CS 341: Algorithms
Lec 09:Minimum Spanning Trees

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Based on lecture notes by Éric Schost

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Spanning trees

Definition:

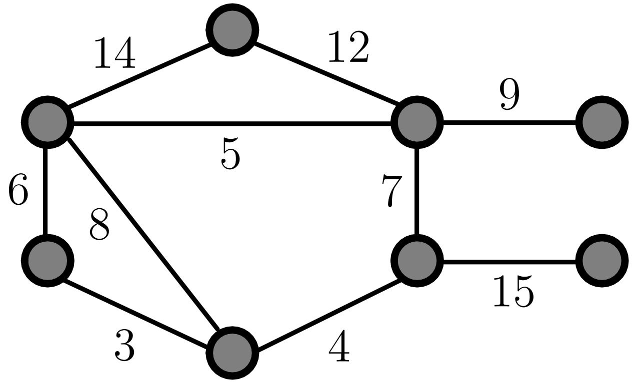
- $G = (V, E)$ is a connected graph
- a **spanning tree** in G is a tree of the form (V, A), with A a subset of E
- in other words: a tree with edges from E that covers all vertices
- examples: BFS tree, DFS tree

Now, suppose the edges have **weights** $w(e_i)$

Goal:

- a spanning tree with **minimal weight**

Example



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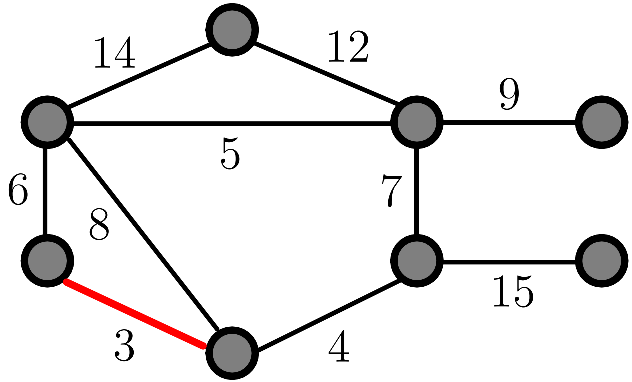

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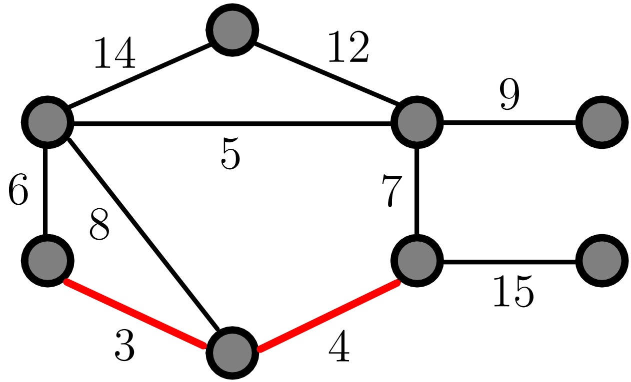

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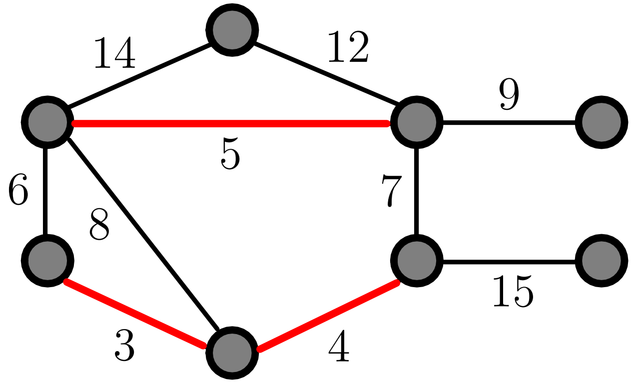

