CS 341: ALGORITHMS

Lecture 10: dynamic programming II

Readings: see website

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DP SOLUTION TO 0-1 KNAPSACK
Suppose the optimal solution $O$ does not include this. Then with the $O$ must achieve the best possible value using only items 1-3.

Subproblem: output max value for $\leq 7$kg out of these three items

Problem: output maximum value one can get from taking $\leq 7$kg, out of these four items.

This is a smaller subproblem: reduced # of items

Goal: create recurrence relation to describe optimal solution in terms of subproblems

Let $P[i, m] =$ maximum profit using any subset of the items $1..i$, with weight limit $m$

Note: $P[n, M] (= P[4, 7])$ is the optimal profit

If $O$ does not include the camera, then $P[4, 7] =$ best we can do with the first three items and weight limit $7$kg

That is, $P[4, 7] = P[3, 7]$

What if the camera IS included in $O$?
Suppose the optimal solution $O$ includes this subproblem:

Subproblem: output max value for $\leq 6$kg out of these three items.

Problem: output maximum value one can get from taking $\leq 7$kg, out of these four items.

This is a smaller subproblem: reduced weight and # of items.

Recall: $P[i, m] = \text{maximum profit using any subset of the items } 1..i, \text{ with weight limit } m$

If $O$ includes the camera, then

$$P[4, 7] = p_4 + \text{best we can do with the first three items and weight limit } 7\text{kg} - w_4 = 6\text{kg}$$

That is, $P[4, 7] = p_4 + P[3, 6]$

How to evaluate both possibilities: in & not in $O$?

Then with the remaining $7\text{kg} - w_4 = 6\text{kg}$, and items 1-3, $O$ must achieve the best possible value.
Recall: $P[i, m] = \text{maximum profit using any subset of the items } 1 \ldots i, \text{ with weight limit } m$

<table>
<thead>
<tr>
<th>If O does not include the camera, then $P[4, 7] = \text{best we can do with the first three items and weight limit } 7\text{kg}</th>
<th>P[4, 7] = P[3, 7]</th>
<th>P[i, m] = P[i - 1, m]</th>
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<tbody>
<tr>
<td>If O includes the camera, then $P[4, 7] = p_4 + \text{best we can do with the first three items and weight limit } 7\text{kg} - w_4 = 6\text{kg}</td>
<td>P[4, 7] = p_4 + P[3, 7 - w_4]</td>
<td>P[i, m] = p_i + P[i - 1, m - w_i]</td>
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</table>

**Try both** and take the better result! *(How?)*

$P[4, 7] = \max\{P[3, 7], p_4 + P[3, 7 - w_4]\}$

$P[i, m] = \max\{P[i - 1, m], p_i + P[i - 1, m - w_i]\}$

Note that $\max\{P[i - 1, m], p_i + P[i - 1, m - w_i]\}$ is **only valid** if $i \geq 2$ and $m \geq w_i$

**What to do when $i = 1$ or $m < w_i$?** These are **special cases.**
Special case 3: $i = 1$ and $m < w_i$

Since $i \leq 1$, we can only use item 1.
Since $m < w_i$, we cannot carry item 1.
So, $P[i, m] = 0$.

Special case 2: $i = 1$ and $m \geq w_i$

Since $i \leq 1$, we can only use item 1.
Since $m \geq w_i$, we can carry item 1.
So, $P[i, m] = p_i$.

General case: $i \geq 2$ and $m \geq w_i$

Since $m \geq w_i$, we can carry item $i$.

$$P[i, m] = \max\{P[i - 1, m], p_i + P[i - 1, m - w_i]\}$$

Special case 1: $i \geq 2$ and $m < w_i$

Since $m < w_i$, we cannot carry item $i$.
So, $P[i, m] = P[i - 1, m]$.

Recurrence Relation:

$$P[i, m] = \begin{cases} 
  \max\{P[i - 1, m], p_i + P[i - 1, m - w_i]\} & \text{if } i \geq 2, \ m \geq w_i \\
  P[i - 1, m] & \text{if } i \geq 2, \ m < w_i \\
  p_i & \text{if } i = 1, \ m \geq w_1 \\
  0 & \text{if } i = 1, \ m < w_1.
\end{cases}$$
FILLING THE ARRAY:

No data dependencies on any other array cells.

