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1 Syntax vs. semantics, truth valuations and truth tables

2 Satisfiability, tautologies, and contradictions

3 Proving argument validity (or invalidity) semantically: Tautological consequence $\models$

4 Proving tautological consequence by truth tables

5 General method for proving argument validity (semantically)
Syntax vs. semantics

- **Syntax** is concerned with the rules used for constructing the formulas in $\text{Form}(L^p)$.
- This is similar to computer science, where the term **syntax** refers to the rules governing the composition of well-formed expressions in a programming language.
- The logic equivalent of a “syntax error” is an expression in $L^p$ that does not belong to $\text{Form}(L^p)$.
- **Semantics** is concerned with meaning:
  - Atoms (proposition symbols) are intended to express simple propositions (sentences), which can be true or false;
  - The connectives take their intended meanings: $\neg, \land, \lor, \rightarrow, \leftrightarrow$ express “not”, “and”, “(inclusive) or”, “if, then”, and “iff”;
  - The “meaning” of a non-atomic formula, that is, its truth value (true or false) is derived from the truth values of its constituent atomic formulas, and the meanings (definitions) of connectives.
Example

Before finding the “meaning” of a non-atomic formula (its truth value) the formula must be parsed; that is, all subformulas of the formula must be found.

Example: If you take a class in computers and if you do not understand recursion, you will not pass.

We want to know exactly when this statement is true and when it is false.

Define:

\( p \): “You take a class in computers.”
\( q \): “You understand recursion.”
\( r \): “You pass.”

The statement becomes \((p \land \neg q) \rightarrow \neg r\)
“Meaning” of a formula - truth table

Truth table for \((p \land \lnot q) \rightarrow \lnot r\)

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There are two identical twins, Jenny and Susan. One of them is a knave (always lies) and one is a knight (always tells the truth). You don’t know which one is which. You meet one of them and you want to find out if she is Jenny or Susan by asking one yes/no question. What is the question and why does it work?

**Question** (for the person you met): “Is Susan a knave?”

**Claim:** The answer to this question is “yes” iff you met Jenny.

**Hint.** Take $J$ to be “Jenny is a knight”, and similarly for Susan, and $M_J$ to be “You met Jenny”. Determine the truth value of “The person you met answers yes to the question Is Susan a knave?” based on the values of $J$, $S$ and $M_J$ (note that $J$ and $S$ cannot have the same truth value, so not all rows of the truth table are needed). For each row, compare this truth value to the truth value of $M_J$. 


Truth valuations and truth tables

• Fix a set \{0, 1\} of truth values. We interpret 0 as false and 1 as true.

• Definition. A truth valuation is a function \( t : \text{Atom}(\mathcal{L}^p) \rightarrow \{0, 1\} \), with the set of all proposition symbols as domain and \{0, 1\} as range.

• Convention: For \( A \in \text{Atom}(\mathcal{L}^p) \) we denote by \( A^t \) the value \( t(A) \in \{0, 1\} \) that \( A \) takes under truth valuation \( t \).

• In practice, we restrict the truth valuation to the set of proposition symbols in the formulas under consideration.

• A truth table list the values of a formula under all possible truth valuations: A truth valuation corresponds to a single row in the truth table.
Value of formulas under a truth valuation

Definition. Let \( t \) be a truth valuation. The value of a formula in \( \text{Form}(L^p) \) with respect to the given truth valuation \( t \) is defined recursively as follows:

1. If the formula is a proposition symbol \( p \), then \( p^t \in \{0, 1\} \) is given by the definition of \( t \).

2. \( (\neg A)^t = \begin{cases} 1 & \text{if } A^t = 0 \\ 0 & \text{if } A^t = 1 \end{cases} \)

3. \( (A \land B)^t = \begin{cases} 1 & \text{if } A^t = B^t = 1 \\ 0 & \text{otherwise} \end{cases} \)

