1 Structural Induction

1. **Problem**: Use induction on the structure of propositional formulas to prove the following (Refer to Exercise 2.2.2 in Lu).

   Let $A$ be a formula of $\text{Form}(\mathcal{L}_P)$.
   Let $m_A$ be the number of proposition symbols in $A$.
   Let $n_A$ be the number of occurrences of the binary connectives $\land$, $\lor$, $\to$, $\leftrightarrow$ in $A$.
   Then $m_A = n_A + 1$.

   **Remark**: The solution uses a check mark (✓) to indicate where a mark is typically assigned in the marking scheme for a similar question about structural induction on an assignment or an exam.

   **Solution**: Let $R(A)$ be the property that $m_A = n_A + 1$ ✓ (given the above definitions for $m_A$ and $n_A$).

   **Basis** ($A$ is $p$, for some proposition symbol $p$):
   For this formula $A = p$, we have $m_A = 1$ and $n_A = 0$. Therefore,
   
   $$m_A = 1 = 0 + 1 = n_A + 1, \checkmark$$
   
   as required.

   The induction step has two sub-cases.

   A is $(\neg B)$, for some formula $B \in \text{Form}(\mathcal{L}_P)$:
   The induction hypothesis is $R(B)$, i.e. $m_B = n_B + 1$. ✓ Also, the formula has $m_A = m_B$ and $n_A = n_B$. Therefore,
   
   $$m_A = m_B = n_B + 1 = n_A + 1, \checkmark$$
   
   as required.

   A is $(B \star C)$, for some formulas $B, C \in \text{Form}(\mathcal{L}_P)$, and some binary connective $\star$:
   The induction hypothesis is that $m_B = n_B + 1$ ✓ and $m_C = n_C + 1$. ✓ Also, we have $m_A = m_B + m_C$ and $n_A = n_B + n_C + 1$. Therefore,
   
   $$m_A = m_B + m_C = (n_B + 1) + (n_C + 1) = n_A + 1, \checkmark$$
   
   as required.
By the Principle of Structural Induction ✓, we have shown that R(A) holds for all formulas A of Form(ℒ^P).

2 Semantics of Propositional Logic

1. Problem: Give the complete truth table for each of the following formulas. For each formula, state whether it is a tautology, a contradiction, or satisfiable.

(a) \((p \to (\neg p))\)

Solution:

\[
\begin{array}{c|c|c}
 p & \neg p & (p \to (\neg p)) \\
 1 & 0 & 0 \\
 0 & 1 & 1 \\
\end{array}
\]

This formula is satisfiable because the 2nd truth valuation in the truth table satisfies it; it is not a tautology because the 1st truth valuation in the truth table does not satisfy it; it is not a contradiction because 2nd truth valuation in the truth table satisfies it.

(b) \(((p \to q) \land r)\)

Solution:

\[
\begin{array}{c|c|c|c|c}
 p & q & r & (p \to q) & ((p \to q) \land r) \\
 1 & 1 & 1 & 1 & 1 \\
 1 & 1 & 0 & 1 & 0 \\
 1 & 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 1 & 1 & 1 \\
 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

This formula is satisfiable because the 1st truth valuation in the truth table satisfies it; it is not a tautology because the 2nd truth valuation in the truth table does not satisfy it; it is not a contradiction because 1st truth valuation in the truth table satisfies it.

(c) \(((p \land r) \lor ((\neg r) \to q))\)

Solution:
This formula is satisfiable because the 1st truth valuation in the truth table satisfies it; it is not a tautology because the 4th truth valuation in the truth table does not satisfy it; it is not a contradiction because 1st truth valuation in the truth table satisfies it.

(d) the propositional formula \( A \), which equals \( ((p \rightarrow (q \land r)) \land (\neg q \land \neg r)) \rightarrow \neg p \)

Solution:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>(p \land r)</th>
<th>((\neg r) \rightarrow q)</th>
<th>((p \land r) \lor ((\neg r) \rightarrow q))</th>
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The formula \( A \) is a tautology because all truth valuations satisfy it; it is satisfiable because at least one truth valuation satisfies it, and it is not a contradiction because there is a truth valuation satisfies it.

2. *(IAAs: likely skip this question, for time)* There is an island in which certain inhabitants, called *knights*, always tell the truth, and others, called *knaves*, always lie. It is assumed that every inhabitant of this island is either a knight or a knave.

