**Problem 1 (10 Points) - Asymptotics**

Determine the tightest asymptotic complexity $(O, \Omega, \Theta, o, \omega)$ of each pair of functions. You must prove that the asymptotic you claim is correct.

1. $f(n) = 3n^2 + 4n + 5$ vs. $g(n) = 2n^2 + 6n \log n$
2. $f(n) = 2^n$ vs. $g(n) = n^3$
3. $f(n) = 1000$ vs. $g(n) = \log n$
4. $f(n) = \log^\gamma n$ vs. $g(n) = 2n$, where $\gamma > 0$ is a constant
5. $f(n) = (n-1)!$ vs. $g(n) = n!$

**Problem 2 (10 Points) - Counting 3SUM**

Consider the COUNT3SUM problem defined as follows:

**Input:** an array of $n$ distinct integers $a := [a_1, \dots, a_n]$ and a target number $c \in \mathbb{N}$

**Output:** return the number of *all* triples of indices $(i, j, k)$ such that $i < j < k$ and $a_i + a_j + a_k = c$

Give an algorithm algorithm that solves the COUNT3SUM problem in time $O(n^2)$. Prove that your algorithm is correct and analyze its time complexity.

**Problem 3 (10 Points) - Recurrences**

Solve the following recurrences. Justify each of your answers using either the Master Theorem (when applicable) or a proof of the upper bound using a guess-and-check inductive argument. (You may include a diagram of the Recurrence Tree if you use that method to guess the correct closed-form expression for $T(n)$, but you are not required to do so.)

1. $T(n) = T(n-10) + n$
2. $T(n) = 9 \cdot T(n/3) + n^2$
3. $T(n) = T(n/2) + T(n/3) + n$
4. $T(n) = 5 \cdot T(n/3) + \log^2 n$
5. $T(n) = 2 \cdot T(n-2) + 1$

**Problem 4 (10 Points) - Median of Arrays**

Suppose you have two sorted arrays $A$ and $B$ each containing $n$ numbers. Design a divide-and-conquer algorithm to find the median of all the $2n$ numbers in $O(\log n)$ time.

Prove that your algorithm is correct and that its time complexity is $O(\log n)$.

**Problem 5 (10 Points) - Maximum Ad Space**

An advertising company wants to post a huge poster in Toronto's harbour front. There are $n$ consecutive buildings there, with positive integer height $h_1, \ldots, h_n \in [1, n^{100}]$ and unit width as shown in the figure:

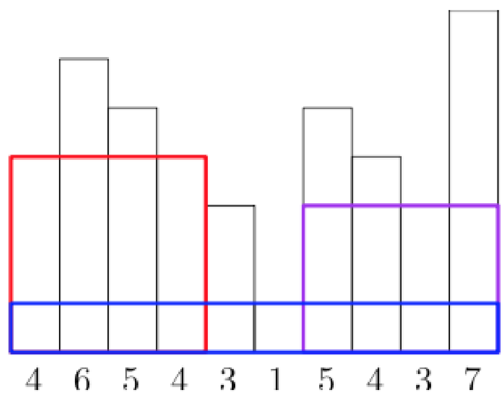


Figure 1: These are three possible solutions. The largest one has area $4 \times 4 = 16$.

Design a divide and conquer algorithm to find the maximum rectangular space to post their poster. Stated mathematically, find $1 \le i \le j \le n$ to maximize

$$(j - i + 1) \cdot \min_{i \le k \le j} h_k.$$

You will get full marks if your algorithm is correct, you present a (correct) proof of correctness, and the runtime is $O(n \log n)$.