4. \( (A \lor B)^t = \begin{cases} 1 & \text{if } A^t = 1 \text{ or } B^t = 1 \text{ (or both)} \\ 0 & \text{otherwise} \end{cases} \)

5. \( (A \rightarrow B)^t = \begin{cases} 1 & \text{if } A^t = 0 \text{ or } B^t = 1 \text{ (or both)} \\ 0 & \text{otherwise} \end{cases} \)

6. \( (A \leftrightarrow B)^t = \begin{cases} 1 & \text{if } A^t = B^t \\ 0 & \text{otherwise} \end{cases} \)
Example

Suppose $A$ is the formula $p \lor q \rightarrow q \land r$, and $t$ is a truth valuation such that $p^t = q^t = r^t = 1$.

Then we have $(p \lor q)^t = 1$, $(q \land r)^t = 1$ and therefore $A^t = 1$.

Suppose $t_1$ is another truth valuation, $p^{t_1} = q^{t_1} = r^{t_1} = 0$.
Then we have $(p \lor q)^{t_1} = 0$, $(q \land r)^{t_1} = 0$ and therefore $A^{t_1} = 1$.

If $t_2$ is yet another value truth valuation, with $p^{t_2} = 1$ and $r^{t_2} = q^{t_2} = 0$, then $A^{t_2} = 0$.

The above example illustrates that, for a particular formula, its value under one truth valuation may (or may not) differ from its value under a different truth valuation.
1. Syntax vs. semantics, truth valuations and truth tables

2. Satisfiability, tautologies, and contradictions

3. Proving argument validity (or invalidity) semantically:
   Tautological consequence $\models$

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5. General method for proving argument validity (semantically)
**Satisfiability**

**Definition.** We say that a truth valuation \( t \) satisfies a formula \( A \) in \( \text{Form}(\mathcal{L}^p) \) iff \( A^t = 1 \).

We use the capital Greek letter \( \Sigma \) to denote any set of formulas.

**Definition.** The value of a set of formulas \( \Sigma \) under truth valuation \( t \) is defined as:

\[
\Sigma^t = \begin{cases} 
1 & \text{if for each formula } B \in \Sigma, \ B^t = 1, \\
0 & \text{otherwise}
\end{cases}
\]

**Definition.** A set of formulas \( \Sigma \subseteq \text{Form}(\mathcal{L}^p) \) is satisfiable iff there exists a truth valuation \( t \) such that \( \Sigma^t = 1 \). If, in the other hand, there is no truth valuation \( t \) such that \( \Sigma^t = 1 \) (or, equivalently, if \( \Sigma^t = 0 \) for all truth valuations \( t \)), then the set \( \Sigma \) is called **unsatisfiable**.
Observations

1. If for a truth valuation $t$ we have that $\Sigma^t = 1$, then $t$ is said to satisfy $\Sigma$, and $\Sigma$ is said to be satisfied by (under) $t$.

2. Note that $\Sigma^t = 1$ means that under the truth valuation $t$, all the formulas of $\Sigma$ are true.

3. On the other hand, $\Sigma^t = 0$ means that for at least one formula $B \in \Sigma$, we have that $B^t = 0$.

4. In particular, $\Sigma^t = 0$ does not necessarily mean that $C^t = 0$ for every formula $C$ in $\Sigma$. 
Sudoku as a Satisfiability (SAT) Problem

In a 4 x 4 Sudoku puzzle, each value in \{1, 2, 3, 4\} must appear exactly once in each row, column, and 2x2 block.