Someone asks X: “Are you a knight?”

X replies: “*If I am a knight, then I will eat my hat.*”

Prove that X will eat his hat.

Solution: Define proposition symbols

- \( p \)  X is a knight.
- \( q \)  X will eat his hat.

Then X’s statement translates to the formula \( (p \rightarrow q) \).

Let \( t \) be a truth valuation. We prove by cases:

(a) \( X \) is a knight. Therefore \( p^t = 1 \) and \( (p \rightarrow q)^t = 1 \) (because X tells the truth). By the rule for the value of a \( \rightarrow \)-formula, this further implies \( q^t = 1 \). Thus in this case \( X \) has to eat his hat.

(b) \( X \) is a knave. Therefore \( p^t = 0 \) and \( (p \rightarrow q)^t = 0 \) (because X lies). But then by the
rule for the value of a $\rightarrow$-formula, $(p \rightarrow q)^f = 1$ no matter what the value of $q^f$ is. Hence we have reached a contradiction (because $(p \rightarrow q)^f = 0 = 1$). Therefore this case cannot occur.

Consequently, only Case ?? can hold, that is, $X$ will eat his hat.

3 Valid and Invalid Arguments

1. Translate the following argument in the language of propositional logic by using the given proposition symbols.
   Determine, with proof, whether the argument is valid (sound).

Premise 1 – If knowing is a state of mind (like feeling a pain), then I could always tell by introspection whether I know.

Premise 2 – If I could always tell by introspection whether I know, then I’d never mistakenly think that I know.

Premise 3 – I sometimes mistakenly think that I know.

Conclusion – Therefore, knowing isn’t a state of mind.

Define proposition symbols
- $p$ Knowing is a state of mind.
- $q$ I could always tell by introspection whether I know.
- $r$ I sometimes mistakenly think that I know.

Solution With the given notation, our premises and conclusion translate as

Premise 1: $(p \rightarrow q)$
Premise 2: $(q \rightarrow \neg r)$
Premise 3: $r$

Conclusion: $(\neg p)$

We can prove that the argument is valid, for example, by using a truth table:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$(p \rightarrow q)$</th>
<th>$(q \rightarrow \neg r)$</th>
<th>$(\neg p)$</th>
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By observation, we note that in all rows where all three premises are true (in this case there is only such row, namely row 7), the conclusion is also true. This completes the proof that the argument is valid.

2. Determine the validity of the following argument:

Premise 1 – If the dog is barking, then the dog is not in the house.

Premise 2 – If the dog is in the house, then someone is at the front door if the dog is barking.

Premise 3 – A necessary condition for the dog to be in the house is that the dog is barking.
Conclusion – Therefore, someone is not at the front of the house unless the dog is barking.

**Solution:** Define the following proposition symbols:

- **p:** The dog is barking.
- **q:** The dog is in the house.
- **r:** Someone is at the front door.

Translating the argument into propositional logic, we obtain:

**Premises:**

- \( p \rightarrow \neg q \)
- \( q \rightarrow (p \rightarrow r) \)
- \( q \rightarrow p \)

**Conclusion:** \( \neg p \rightarrow \neg r \)

We shall try to assign truth values to the proposition symbols so that the premises are true and the conclusion is false. In order for \( \neg p \rightarrow \neg r \) to be false, \( p \) must be false and \( r \) true. (A conditional statement is false if and only if the antecedent is true and the consequent false.) Moreover, we see that if \( q \) is false, all three premises are true (as implications with false antecedents). Thus, the one line of the truth table

<table>
<thead>
<tr>
<th>( p )</th>
<th>( r )</th>
<th>( q )</th>
<th>( (p \rightarrow \neg q) )</th>
<th>( q \rightarrow (p \rightarrow r) )</th>
<th>( (q \rightarrow p) )</th>
<th>( (\neg p \rightarrow \neg r) )</th>
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shows that it is possible to assign truth values to the proposition symbols in such a way that the premises are true but the conclusion is false. Such a truth value assignment is sometimes called a *counter-example*. Consequently, the argument is **not** valid.