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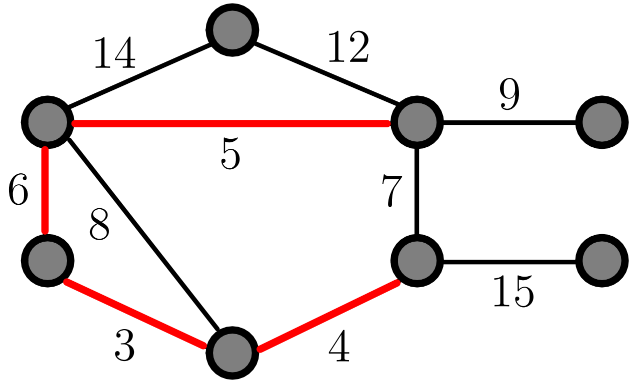

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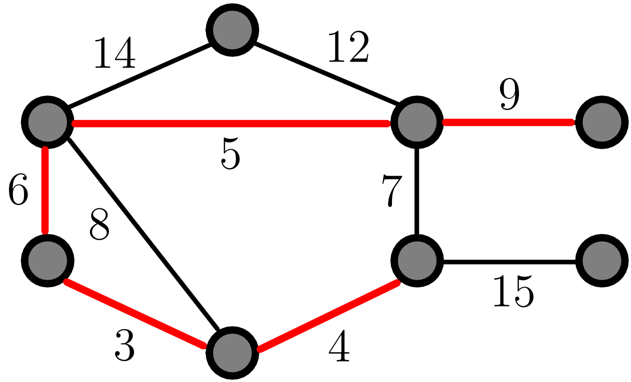

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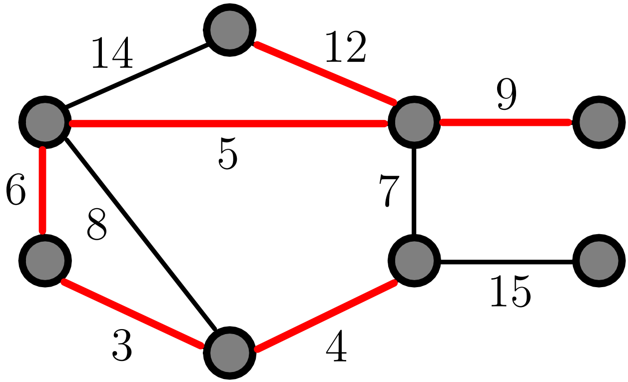

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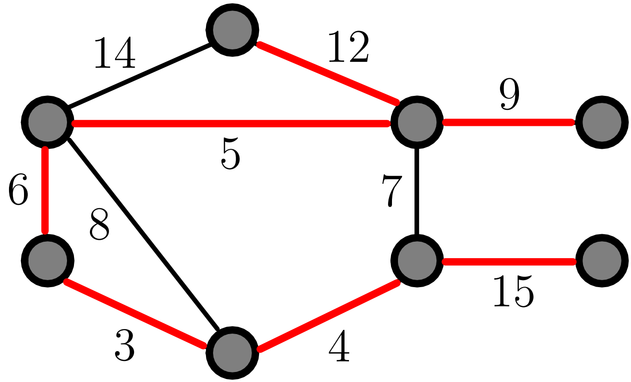

Kruskal's algorithm

GreedyMST(G)

1. $A \leftarrow [\ ]$
2. sort edges by increasing weight
3. **for** $k = 1, \ldots, m$ **do**
4.   **if** e_k does not create a cycle in A **then**
5.   append e_k to A

Augmenting sets without cycles

Claim

Let G be a connected graph, and let A be a subset of the edges of G.

If (V, A) has no cycle and $|A| < n - 1$, then one can find an edge e not in A such that $A \cup \{e\}$ still has no cycle.

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Proof

- in any graph, #vertices $-$ #con. comp. $\leq$ #edges
- for (V, A), this gives $\boldsymbol{n - c < n - 1}$ so $\boldsymbol{c > 1}$
- take any edge on a path that connects two components.