Write a formula that requires all the following to be true:

- (a) Solution must be consistent with the starting grid
- (b) At most one digit per square
- (c) In each row, each digit must appear exactly once
- (d) In each column, each digit must appear exactly once
- (e) In each block, each digit must appear exactly once
Sudoku as SAT problem

Set up proposition symbols \( v_{ijk} \), \( 1 \leq i, j, k \leq 4 \), so that:

- \( v_{ijk} = 1 \), iff the cell at position \((i, j)\) equals number \(k\), and
- \( v_{ijk} = 0 \), otherwise

Claim. A solution to a Sudoku puzzle exists iff the formula obtained by taking the conjunction of all formulas defined below is satisfiable:

- (a) **Consistency with starting grid:**
  If the cell \((1, 1)\) has digit 3 in it, then add to the conjunction the formula \( v_{113} \), etc. Do not add anything for unfilled cells.

- (b) **At most one digit per cell:** For every cell \((i, j)\), and each pair of different digits \(k, k'\), add to the conjunction the formula \( (v_{ijk} \rightarrow \neg v_{ijk'}) \)
(c) **In each row each digit appears exactly once:**
- **“at least once”:** For each row \( i \) and digit \( k \), add the formula
  \[
  (v_{i1k} \lor v_{i2k} \lor v_{i3k} \lor v_{i4k})
  \]
- **“at most once”:** Look at every pair of cells in a row, \((i,j)\) and \((i,j')\), \( j \neq j' \), and require that they do not both contain \( k \) by adding \((v_{ijk} \rightarrow \neg v_{ij'k})\)

(d) **In each column each digit appears exactly once:** Like rows, but fixes the column \( j \) and digit \( k \).

(e) **In each block, each digit appears exactly once:** Same pattern as rows and columns, but the row and column indexes must vary over the cells within a given block.
Tautologies and contradictions

- **Definition.** A formula $A$ is a **tautology** iff it is true under all possible truth valuations, i.e. iff for any truth valuation $t$, we have that $A^t = 1$.

- **Definition.** A formula $A$ is a **contradiction** iff it is false under all possible truth valuations, i.e., iff for every truth valuation $t$, we have that $A^t = 0$.

- **Definition.** A formula that is neither a tautology nor a contradiction is called **contingent**.
Consider the formula \( \neg(p \land q) \lor q \). Is this formula a tautology?

The truth table below shows that the given formula is a tautology:

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<thead>
<tr>
<th></th>
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<th>( p \land q )</th>
<th>( \neg(p \land q) )</th>
<th>( \neg(p \land q) \lor q )</th>
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Important tautology

Law of the excluded middle ("tertium non datur") states that $p \lor \neg p$ is a tautology. In other words, $p$ is either true or false, everything else is excluded.

The truth table below proves this law by showing that $p \lor \neg p$ is a tautology:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$p \lor \neg p$</th>
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<tbody>
<tr>
<td>1</td>
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</table>
If $A$ is a tautology that contains the proposition symbol $p$, one can determine a new expression by replacing all instances of $p$ by an arbitrary formula. The resulting formula $A'$ is also a tautology.

For example, $p \lor \neg p$ is a tautology. Replace all instances of $p$ by any formula we like, say by $p \land q$. The resulting formula $A' = (p \land q) \lor \neg(p \land q)$ is again a tautology.

**Theorem.** Let $A$ be a tautology and let $p_1, p_2, \ldots, p_n$ be the proposition symbols of $A$. Suppose that $B_1, B_2, \ldots, B_n$ are arbitrary formulas. Then, the formula obtained by replacing $p_1$ by $B_1$, $p_2$ by $B_2$, ..., $p_n$ by $B_n$, is a tautology.

**Example.** Use the fact that $\neg(p \land q) \lor q$ is a tautology to prove that $\neg((p \lor q) \land r) \lor r$ is a tautology.

**Solution:** Replace in the original formula $p$ by $p \lor q$, and $q$ by $r$. 
Important contradiction

Law of contradiction: “Nothing can both be, and not be”, that is, \( \neg(p \land \neg p) \) is a tautology; equivalently, \((p \land \neg p)\) is a contradiction.

