University of Waterloo
CS 341 Winter 2025
Written Assignment 1

Elena Grigorescu, Mark Petrick, Luke Schaeffer

Due Date: Friday, January 23 at 11:59pm to Crowdmark
All work submitted must be the student's own.

- Make sure to read the Assignments section on the course webpage for instructions on submission and question expectations ("Instructions for Assignments"): `https://student.cs.uwaterloo.ca/~cs341/#Assignments`

Question 1 [12 marks] Asymptotic Notation

For each pair of functions $f(n)$ and $g(n)$, fill in the correct asymptotic notation among Θ, o, and ω in the statement $f(n) \in \sqcup\,(g(n))$. If none of these are appropriate, state "None apply". Formal proofs are not necessary, but provide brief justifications for all of your answers. (The default base in logarithms is 2.)

a) $f(n) = n^3(\log n)^2$ vs. $g(n) = n^2(\log n)^3$

b) $f(n) = n^{341} + 2024^n$ vs. $g(n) = n^{240} + 2025^n$

c) $f(n) = 3^{\log_9 n}$ vs. $g(n) = n^{1/4} + \sqrt{n} + \log n$

d) $f(n) = (\log n)^{\log n}$ vs. $g(n) = n^2$

e) $f(n) = \sum_{i=0}^{n} 2^i$ vs. $g(n) = 3^n$

f) $f(n) = n^3$ vs. $g(n) = (\lceil \frac{n}{2} \rceil - \frac{n}{2})n^3$

Question 2 [12 marks] Recursion Tree

Solve the following recurrence relation, use the recursion tree method. Express your solution in terms of a Θ bound on $T(n)$. Show your work clearly.

- Draw the final tree showing at least 4 levels (including the root and leaves). Show the work done at each node (do not simply give a total for the level).
- Give a mathematical expression for the sum of work in the recursion tree identifying the work done in the base cases and the recursive cases (leave this as a summation) - an induction proof is not required.

- Simplify the expression (show your work) to give a closed form and derive a Θ bound on $T(n)$.

Note: You may assume that n is a power of 3. You may use the Master Theorem to verify your result.

$$T(n) = \begin{cases} 4, & n = 1, \\ 5T(n/3) + n\sqrt{n}, & n > 1. \end{cases}$$

Question 3 [12 marks] (Lucky) Guess and Check

Use induction to verify the following recurrence with the corresponding guess:

a) $T(n) = 3T(\lfloor n/3 \rfloor) + 2n$ for $n > 2$ and $T(n) = 1$ for $n \leq 2$
Guess: $T(n) \in O(n \log n)$.

b) $T(n) = 3T(\lfloor n/3 \rfloor) + 10$ for $n > 2$ and $T(n) = 2$ for $n \leq 2$
Guess: $T(n) \in O(n)$

Clearly indicate the following components: Basis, Induction Hypothesis, Induction Step and Concluding Statement. You should also clearly label where you are using the induction hypothesis in the induction step.
Hint: You may use the fact that $\lfloor n/3 \rfloor \leq (n/3)$ to simplify the floors away.

Question 4 [10 marks] Divide and Conquer I

a) Researchers are often ranked by their h-index which is the maximum integer h such that the researcher has at least h papers that have been cited at least h times. Suppose Professor X has written n papers and paper i has been cited a_i times and you have the papers sorted with $a_1 > a_2 > \cdots > a_n$. Design a $O(\log n)$ time divide-and-conquer algorithm to find Professor X's h-index.

b) Suppose you have two sorted arrays A and B each containing n numbers. Design a divide-and-conquer algorithm to find the median of all the $2n$ numbers in $O(\log n)$ time.

Question 5 [10 marks] Divide and Conquer II

Suppose you are given a set S of n points in the plane where each point is labelled either "red" or "blue". We want to count the number of pairs (r, b) where r is a red point in S and b is a blue point in S, such that r dominates b. Here, we say that r dominates b if r has larger x-coordinate and larger y-coordinate than b. Design a divide-and-conquer algorithm that divides the points in half using the median x-coordinate and solves this problem in $O(n \log n)$ time. Analyze the runtime of your algorithm.

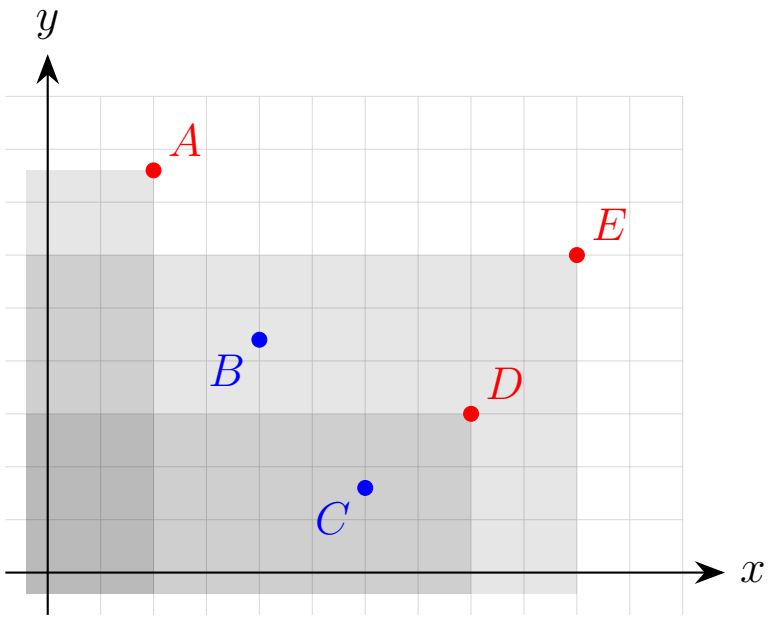

Figure 1: An instance with red points $\{A, D, E\}$ and blue points $\{B, C\}$ where E dominates B and C (and D, technically, but D is red), D dominates C, and A dominates nothing.