Properties of the output

Claim

If the output is $A = [e_1, \ldots, e_r]$, then (V, A) is a spanning tree (and $r = n - 1$)

Proof:

- of course, (V, A) has no cycle.
- suppose (V, A) is not a spanning tree. Then, there exists an edge e not in A, such that $(V, A \cup \{e\})$ still has no cycle.
- **Case 1:** $\boldsymbol{w(e) < w(e_1)}$. Impossible, since e_1 is the element with the smallest weight.
- **Case 2:** $\boldsymbol{w(e_i) < w(e) < w(e_{i+1})}$. Impossible: at the moment we inserted e_{i+1}, we decided not to include e. This means that e created a loop with $e_1, \ldots, e_i$.
- **Case 3:** $\boldsymbol{w(e_r) < w(e)}$. Impossible: we would have included it in A, since there is no loop in $A \cup \{e\}$.

Exchanging edges

Claim

Let (V, A) and (V, T) be two spanning trees, and let e be an edge in T but not in A.

Then there exists an edge e' in A but not in T such that $(V, T + e' - e)$ is still a spanning tree. **Bonus:** e' is on the cycle that e creates in A.

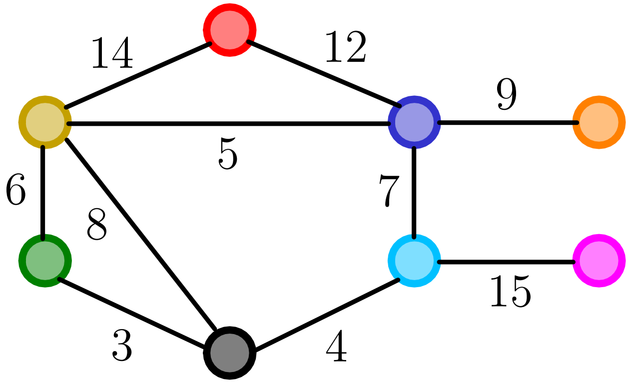
Proof:

- write $e = \{v, w\}$
- $(V, A + e)$ contains a cycle $\boldsymbol{c = v, w, \ldots, v}$
- removing e from T splits $(V, T - e)$ into two connected components T_1, T_2
- $\boldsymbol{c}$ starts in T_1, crosses over to T_2, so it contains another edge e' between T_2 and T_1
- e' is in A, but not in T
- $(V, T + e' - e)$ is a spanning tree

Correctness: exchange argument

- let A be the output of the algorithm
- let (V, T) be **any** spanning tree
- if $T \neq A$, let e be an edge in T but not in A
- so there is an edge e' in A but not in T such that $\boldsymbol{(V, T + e' - e)}$ is a spanning tree, **and** e' is on the cycle that e creates in A
- during the algorithm, we considered e but rejected it, because it created a cycle in A
- all other elements in this cycle have smaller (or equal) weight
- so $\boldsymbol{w(e') \leq w(e)}$
- so $T' = T + e' - e$ has weight $\leq w(T)$, and **one more common element** with A
- keep going

Merging connected sets of vertices



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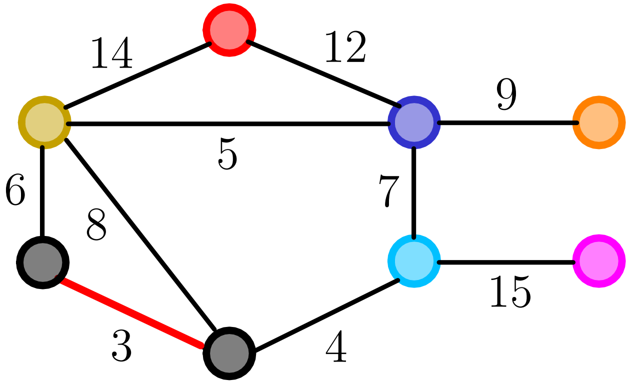

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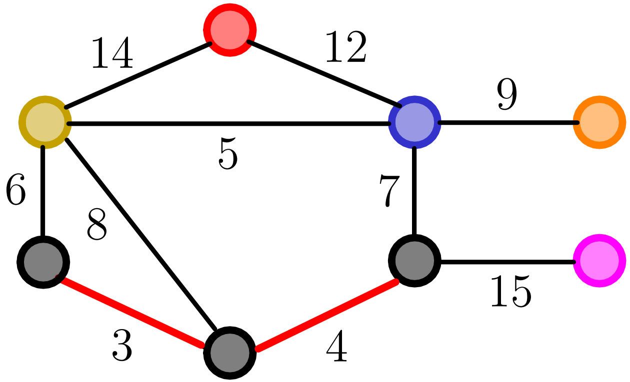