**i-axis** (can use items in 1..i)

Suppose item 1 does not fit until this **m** value (**m** = **w**₁)

**m-axis** (remaining weight limit)

\[
P[i, m] = \begin{cases} 
\max\{P[i-1, m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2, m \geq w_i \\
P[i-1, m] & \text{if } i \geq 2, m < w_i \\
p_1 & \text{if } i = 1, m \geq w_1 \\
0 & \text{if } i = 1, m < w_1.
\end{cases}
\]
FILLING THE ARRAY:

\[ P[i, m] = \begin{cases} 
\max\{P[i-1, m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2, m \geq w_i \\
\max\{P[i-1, m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2, m < w_i \\
p_i & \text{if } i = 1, m \geq w_1 \\
0 & \text{if } i = 1, m < w_1.
\end{cases} \]

Suppose \( m < w_2 \) from here

- **\( i \)-axis** (can use items in 1..\( i \))
- **\( m \)-axis** (remaining weight limit)

Data dependency: need cell above to be computed already...

\( w_1 \)

\( w_2 \)
**FILLING THE ARRAY:**

\[
P[i, m] = \begin{cases} 
    \max\{P[i - 1, m], p_i + P[i - 1, m - w_i]\} & \text{if } i \geq 2, \ m \geq w_i \\
    P[i - 1, m] & \text{if } i \geq 2, \ m < w_i \\
    p_i & \text{if } i = 1, \ m \geq w_1 \\
    0 & \text{if } i = 1, \ m < w_1.
\end{cases}
\]

- **𝑖-axis** (can use items in 1..𝑖)

Where is slot \([𝑖 − 1, 𝑚 − 𝑤_𝑖]\)?

Consider this entry where \(m \geq w_2\)

Data dependency: need this to be computed already

So, what value should be stored in this entry?

\[\max\{p_1, p_2 + 0\}\]

\(w_2\)

- **𝑚-axis** (remaining weight limit)
**FILLING THE ARRAY:**

- **i-axis** (can use items in $1..i$)
  - We only ever look at the previous row!

- **m-axis** (remaining weight limit)
  - Depending how many zeros we have in the top row, and how far back we’re looking, might start to get cells containing $\max\{p_1, p_2 + p_1\}$

---

**Formula:**

\[
P[i, m] = \\
\begin{cases} 
\max\{P[i - 1, m], p_i + P[i - 1, m - w_i]\} & \text{if } i \geq 2, \ m \geq w_i \\
\max\{P[i - 1, m], p_i\} & \text{if } i \geq 2, \ m < w_i \\
p_1 & \text{if } i = 1, \ m \geq w_1 \\
0 & \text{if } i = 1, \ m < w_1.
\end{cases}
\]

---

Would the following fill-order work?

\[
\text{for } (i = 1..n), \text{ for } (m = M..0)
\]
Suppose we have profits 1, 2, 3, 5, 7, 10, weights 2, 3, 5, 8, 13, 16, and capacity 30.

The following table is computed:

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 2 | 0 | 0 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
| 3 | 0 | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
| 4 | 0 | 0 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  |

\[ P[3, 16] = \text{?} \]

? What do you think?
EXERCISE

Suppose we have profits 1, 2, 3, 5, 7, 10, weights 2, 3, 5, 8, 13, 16, and capacity 30.

The following table is computed:

| i-axis (items) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1             |   | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2             |   | 0 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3             |   | 0 | 1 | 2 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 4             |   | 0 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 8 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| 5             |   | 0 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 8 | 8 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| 6             |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

$P[3, 16] = \max\{P[2, 16], P[2, 11] + 3\} = \max\{3, 3 + 3\} = 6.$
Recall: To satisfy data dependencies, we can fill entries in the order:
for \( i = 1 \ldots n \), for \( m = 0 \ldots M \)

\[
P[i, m] = \begin{cases} 
\max\{P[i-1, m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2, \ m \geq w_i \\
P[i-1, m] & \text{if } i \geq 2, \ m < w_i \\
p_1 & \text{if } i = 1, \ m \geq w_1 \\
0 & \text{if } i = 1, \ m < w_1.
\end{cases}
\]

```java
Knapsack01(p[1..n], w[1..n], M)
    P = new table[1..n][0..M]

    // base cases where i=1
    for m = 0..M
        if m < w[1] then
            P[1][m] = 0
        else
            P[1][m] = p[1]

    // general cases where i>=2
    for i = 2..n
        for m = 0..M
            if m < w[i] then
                P[i][m] = P[i-1][m]
            else
                P[i][m] = max(P[i-1][m], p[i] + P[i-1][m-w[i]])

    return P[n][M]
```