### 4 Tautological Consequence

1. Prove the following tautological consequence.

\[ \{p \rightarrow (q \rightarrow r), q\} \models p \rightarrow r. \]

**Solution 1:** The definition of tautological consequence states that the premises tautologically imply the conclusion iff every truth valuation which satisfies the premises also satisfies the conclusion. We can construct a truth table which lists every possible truth valuation (assignment of true or false to each proposition symbol) and check this definition. We see below that whenever all of the premises are true, the conclusion \( p \rightarrow r \) is also true (see rows marked with *). Therefore the premises tautologically imply \( p \rightarrow r \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( (q \rightarrow r) )</th>
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**Solution 2** *(Present only if time permits):*

- Let \( \Sigma \) be \( \{p \rightarrow (q \rightarrow r), q\} \).
- I claim that \( \Sigma \models p \rightarrow r \).
- Let \( t \) be any truth valuation such that \( \Sigma^t = 1 \).
- We have the following cases for \( p^t \).
- $p^t = 0$:
  * Then by $\to$ properties, $(p \to r)^t = 1$.
- $p^t = 1$:
  * Since $\Sigma^t = 1$, therefore $(p \to (q \to r))^t = 1$.
  * Then by $\to$ properties, therefore $(q \to r)^t = 1$.
  * Since $\Sigma^t = 1$, therefore $q^t = 1$.
  * Then by $\to$ properties, therefore $r^t = 1$.
  * Then by $\to$ properties, $(p \to r)^t = 1$.

• In either case, $(p \to r)^t = 1$.
• This completes the proof.

2. The left side of the $\models$ relation can be very large, even infinite.
   (a) Show that
   \[
   \{(p_i \to p_{i+1}) : i \in \mathbb{N}\} \models (p_{77} \to p_{79}).
   \]
   **Solution:** We will prove this by arguing over all possible truth valuations. Let $t$ be any truth valuation such that for every $i \in \mathbb{N}$, $(p_i \to p_{i+1})^t = 1$.
   Now, if $(p_{77})^t = 0$, then $(p_{77} \to p_{79})^t = 1$ regardless of the value of $(p_{79})^t$.
   Otherwise, if $(p_{77})^t = 1$, since $(p_{77} \to p_{79})^t = 1$, it must be that $(p_{78})^t = 1$. Similarly, since $(p_{78})^t = 1$ and $(p_{79} \to p_{79})^t = 1$, $(p_{79})^t = 1$. Lastly, since $(p_{77})^t = 1$ and $(p_{79})^t = 1$, therefore $(p_{77} \to p_{79})^t = 1$.
   Hence, for every truth valuation $t$ such that for every $i \in \mathbb{N}$, we have $(p_i \to p_{i+1})^t = 1, (p_{77} \to p_{79})^t = 1$. Thus from the definition of tautological consequence $\{(p_i \to p_{i+1}) : i \in \mathbb{N}\} \models (p_{77} \to p_{79})$.
   (b) Show that for all $j \in \mathbb{N}$,
   \[
   \{(p_i \to p_{i+1}) : i \in \mathbb{N}\} \models (p_j \to p_{j+2}).
   \]
   **Solution:**
   Let $j \in \mathbb{N}$ be arbitrary. Repeat above proof, replacing 77, 78, 79 with $j, j + 1, j + 2$ respectively.

3. Recall that a logical argument is called **valid** if whenever all of its premises are true, its conclusion is also true. Use this definition, and truth tables, to determine whether or not each of the following arguments are valid.
   (a) **Premises:**
   \[
   \Sigma = \{p \to q, r \to s, (r \to p) \lor (\neg s), r \lor (\neg s), s \lor (\neg q)\}
   \]
   **Conclusion:** $p \leftrightarrow r$
   **Solution 1:** We construct the following truth table for all the formulas involved.
p | q | r | s | p → q | r → s | (r → p) ∨ (¬s) | r ∨ (¬s) | s ∨ (¬q) | p ↔ r
---|---|---|---|---|---|---|---|---|---
1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1
1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1
1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1
1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1
1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1
1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1
1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1
0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1
0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1
0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0
0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1
0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1

The only rows that matters for assessing the argument (i.e. the only rows on which all the premise formulas are true) are the first and last ones, marked with *. Since the conclusion formula is also true on those rows, therefore the argument is valid.