The truth table below shows that the formula \((p \land \neg p)\) is a contradiction:

<table>
<thead>
<tr>
<th></th>
<th>(p)</th>
<th>(\neg p)</th>
<th>(p \land \neg p)</th>
<th>(\neg(p \land \neg p))</th>
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Contradictions are related to tautologies: \(A\) is a tautology if and only if \(\neg A\) is a contradiction.
Three essential laws of thought - Plato

1. **Law of identity:**
   “Whatever is, is.” \( p = p \)

2. **Law of contradiction:**
   “Nothing can both be and not be.” \( \neg(p \land \neg p) \)

3. **Law of excluded middle:**
   “Everything must either be, or not be.” \( (p \lor \neg p) \)

\( \neg(p \land \neg p) \)
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Logical arguments: valid or invalid

Logical argument:

Premise 1
Premise 2
...
Premise \( n \)

Conclusion

Logical arguments can be
- Correct (valid, sound)
- Incorrect (invalid, unsound)

Note: Any proved mathematical theorem is a valid logical argument, with several assumptions (hypotheses, premises) and a conclusion.
Formalizing the notion of argument validity: Tautological consequence (denoted by $\vdash$)

**Definition** Suppose $\Sigma \subseteq \text{Form}(\mathcal{L}^p)$ and $A \in \text{Form}(\mathcal{L}^p)$.
$A$ is a **tautological consequence** of $\Sigma$ (that is, of the formulas in $\Sigma$), written as $\Sigma \vdash A$, iff for any truth valuation $t$, we have that $\Sigma^t = 1$ implies $A^t = 1$.

**Observations**

- $\vdash$ is not a symbol of the formal propositional language and $\Sigma \vdash A$ is not a formula.
- $\Sigma \vdash A$ is a statement (in the metalanguage) about $\Sigma$ and $A$.
- We write $\Sigma \not\vdash A$ for “not $\Sigma \vdash A$”.
- If $\Sigma \vdash A$, we say that the formulas in $\Sigma$ (tauto)logically imply formula $A$. 
A special case:  $\emptyset \models A$

When $\Sigma$ is the empty set, we obtain the important special case of tautological consequence, $\emptyset \models A$

By definition, $\emptyset \models A$ means that the following holds:

“For any truth valuation $t$, if $\emptyset^t = 1$ then $A^t = 1$.”

where $\emptyset^t = 1$ means “For any $B$, if $B \in \emptyset$ then $B^t = 1$”

Because $B \in \emptyset$ is false, “$\emptyset^t = 1$” is always (vacuously) true. Consequently, $\emptyset \models A$ means that $A$ is always true (is a tautology).

Intuitively speaking, $\Sigma \models A$ means that the truth of the formulas in $\Sigma$ is a sufficient condition for the truth of $A$.

Since $\emptyset$ has no formulas, $\emptyset \models A$ means that the truth of $A$ is unconditional, hence $A$ is a tautology.
Validity of arguments ($\models$) and satisfiability

Let $\Sigma = \{ A_1, A_2, \ldots, A_n \} \subseteq \text{Form}(\mathcal{L}_p)$ be a set of formulas (premises) and $C \in \text{Form}(\mathcal{L}_p)$ be a formula (conclusion). The following are equivalent:

- The argument with premises $A_1, A_2, \ldots, A_n$ and conclusion $C$ is valid.
- $C$ is a tautological consequence of $\Sigma$, i.e. $\{ A_1, A_2, \ldots, A_n \} \models C$.
- The formula $(A_1 \land A_2 \land \cdots \land A_n) \rightarrow C$ is a tautology.
- The formula $(A_1 \land A_2 \land \cdots \land A_n \land \neg C)$ is a contradiction.
- The formula $(A_1 \land A_2 \land \cdots \land A_n \land \neg C)$ is unsatisfiable.
- The set $\{ A_1, A_2, \ldots, A_n, \neg C \}$ is unsatisfiable.
Observations on valid arguments