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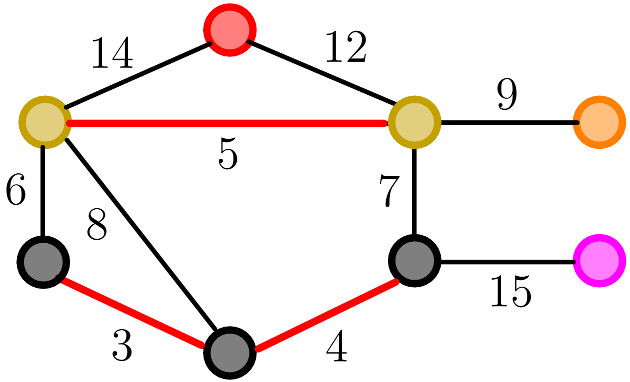

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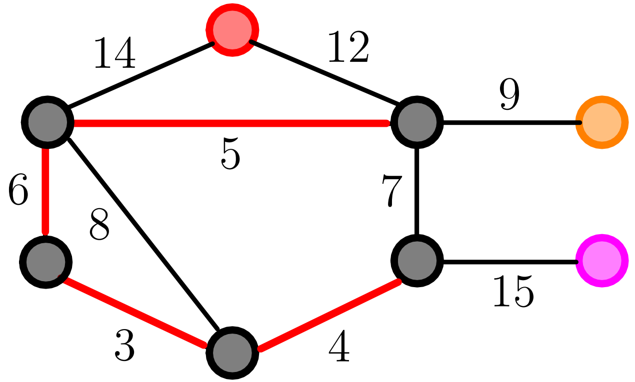

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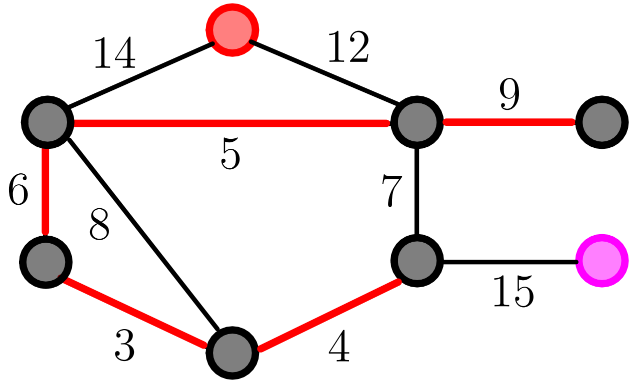

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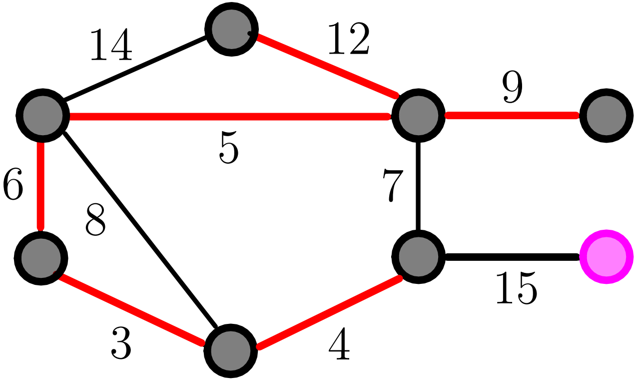

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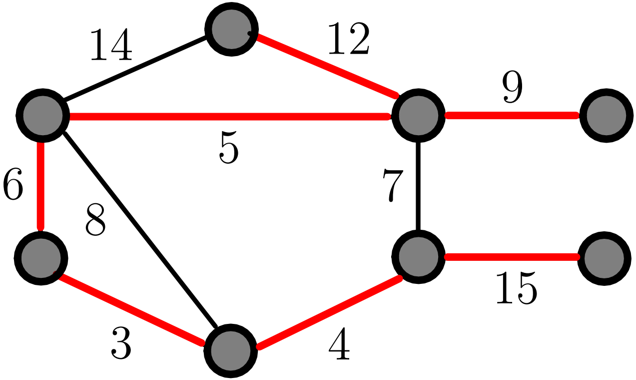