**Solution 2 (Present only if time permits):** Let \( t \) be any truth valuation such that \( \Sigma^t = 1 \). We have the following cases for \( p^t \):

- \( p^t = 0 \):
  - Since \( ((r \rightarrow p) \lor (\neg s))^t = 1 \), therefore we have the following sub-cases.
    - \( (r \rightarrow p)^t = 1 \):
      - Then \( r^t = 0 \), by \( \rightarrow \) properties.
      - Therefore \( (p \leftrightarrow r)^t = 1 \), completing this case.
    - \( (\neg s)^t = 1 \):
      - Then \( s^t = 0 \), by \( \neg \) properties.
      - Then since \( (r \rightarrow s)^t = 1 \), therefore \( r^t = 0 \) by \( \rightarrow \) properties.
      - Therefore \( (p \leftrightarrow r)^t = 1 \), completing this case.
    - It follows that \( q^t = s^t = 0 \). It is an exercise to prove this for yourself.

- \( p^t = 1 \):
  - Then since \( (p \rightarrow r)^t = 1 \), therefore \( r^t = 1 \).
  - Therefore \( (p \leftrightarrow r)^t = 1 \), completing this case.
  - It follows that \( q^t = s^t = 1 \). It is an exercise to prove this for yourself.

**Solution 3 (Present only if time permits):** Towards a contradiction, suppose that there exists a truth valuation, \( t \), such that \( \Sigma^t = 1 \) and \( (p \leftrightarrow r)^t = 0 \). Then since \( (p \rightarrow r)^t = 0 \), we have that \( p \) and \( r \) take opposite values under \( t \). We have these cases for \( p^t \):

i. \( p^t = 0 \):
  - Then \( r^t = 1 \).
  - Then \( (r \rightarrow s)^t = 1 \) implies \( s^t = 1 \).
  - Then \( ((r \rightarrow p) \lor (\neg s))^t = 1 \) implies \( (r \rightarrow p)^t = 1 \).
• Therefore $p' = 1$.
• This is a contradiction.

ii. $p' = 1$:
• Then $r' = 0$.
• Then $(r \lor (\neg s))' = 1$ implies $(\neg s)' = 1$, i.e. $s' = 0$.
• Then $(s \lor (\neg q))' = 1$ implies $(\neg q)' = 1$, i.e. $q' = 0$.
• Then $(p \rightarrow q)' = 1$ implies $p' = 0$.
• This is a contradiction.

All possibilities lead to contradiction. This shows that the argument is valid.

(b) **Premises:**

$$\Sigma = \{(\neg p) \rightarrow (q \lor r), (\neg q) \rightarrow ((\neg p) \land s), (s \rightarrow (q \lor r))\}$$

**Conclusion:** $q$

**Solution 1:** We construct the following truth table for all the formulas involved.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>$s$</th>
<th>$\neg p \rightarrow (q \lor r)$</th>
<th>$\neg q \rightarrow ((\neg p) \land s)$</th>
<th>$s \rightarrow (q \lor r)$</th>
<th>$q$</th>
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</table>

The rows that matter for assessing the argument (i.e. the rows on which all the premise formulas are true) are marked with $\ast$ or $\diamond$. The conclusion formula is false on the last such row (the row marked with $\diamond$). The presence of such a row witnesses that the argument is **not valid**.

**Solution 2 (Present only if time permits):** It is enough for us to exhibit a truth valuation, $t$, such that $\Sigma^t = 1$ and $q^t = 0$. Let’s suppose that such a $t$ exists, and determine what properties $t$ must have.

- Since $q^t = 0$, therefore $(\neg q)^t = 1$.
- Then since $((\neg q) \rightarrow ((\neg p) \land s))^t = 1$, by $\rightarrow$ properties, we have $((\neg p) \land s)^t = 1$.
- Then by $\land$ properties, we have $(\neg p)^t = 1$ (i.e. $p^t = 0$) and $s^t = 1$.
- Now since $((\neg p) \rightarrow (q \lor r))^t = 1$, by $\rightarrow$ properties, we have $(q \lor r)^t = 1$.
- Because $q^t = 0$, by $\lor$ properties, we have $r^t = 1$.

We have determined that the truth valuation $p^t = q^t = 0; r^t = s^t = 1$ has the
required properties to witness that the argument is not valid (as we saw in the above truth table).