- Consider an argument with premises $A_1, A_2, \ldots, A_n$ and conclusion $C$.
- The conclusion $C$ is true, if the following two conditions hold:
  - The argument with premises $A_1, A_2, \ldots, A_n$ and conclusion $C$ is valid (sound, correct),
  - The premises $A_1, A_2, \ldots, A_n$ are all true.
- The validity of an argument does not guarantee the truth of the conclusion.
- Only when the argument is valid AND the premises are all true, is the conclusion guaranteed to be true.
- In other words, sound reasoning (argument validity) only guarantees the truth of the conclusion if, in addition to being valid, the argument is based on true premises (assumptions).
Tautological equivalence (denoted by $\models$)

Definition: For two formulas we write

$$A \models B$$

to denote “$A \models B$ and $B \models A$.”

(Here, the sets of premises each consists of a single formula.)

$A$ and $B$ are said to be tautologically equivalent (or simply equivalent) iff $A \models B$ holds.

Tautologically equivalent formulas are assigned the same truth values by any truth valuation.

Note: Tautological equivalence is weaker than equality of formulas. For example, if $A = \neg(p \land q)$ and $B = (\neg p \lor \neg q)$ then $A \models B$, as can be proved by a truth table, but $A \neq B$. 
The meaning of “(tauto)logically implies” ($A \models B$) is different from the meaning of the connective “implies” ($A \rightarrow B$).

$A \models B$ if and only if $A \rightarrow B$ is a tautology (always true).
$A \rightarrow B$ is a formula, which can be true or false.
$\emptyset \models A \rightarrow B$ means that $A \rightarrow B$ is a tautology.

The meaning of “(tauto)logically equivalent” ($A \equiv B$), is different from the meaning of the connective “equivalent” ($A \leftrightarrow B$).

$A \equiv B$ if and only if $A \leftrightarrow B$ is a tautology (always true).
$A \leftrightarrow B$ is a formula, which can be true or false.
$\emptyset \models A \leftrightarrow B$ means that $A \leftrightarrow B$ is a tautology.
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5 General method for proving argument validity (semantically)
To prove the tautological consequence $\Sigma \models A$ (that is, to prove the validity of the argument with premises $\Sigma$ and conclusion $A$) we must show that any truth valuation $t$ satisfying $\Sigma$ also satisfies $A$. One way to show this is by using truth tables.

Example: Show that \( \{p \rightarrow q, q \rightarrow r\} \models (p \rightarrow r) \)

The premises are $A_1 = p \rightarrow q$ and $A_2 = q \rightarrow r$
The conclusion is $p \rightarrow r$. 
Proving that \( \{ p \rightarrow q, q \rightarrow r \} \models (p \rightarrow r) \)

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Proving argument validity by truth tables

The truth valuations in rows 1, 5, 7, 8 (marked with $\leftarrow$) are all the truth valuations which make all premises true, that is, which satisfy $\Sigma = \{p \rightarrow q, q \rightarrow r\}$. For each of these four truth valuations, the conclusion $p \rightarrow r$ is also true (is satisfied).

This shows that

$$\{p \rightarrow q, q \rightarrow r\} \models (p \rightarrow r)$$

This further means that the argument

Premise 1: $p \rightarrow q$
Premise 2: $q \rightarrow r$

............

Conclusion: $p \rightarrow r$

is a valid argument.
How to prove that an argument is invalid: Give a counterexample

Example: Prove that $(p \rightarrow q) \lor (p \rightarrow r) \nmid p \rightarrow (q \land r)$.

Solution: Find at least one row in the truth table in which the premises are true but the conclusion is false.

The row in the truth table that corresponds to the truth valuation $t$ which assigns $p^t = 1$, $q^t = 1$, $r^t = 0$ is one such counterexample.