Data structures

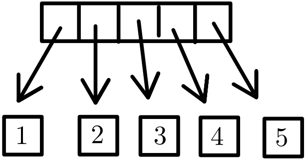
Operations on **disjoint sets of vertices:**

- **Find:** identify which set contains a given vertex
- **Union:** replace two sets by their union

```
GreedyMST_UnionFind(G)
    T ← [ ]
    U ← {{v_1}, …, {v_n}}
    sort edges by increasing weight
    for k = 1, …, m do
        if U.Find(e_k.1) ≠ U.Find(e_k.2) then
            U.Union(U.Find(e_k.1), U.Find(e_k.2))
            append e_k to T
```

An OK solution

- U is an **array of linked lists**



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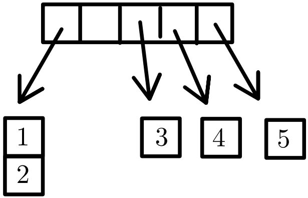

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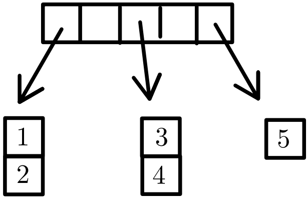

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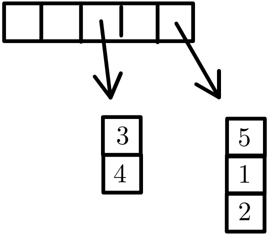

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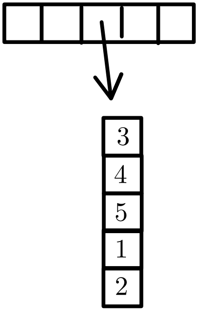

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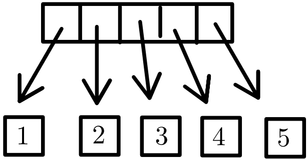

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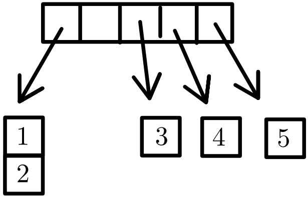

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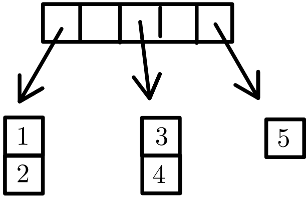

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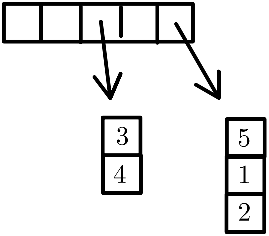

$$X = [5, 5, 3, 3, 5]$$

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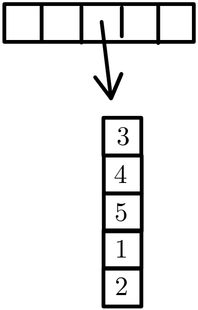

$$X = [3, 3, 3, 3, 3]$$

Analysis

Worst case:

- **Find** is $O(1)$
- **Union** traverses one of the linked lists, updates corresponding entries of X, concatenates two linked lists. Worst case $\Theta(n)$

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Kruskal's algorithm:

- sorting edges $\boldsymbol{O(m\log(m))}$
- $O(m)$ **Find**
- $O(n^2)$ **Union**

Worst case $\boldsymbol{O(m\log(m)+n^2)}$

A simple heuristics for Union

Modified Union

- each set in U keeps track of its size
- only traverse the **smaller list**
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Key observation: worst case for **one** union **still** $\Theta(n)$, but better total time.

- for any given vertex v, the size of the set containing v **at least doubles** when we update $X[v]$
- so $X[v]$ updated at most $\log(n)$ times
- so the **total** cost of union per vertex is $O(\log(n))$

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Conclusion: $\boldsymbol{O(n \log(n))}$ for all unions and $\boldsymbol{O(m \log(m))}$ total