Read & return optimal profit

How about the optimal items?
The optimal solution is computed by tracing back through the table. For the previous example, consisting of profits 1, 2, 3, 5, 7, 10, weights 2, 3, 5, 8, 13, 16, and capacity 30, the optimal solution is ???

| Items you can take | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|--------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1                  | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2                  | 0 | 0 | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 3                  | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 4                  | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 7 | 7 | 8 | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| 5                  | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 7 | 7 | 8 | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 11 | 12 | 12 | 13 | 14 | 14 | 15 | 15 | 15 | 15 |
| 6                  | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 7 | 7 | 8 | 8 | 9 | 10 | 10 | 11 | 11 | 11 | 11 | 12 | 12 | 13 | 14 | 14 | 15 | 15 | 16 | 17 |

- **8 > 6 so O must take item 4**
- **Same profit using items 1..4 or 1..5. So, there exists an optimal solution O that does not use item 5! Consider O.**
- **Best profit for remaining items + weight**
- **18 > 17, so any optimal solution must take item 6**
- **Start at optimal profit remaining weight = 14**
- **Exercise: continue, and determine which other items are in O**
The optimal solution is computed by tracing back through the table.

For the previous example, consisting of profits 1, 2, 3, 5, 7, 10, weights 2, 3, 5, 8, 13, 16, and capacity 30, the optimal solution is \([1, 1, 0, 1, 0, 1]\).

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</table>
Knapsack01_Items(p[1..n], w[1..n], M, P)

x = new array[1..n]
i = n
m = M

while i > 1
    if P[i][m] == P[i-1][m]
        x[i] = 0
        i = i - 1
    else
        x[i] = 1
        m = m - w[i]
        i = i - 1
x[1] = (P[i][m] > 0) ? 1 : 0
return x

Runtime given P?
Θ(n)
Is this linear time?
More on this soon...
Complexity of the Algorithm

Suppose we assume the unit cost model, so additions / subtractions take time $O(1)$.

The complexity to construct the table is $\Theta(nM)$

Is this a polynomial-time algorithm, as a function of the size of the problem instance?

We have

$$\text{size}(I) = \log_2 M + \sum_{i=1}^{n} \log_2 w_i + \sum_{i=1}^{n} \log_2 p_i.$$ 

Note in particular that $M$ is exponentially large compared to $\log_2 M$. So constructing the table is not a polynomial-time algorithm, even in the unit cost model.

What would the complexity of a recursive algorithm be?

So the DP alg is faster when there are many item types, but small profit values.

Huge $n$ is fine, but $M$ should be in $\text{poly}(n)$ to get an asymptotic improvement.

DP takes $\Theta(nM)$ time, which could be $\Theta(n2^n)$ for huge $M$.

$n$ must be very small.

A recursive algorithm would take $\sim\Theta(2^n)$ time.
**SIMPLIFYING BASE CASES**

$$P[i, m] = \begin{cases} 
\max\{P[i-1, m], p_i + P[i-1, m-w_i]\} & \text{if } i \geq 2, m \geq w_i \\
P[i-1, m] & \text{if } i \geq 2, m < w_i \\
p_i & \text{if } i = 1, m \geq w_1 \\
0 & \text{if } i = 1, m < w_1.
\end{cases}$$

For $i = 1, m < w_i$, we have $P[i-1, m]$ which is 0.

For $i = 1, m \geq w_i$, we have $p_i + P[i-1, m-w_i]$ which is $p_i + 0$. 

---

**$i$-axis** (can use items in 1...$i$)

**$m$-axis** (remaining weight limit)
We get much simpler code!

Compare:

```python
Knapsack01(p[1..n], w[1..n], M)
    P = new table[0..n][0..M] containing zeros

    for i = 1..n
        for m = 0..M
            if m < w[i] then
                P[i][m] = P[i-1][m]
            else
                P[i][m] = max(P[i-1][m],
                               p[i] + P[i-1][m-w[i]])

    return P[n][M]
```
We never look at $P[i-2][...].$

Just keep two arrays representing $P[i]$ and $P[i-1]$

Space complexity changes from $O(mn)$ to $O(m)$
DP SOLUTION TO COIN CHANGING
Problem 5.2
Coin Changing

Instance: A list of coin denominations, \(1 = d_1, d_2, \ldots, d_n\), and a positive integer \(T\), which is called the target sum.

Find: An \(n\)-tuple of non-negative integers, say \(A = [a_1, \ldots, a_n]\), such that \(T = \sum_{i=1}^{n} a_i d_i\) and such that \(N = \sum_{i=1}^{n} a_i\) is minimized.