- Note that several such truth valuations (counterexamples) may exist.
- We only need one such truth valuation (that makes all premises true but the conclusion false), in order to prove that an argument is not valid.
Problem with the truth table method:
Truth table size

If the formula has $n$ proposition symbols and $m$ occurrences of connectives:

- How many rows does the truth table have? $2^n$
- How many columns does the truth table have? $\leq n + m$

We need another method for proving argument validity when the number of proposition symbols is too large.
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Proving validity without truth tables

We use the proof method called “proof by contradiction”. 

ATTENTION: Please differentiate the construct “proof by contradiction”, from the specific usage of the word “contradiction” in propositional logic, wherein it defines a formula that is always false.

Example: Show that \( \{A \rightarrow B, B \rightarrow C\} \vdash (A \rightarrow C) \).

Proof: Assume the contrary, that is \( \{A \rightarrow B, B \rightarrow C\} \nvdash (A \rightarrow C) \).

This means that there is a truth valuation \( t \) that makes all premises true but the conclusion false, that is,

\[
\begin{align*}
(1) \quad (A \rightarrow B)^t &= 1, \\
(2) \quad (B \rightarrow C)^t &= 1, \\
(3) \quad (A \rightarrow C)^t &= 0.
\end{align*}
\]
Example \( \{ A \rightarrow B, B \rightarrow C \} \models (A \rightarrow C) \) cont’d

Recall our assumption that:

1. \((A \rightarrow B)^t = 1\),
2. \((B \rightarrow C)^t = 1\),
3. \((A \rightarrow C)^t = 0\).

By (3), we have that \(A^t = 1\) and \(C^t = 0\) \(\text{(4)}\)

By (1) and the fact that \(A^t = 1\), we have \(B^t = 1\) \(\text{(5)}\)

From \(B^t = 1\) and (2), we deduce \(C^t = 1\), which contradicts (4).

As we reached a contradiction, our assumption that the argument was invalid was false, hence the opposite is true: The argument is valid.
A patient is administered three different tests.

Test $A$ will give a positive result if and only if either virus $X$ or virus $Y$ is present. Test $B$ will give a positive result if and only if virus $Y$ or $Z$ is present. If test $C$ is positive, virus $Y$ can be excluded. The patient reacts positively to all three tests.

Prove that the patient has virus $X$ and virus $Z$ but not virus $Y$.

*Hint.* Take propositional symbol $a$ to denote “Test $A$ gives a positive result”, and similarly for $b$ and $c$. Take propositional symbol $x$ to denote “Virus $X$ is present”, and similarly for $y$ and $z$. Translate the text into propositional logic, as a logical argument with four premises and one conclusion. Use “proof by contradiction” to show that the argument is valid.
Argument invalidity without truth tables

To prove $\Sigma \nvdash A$, we must construct a counterexample: A truth valuation $t$ satisfying $\Sigma$ but not satisfying $A$.

Example: Show that

$\{(p \rightarrow \neg q) \lor r, q \land \neg r, p \leftrightarrow r\} \nvdash (\neg p \land (q \rightarrow r))$.

Proof: Find a counterexample (using any method you can think of).

Let $t$ be the truth valuation $p^t = 0, q^t = 1$ and $r^t = 0$.

Then we have

$((p \rightarrow \neg q) \lor r)^t = 1$

$(q \land \neg r)^t = 1$.

$(p \leftrightarrow r)^t = 1$

$(\neg p \land (q \rightarrow r))^t = 0$.

We found a counterexample (a truth valuation that makes all premises true but the conclusion false), hence the argument is invalid.
Important (tautolog)ical equivalences —
De Morgan’s Laws

Consider the following two statements:

It is not true that he is informed and honest.
He is either not informed, or he is not honest.

Intuitively, these two statements are logically equivalent.

We prove this now. Define $p$ and $q$ to be the statements that “he is informed” and that “he is honest” respectively.

