierce predators.

Why study logic?

- Most people find logic enjoyable
- Logic improves one's general powers of analytical thinking
- Logic is fundamental to Computer Science

"I expect that digital computing machines will eventually stimulate a considerable interest in symbolic logic . . . . The language in which one communicates with these machines . . . forms a sort of symbolic logic."(Alan Turing, 1947)



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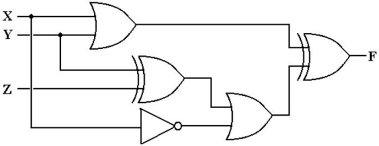
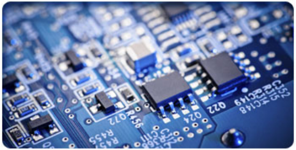
Applications of logic to computer science

- The underpinnings of all electronic computers are logic gates



- Electronic digital circuits are formed out of logic gates
- Logic can be used for the minimization of the number of components of electronic circuits



Applications of logic to computer science

- Artificial Intelligence - Expert systems (knowledge base + inference engine)
  - DENDRAL (Stanford University, 1960s) - an expert system to aid the identification of unknown organic molecules.
  - MYCIN (Stanford University, 1972), an expert system for treating blood infections (medical diagnosis systems: The user describes their symptoms to the computer, as they would to a human doctor, and the computer returns a medical diagnosis.) It gave acceptable therapy in about 69% of cases, which was roughly the same level of competence as human specialists in blood infections and rather better than general practitioners.
  - MISTRAL (Italy, 1990s) - an expert system for monitoring dam safety (still operational).

Applications of logic to computer science

- Automated theorem proving; automated proof verifiers
- Fully automated and semi-automated techniques for analyzing the behaviour of reactive programs
- Databases: core of modern database systems (e.g. SQL) uses first-order logic
- Programming:
    - Program specification
    - Formal verification (how do we know that a program does what it is supposed to do)

Applications of logic to computer science

- The programming language PROLOG (PROgramming with LOGic)
- DNA Computing

- Etc.

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Propositional logic

- Propositional logic is a branch of mathematical logic which studies logical relationships between propositions (true/false statements, sentences, assertions)
- Propositional logic is also known by the names sentential logic, propositional calculus, and sentential calculus
- The basic building block is a proposition, that is, a declarative statement that is either true or false. Propositions can be atomic, or compound (compound propositions are formed by using logical connectives to combine atomic propositions)
- Propositional logic analyzes logical arguments (argument = a claim that contains several premises which support a conclusion)
- It provides methods to determine whether a logical argument is valid (correct, sound), or invalid (incorrect, unsound)

Logical arguments, valid or invalid

- A (logical) argument is a set of statements, namely one or several premises and a conclusion, usually connected by "therefore".
- An argument here is not a quarrel or fight. Rather it is the verbal expression of a reasoning process.
- Consider this argument

  No pure water is burnable.
  Some Cuyahoga River water is burnable.

  Some Cuyahoga River water is not pure.

  (the horizontal line is short for *therefore*.)

This argument is valid.

A valid (correct, sound) argument is one in which, whenever the premises are true, the conclusion is also true.

Cuyahoga River Fire, Cleveland, Ohio, 1969

- The Cuyahoga River is a river in the United States, located in Northeast Ohio, that feeds into Lake Erie.
- The river is famous for having been so polluted that it "caught fire" in 1969.
- The event spurred the environmental movement in the US.

Logical arguments, valid or invalid

- Let us take another argument:

  No pure water is burnable.
  Some Cuyahoga River water is not burnable.

  Some Cuyahoga River water is pure water.

This argument is invalid (incorrect, unsound).
(The whole Cuyahoga river could be polluted by non-burnables.)

Note: Logic studies forms of reasoning. The content might deal with anything - water purity, mathematics, cooking, nuclear physics, ethics, or whatever. When we learn logic, we are learning tools of reasoning that can be applied to any subject.

Logical arguments

Example of a logical argument (A)

1. If the demand rises, then companies expand.
2. If companies expand, then companies hire workers.

---

3. If the demand rises, then companies hire workers.

The argument consists of two premises (1, 2) and the conclusion (3).

If, whenever the premises are accepted as true it follows that the conclusion is also true, we say that the conclusion logically follows from the premises, or that the argument is valid (correct, sound).

IMPORTANT NOTES

- The fact that an argument is valid does not mean that the conclusion is necessarily true. For example, in argument (A) one can argue against the conclusion, and claim that it is false.
- Such a situation (valid argument, but false conclusion) occurs if one of the premises is false (in argument (A), premise 2 could be false, with companies developing AI solutions instead of hiring workers, to meet rising demand)
- The fact that an argument is valid only guarantees that if all premises are true then the conclusion is true.
- Argument validity does not say anything about the conclusion in the cases when at least one of the premises is false.
- In other words, the conclusion being false does not necessarily prove that an argument is invalid. An argument where one of the premises is false and the conclusion is also false could still be valid (the "if" statement above is vacuously true.)

Logical arguments

Example of logical argument (B)

1. This computer program has a bug, or the input is erroneous.
2. The input is not erroneous.

---

3. This computer program has a bug.

- Statements can be atomic - cannot be further subdivided - (““This computer program has a bug.”), or compound (“The input is not erroneous.”).
- Compound statements consist of several parts, each of which is a statement in its own right.
- In Example (B), premise 1: “this computer program has a bug”, “the input is erroneous” are connected by or.

Some important logical arguments

To see which arguments are correct and which not, we abbreviate the essential statements by substituting letters p, q, r.

The letter p may express the statement that “demand rises”,
The letter q may express the statement “companies expand”,
The letter r may express the statement “companies hire workers”

Then the logical argument in Example (A) becomes:

1. If p then q.
2. If q then r.

---

3. If p then r.

This type of argument is called a hypothetical syllogism.

Some important logical arguments

1. p or q.
2. Not q.

---

3. p.

Example:

1. It is either day or night.
2. It is not night.

---

3. It is day.

The above type of argument is called disjunctive syllogism.

1. If p then q.
2. p.

---

3. q.

Example:

1. If it rains I will get wet.
2. It is raining.

---

3. I will get wet.

The above type of argument is called modus ponens.

Some important logical arguments (contd.)

1. If p then q.	Example:	1. If I do all my chores I will get a puppy.
2. Not q.		2. I did not get a puppy.
3. Not p.		3. I did not do all my chores.

The above type of argument is called modus tollens.

Propositions

Definition: Any statement that is either true or false is called a proposition.

Meaningless statements, commands or questions are not propositions.

- p, q, r are called propositional symbols.
- True (denoted by 1) and false (denoted by 0) are propositional constants.
- Any propositional symbol can be assigned the value 1 or 0.

Proposition examples

Which of the following sentences are propositions? What are the truth values of those that are propositions?

1. Waterloo is the capital of Ontario.
2. Montreal is the capital of Canada.
3. $2 + 3 = 5$.
4. $5 + 7 = 10$.
5. $x + 2 = 11$.
6. Answer this question.
7. $x + y = y + x$ for every pair of real numbers x and y.
8. Do not pass go.
9. What time is it?

Atomic and compound propositions

- Propositional symbols are atomic propositions, that is, they cannot be further subdivided
- Compound propositions are obtained by combining several atomic propositions
- The function of the words or, and, not, if-then is to combine propositions, and they are therefore called logical connectives.

Logical connectives

Statements formulated in natural languages are frequently ambiguous because the words can have more than one meaning. We want to avoid this. Therefore we introduce new mathematical symbols to take the role of connectives.

Convention: Stating a proposition in English implies that this proposition is true.

"it is true that cats eat fish" $=$ "cats eat fish".

Similarly, if p is a proposition, then "p" means "p is true" or that "p holds".

Logical connective: Negation (not)

Definition Let p be a proposition. The compound proposition $\neg p$, pronounced "not p", is the proposition that is true when p is false, and that is false when p is true.

- $\neg p$ is called the negation of p.
- The connective $\neg$ may be translated into English as "It is not the case that," or simply by the word "not".

Truth table for negation

p	$\neg p$
1	0
0	1

Logical connective: Conjunction (and)

Definition. Let p and q be two propositions. The proposition $p \wedge q$ is true when both p and q are true, and false otherwise.

- $p \wedge q$ is called the conjunction of p and q.
- the connective $\wedge$ is pronounced "and" and may be translated by the English word "and".