The first statement translates into $\neg(p \land q)$, whereas the second into $\neg p \lor \neg q$. 
De Morgan’s Laws

De Morgan’s law: \( \neg(p \land q) \equiv (\neg p \lor \neg q) \)

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<th>( p )</th>
<th>( q )</th>
<th>( p \land q )</th>
<th>( \neg(p \land q) )</th>
<th>( \neg p \lor \neg q )</th>
<th>( \neg(p \land q) \leftrightarrow (\neg p \lor \neg q) )</th>
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Dual De Morgan’s Law: \( \neg(p \lor q) \equiv (\neg p \land \neg q) \)

De Morgan’s Laws are used to negate conjunctions and disjunctions, and show how to distribute \( \neg \) over \( \land \), and over \( \lor \).

To negate a conjunction, take the disjunction of the negations of the conjuncts.

To negate a disjunction, take the conjunction of the negations of the disjuncts.
Consider the following pair of statements about natural numbers:

If a number \( n \) is even, then \( n^2 \) is even.
If \( n^2 \) is not even, then \( n \) is not even.

Intuitively, these two statements are logically equivalent.

If \( p \) stands for “The number \( n \) is even” and \( q \) stands for “The number \( n^2 \) is even” respectively, then these two statements translate into \( p \rightarrow q \) and \( \neg q \rightarrow \neg p \).

**Definition:** Given an implication of the form \((p \rightarrow q)\), the formula \((\neg q \rightarrow \neg p)\) is called the contrapositive of \((p \rightarrow q)\), and the formula \((q \rightarrow p)\) is called the converse of \((p \rightarrow q)\).
The table below shows that contrapositives are equivalent, that is

\[ p \to q \iff \neg q \to \neg p \]

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Why “proof by contradiction” works

- Consider a theorem \((A_1 \land A_2 \land \ldots \land A_n) \rightarrow C\), where \(A_1, \ldots A_n\) are the hypotheses, and \(C\) is the conclusion.

- Instead of a “direct proof” we can use the fact that an implication is equivalent to its contrapositive, that is,

\[
(A_1 \land A_2 \land \ldots \land A_n) \rightarrow C \iff \neg C \rightarrow \neg (A_1 \land A_2 \land \ldots \land A_n)
\]

In other words, if by assuming that the conclusion is not true (\(\neg C\) holds) this leads to contradicting one of the hypotheses (\(\neg A_i\) holds, for some \(i\)), then this proves the original theorem.

- In fact, we do not even need to contradict one of the hypotheses, as reaching any logical contradiction suffices, due to:

\[
(A \rightarrow C) \iff (\neg A \lor C) \iff (A \land \neg C) \iff ((A \land \neg C) \rightarrow (p \land \neg p))
\]

where \(A = (A_1 \land A_2 \ldots \land A_n)\).
Logic humour

Rene Descartes (1596-1650), French mathematician and philosopher, famous for his statement “I think, therefore I am.”

Rene Descartes is sitting in a bar, having a drink. The bartender asks him if he would like another. “I think not,” he says, and disappears in a puff of smoke.

What is this joke funny?
Converse vs. contrapositive

- Suppose we have an implication of the form \((P \rightarrow Q)\).
- Its converse is \((Q \rightarrow P)\), and its contrapositive is \((\neg Q \rightarrow \neg P)\).
- Contrapositives are equivalent, \((P \rightarrow Q) \equiv (\neg Q \rightarrow \neg P)\), but the converse of an implication is NOT equivalent to it.
- In the joke, let \(T\) denote ‘‘I think’’ and \(A\) denote ‘‘I am’’.
- Stating ‘‘I think, therefore I am’’ becomes \((T \rightarrow A) = 1\).
- ‘‘If I think not, then I vanish (I am not)’’ is \((\neg T \rightarrow \neg A) \equiv A \rightarrow T\), which is the converse of the initial implication, therefore not equivalent to it.
- Thus, from \((T \rightarrow A) = 1\) we cannot deduce \((\neg T \rightarrow \neg A) = 1\), which is why the joke is funny.
- Note: The contrapositive of ‘‘I think, therefore I am,’’ \((T \rightarrow A)\), is ‘‘If I am not, then I think not,’’ which is equivalent to it.
Important (tauto)logical equivalences - biconditional

Consider the following two statements:
(1) \( p \) and \( q \) have the same truth value
(2) If \( p \), then \( q \), and if \( q \) then \( p \).