Truth table for conjunction

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

When writing truth tables, we will use the convention that rows of 0s and 1s are written in decreasing lexicographic order whereby $1 > 0$,

$$11 > 10 > 01 > 00,$$

$$111 > 110 > 101 > 100 > 011 > 010 > 001 > 000\ldots \text{ etc.}$$

Observations on conjunction

- In English we often use shortcuts that are not allowed in logic statements.
  "He eats and drinks." really means "He eats, and he drinks."
- In logic, every statement must have its own subject (e.g., "he") and its own predicate (e.g., "drinks")!
- Taking p: "He eats." and q: "He drinks.", our sentence becomes $p \wedge q$
- Sometimes we use words other than "and" to denote a conjunction such as but, in addition to, and moreover
  Example: "I love icecream, but I also love chocolate", etc
- Not all instances of the word "and" denote conjunctions.
  Example: The word "and" in "Jack and Jill are cousins" is not a conjunction at all!

Logical connective: Disjunction (inclusive or)

Definition. Let p and q be two propositions. The proposition $p \vee q$ is true when either p, or q, or both p and q are true, and is false when both p and q are false.

- $p \vee q$ is called the disjunction of p and q
- the connective $\vee$ is pronounced "or" and can usually be translated into English by the word "(inclusive) or"

Truth table for disjunction

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Observations on disjunction

The English word "or" has two different meanings.

- Exclusive or: "You can either have soup or salad" means you can have soup or salad, but not both
- Inclusive or: "The computer has a bug, or the input is erroneous"
- To avoid ambiguity one should translate $p \vee q$ into English as inclusive or, that is, "p or q, or both"
- Conversely, whenever translating from English into logic, "p or q" should be translated as $p \vee q$ (unless the text explicitly states "p or q, but not both")

Note: When performing the disjunction of two sentences, always make sure that the sentences are complete: each sentence must have its own subject and predicate. "There was an error on line 15 or 16" must first be expanded to "There was an error on line 15, <u>or</u> there was an error on line 16".

Logical connective: Implication (if-then)

Definition. Let p and q be two propositions. Then $p \to q$ is false when p is true and q is false, and true otherwise.

- $p \to q$ is called the implication (or conditional) of p and q.
- The implication of p and q may be translated into English by using the "If...then" construct, as in "If p, then q", or to " It is not the case that p is true and q is false"
- $p \to q$ means that, whenever p is correct, so is q.
- p is called antecedent, q is called consequent

The truth table for implication

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Observations on implication

Generally, if p is false, then "$p \to q$" is vacuously true, since in such case the verification of "if p then q" does not require doing anything to deduce q from p.

Although unusual, this yields no inconsistency with everyday speech.

Example: "If the sun will rise from the West, I will eat my hat."

My statement will never be contradicted (and in that sense it is true) because I know that "the sun will rise from the West" is false.

Equivalent ways of expressing implication

- The following are logically equivalent:

1. $p \rightarrow q$.
2. If p then q.
3. Whenever p, then q.
4. p is sufficient for q.
5. p only if q.
6. p implies q.
7. q if p.
8. q whenever p.
9. q is necessary for p.
10. q is implied by p.

Equivalent ways of expressing implication

Example: Try understanding that the equivalences on the previous slide using the number theory example wherein n is a natural number in $\mathbb{N} = \{0, 1, 2, \ldots, \}$ and

- p stands for "n is divisible by 6"
- q stands for "n is divisible by 3".

Notably,

- "p only if q" is translated as "$p \to q$"
- "p if q" is translated as "$q \to p$"
- "p is sufficient for q" is translated as "$p \to q$"
- "p is necessary for q" is translated as "$q \to p$"

Logical connective: Equivalence (equivalent to)

Definition. Let p and q be two propositions. Then $p \leftrightarrow q$ is true whenever p and q have the same truth values.

- The proposition $p \leftrightarrow q$ is called equivalence (or biconditional),
- It is pronounced "p if and only if q",
- One often uses iff as an abbreviation for "if and only if"
- In the equivalence "p iff q" ($p \leftrightarrow q$), "only if" is the direct implication $p \rightarrow q$, and "if" is the converse implication $q \rightarrow p$.

Truth table for equivalence

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Sorting out equivalence and implication

One should always be aware of the difference between equivalence and implication. In English, it is not always clear which connective is intended, as seen in the example below.

Eating hamburgers at a fast-food bar is equivalent to aiding the destruction of the world's rainforest.

This sentence looks like an equivalence, $\leftrightarrow$, but if we swap the sentence around

Aiding the destruction of the rainforest is equivalent to eating hamburgers at a fast-food bar.

we can see that something is wrong.

In fact, the intended meaning is implication $\rightarrow$, not equivalence $\leftrightarrow$:
If one eats hamburgers at a fast-food bar then one is aiding the destruction of the world's rainforest.

Ambiguity and imprecision

Logic helps to clarify the meanings of descriptions written, for example, in English. After all, one reason for our use of logic is to state precisely the requirements of computer systems.

Descriptions in natural languages can be ambiguous or imprecise.

- An ambiguous sentence can have more than one distinct meaning.

- In contrast, an imprecise or vague sentence has only one meaning, but, as a proposition, the distinction between the circumstances under which is true and the circumstances under which it is false is not clear-cut.

Ambiguous sentences: Examples

- David and John from Toronto are coming for a visit.
  Who is from Toronto? David or John or both? It is impossible to know without further information.
- I know a much funnier man than Bill.
  This may have two meanings: I know a much funnnier man than Bill does, or I know a much funnier man than Bill is.
- Don't leave animals in cars because they rapidly turn into ovens.(From *News Quiz*, BBC Radio 4, 10 October, 1994).
  The immediate reading is far from the intended meaning.

Imprecise sentences: Examples

- John is tall.
  We do not know exactly what tall means. A more precise description is John is over 2 meters tall.
- This computer is fast.
  The meaning of "fast" is imprecise - fast compared to what? A more precise description would be This computer executes 2 million instructions per second.

Dealing with imprecision and ambiguity

- An ambiguous sentence usually has several interpretations. Ambiguity has to be eliminated by querying the author of the sentence or by examining the context.
- Imprecision or vagueness arises from the use of qualitative descriptions. Often we need to introduce some quantitative measures to remove vagueness.

Further remarks on connectives

- $\neg$ is the only unary connective, that is, $\neg p$ negates a single proposition.
- All other connectives are binary connectives (they require two propositions which are joined by the connective)
- The binary connectives $\vee, \wedge, \leftrightarrow$ are symmetric, in the sense that the order of the two propositions joined by the connective does not affect the truth value of the resulting propositions.
  The truth value of $p \wedge q$ is the same as the truth value of $q \wedge p$.
- The connective $\rightarrow$ is not symmetric: $p \rightarrow q$ and $q \rightarrow p$ have different truth values.

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A riddle

Suppose the following two statements are true:

(1) I love Betty or I love Jane.

(2) If I love Betty then I love Jane.

Does it necessarily follow that I love Betty? Does it necessarily follow that I love Jane?

A puzzle: The island of knights and knaves

There is an island in which certain inhabitants, called knights always tell the truth, and others, called knaves always lie. It is assumed that every inhabitant of this island is either a knight or a knave.

Suppose A says, "Either I am a knave or B is a knight."

What are A and B?

Translations from English into logic

(a) Write the truth table for "exclusive or".
(b) Give the truth tables for $p \wedge p$ and $p \vee p$.
(c) Translate the following statements into logic formulas:

1. He is clever and diligent.
2. He is clever but not diligent.
3. He didn't write the letter, or the letter was lost.
4. He must study hard, otherwise he will fail.
5. He will fail, unless he studies hard.
6. He will go home, unless it rains.
7. He will go home only if it rains.
8. If it rains, he will be at home; otherwise, he will go to the market or school.
9. The sum of two integers is even if both integers are even or both integers are odd.

Translations from English into Logic

(d) Translate the following statements into logic formulas.

1. It snows whenever the wind blows from the northeast.
2. The apple trees will bloom if it stays warm for a week.
3. That the Toronto Maple Leafs win the Stanley Cup implies that they beat the Montreal Canadiens.
4. It is necessary to walk 3,500 meters to get to the top of Mount Everest.
5. To get promoted to physics Full Professor at Waterloo, it is sufficient to win the Nobel Prize.
6. If you drive more than 800 kilometers, you will need to buy gasoline.
7. Your guarantee is good only if you bought your iPhone less than 90 days ago.

Learning Objectives

- What is logic? What is propositional logic?
- Know the definitions of (logical) argument, valid (correct, sound) argument, invalid (incorrect, unsound) argument.
- Know the definitions and notations for: (logic) proposition, propositional symbol, propositional constant, atomic proposition, compound proposition, (logical) connective.
- Know the definitions, notations, and the truth tables that define the action of the five standard logical connectives: negation (*not*), conjunction (*and*), disjunction (*inclusive or*), implication (*if-then*, conditional), equivalence (*equivalent to*, biconditional).
- Use appropriate notation to "translate" English language statements into (atomic or compound) logic propositions with the same meaning.