The first statement becomes \( p \iff q \), the second \( (p \rightarrow q) \land (q \rightarrow p) \).

The table below shows that \( p \iff q \vdash (p \rightarrow q) \land (q \rightarrow p) \):

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<th>( p )</th>
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<th>( p \iff q )</th>
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<th>( (p \rightarrow q) \land (q \rightarrow p) )</th>
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How to use this in proofs: If we have to prove \( A \iff B \) (\( A \) iff \( B \)), we must prove both the direct implication \( A \rightarrow B \) (the “only if” part), and the converse implication \( B \rightarrow A \) (the “if” part).
Lemma If $A \models A'$ and $B \models B'$, then

1. $\neg A \models \neg A'$.
2. $A \land B \models A' \land B'$.
3. $A \lor B \models A' \lor B'$.
4. $A \rightarrow B \models A' \rightarrow B'$.
5. $A \leftrightarrow B \models A' \leftrightarrow B'$. 
Theorems: Replaceability and Duality

Theorem (Replaceability of tautologically equivalent formulas) Let $A$ be a formula in $\text{Form}(\mathcal{L}^p)$ which contains a subformula $B$. Assume that $B \vdash C$, and let $A'$ be the formula obtained by simultaneously replacing in $A$ some (but not necessarily all) occurrences of the formula $B$ by formula $C$. Then $A' \vdash A$.

Proof By structural induction.

Theorem (Duality) Suppose $A$ is a formula in $\text{Form}(\mathcal{L}^p)$ composed only of atoms and the connectives $\neg$, $\lor$, $\land$, by the formation rules concerned these three connectives. Suppose $\Delta(A)$ results from simultaneously replacing in $A$ all occurrences of $\land$ with $\lor$, all occurrences of $\lor$ with $\land$, and each atom with its negation. Then $\Delta(A) \vdash \neg A$.

Proof By structural induction.

Example. Let $A = (p \land \neg q) \land (\neg r \land s)$. Find $\neg A$, the negation of $A$. 

Propositional Language - Semantics 

CS245, University of Waterloo Canada
Fuzzy Logic: An alternative approach

Truth values are real numbers in the interval \([0, 1]\)

“true” = 1, “false” = 0, “partially true” = a number between 0 and 1

- \(\text{AND}(x, y) = \min\{x, y\}\)
- \(\text{OR}(x, y) = \max\{x, y\}\)
- \(\text{NOT}(x) = 1 - x\)

If the values of \(x\) and \(y\) are 0 or 1, these definitions coincide with the definitions of classical logic connectives \(\land, \lor, \neg\).

Does the law of excluded middle hold? **No.**
For \(p = 0.2\), we have \(p \lor \neg p = \max\{0.2, 0.8\} = 0.8 \neq 1\).

Does the law of contradiction hold? **No.**
For \(p = 0.2\), we have \(p \land \neg p = \min\{0.2, 0.8\} = 0.2 \neq 0\).
Learning Objectives

- Know the difference between syntax and semantics
- Know the definitions of: truth valuation, truth table, satisfiable formula, unsatisfiable formula, tautology, contradiction
- Know the definition of tautological consequence, $\models$, and its connection to argument validity
- Be able to prove semantically that an argument is valid (i.e., the conclusion is a tautological consequences, $\models$, of the premises):
  - Using the truth table method
  - Using a more general method, such as “proof by contradiction” (the use of direct proof is also allowed)
  - Be able to prove semantically that an argument is invalid, by giving a counterexample
- Know De Morgan’s Laws, and the definition of the converse and the contrapositive of an implication
- Know the Replaceability Theorem and the Duality Theorem