What subproblems should be considered?

What table of values should we fill in?

In 0-1 knapsack, we only considered two subproblems in our recurrence: taking an item, or not.

Here we can do more than use a coin denomination or not.

What do you think?
Let $N[i, t]$ denote the optimal solution to the subproblem consisting of the first $i$ coin denominations $d_1, \ldots, d_i$ and target sum $t$.

<table>
<thead>
<tr>
<th>Exploring: some sensible base case(s)?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General case:</strong></td>
</tr>
<tr>
<td>What are the different ways we could use coin denomination $d_i$?</td>
</tr>
<tr>
<td>What subproblems / solutions should we use?</td>
</tr>
<tr>
<td><strong>Final recurrence relation</strong></td>
</tr>
</tbody>
</table>

Let $N[i, t]$ denote the optimal solution to the subproblem consisting of the first $i$ coin denominations $d_1, \ldots, d_i$ and target sum $t$. Since $d_1 = 1$, we immediately have $N[1, t] = t$ for all $t$.

Also $N[i, 0] = 0$ for all $i$.

General case:
What are the different ways we could use coin denomination $d_i$?
What subproblems / solutions should we use?

Final recurrence relation
Let $N[i, t]$ denote the optimal solution to the subproblem consisting of the first $i$ coin denominations $d_1, \ldots, d_i$ and target sum $t$. Since $d_1 = 1$, we immediately have $N[1, t] = t$ for all $t$.

For $i \geq 2$, the number of coins of denomination $d_i$ is an integer $j$ where $0 \leq j \leq \lfloor t/d_i \rfloor$.

If we use $j$ coins of denomination $d_i$, then the target sum is reduced to $t - jd_i$, which we must achieve using the first $i - 1$ coin denominations.

Thus we have the following recurrence relation:

$$N[i, t] = \begin{cases} 
\min\{j + N[i - 1, t - jd_i] : 0 \leq j \leq \lfloor t/d_i \rfloor\} & \text{if } i \geq 2 \\
 t & \text{if } i = 1 \text{ OR } t = 0
\end{cases}$$

Also $N[i, 0] = 0$ for all $i$.  

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FILLING THE ARRAY $N[1 \ldots n, 0 \ldots T]$:

No data dependencies on any other array cells.

$i$-axis (coin type)
(recall: $N[i, t]$ uses coin types $1 \ldots i$)

$t$-axis (target sum remaining)

$$N[i, t] = \begin{cases} \min \{j + N[i - 1, t - jd_i] : 0 < j < |t/d_i|\} & \text{if } i > 2 \\ t & \text{if } i = 1.\\ \text{OR } t = 0 \end{cases}$$
FILLING THE ARRAY $N[1 \ldots n, 0 \ldots T]$: 

$i$-axis (coin type)  
(recall: $N[i, t]$ uses coin types $1 \ldots i$)

No data dependencies on any other array cells.

$t$-axis (target sum remaining)

$$N[i, t] = \begin{cases} 
\min\{j + N[i-1, t - jd_i] : 0 \leq j \leq \lfloor t/d_i \rfloor\} & \text{if } i \geq 2 \\
\frac{t}{d_i} & \text{if } i = 1.
\end{cases}$$  OR $t = 0$
FILLING THE ARRAY $N[1 \ldots n, 0 \ldots T]$:

$$N[i, t] = \begin{cases} \min\{j + N[i-1, t-jd_i] : 0 \leq j \leq \lfloor t/d_i \rfloor\} & \text{if } i \geq 2 \\ t & \text{if } i = 1 \text{.} \end{cases}$$

OR $t = 0$

$i$-axis (coin type)
(recall: $N[i, t]$ uses coin types $1 \ldots i$)

$t$-axis (target sum remaining)

Consider cell $N[i, t]$

We only look at the previous $i$-row!

It is sufficient to fill:
row $i=1$ (base case), then for $(i = 2 \ldots n)$, for $(t = 0 \ldots T)$

$d_i$
using other coin types

\[ N[i, t] = \begin{cases} \min\{j + N[i-1, t-jd_i] : 0 \leq j \leq \lfloor t/d_i \rfloor\} & \text{if } i \geq 2 \\ t & \text{if } i = 1. \end{cases} \]

### Numeric Values

- \( N[1][t] = t \)
- \( J[1][t] = t \)

### For General Cases

1. \( i = 2 \ldots n \)
2. \( t = 0 \ldots T \)
3. Initially best solution is 0 of \( d[i] \)
   - \( N[i][t] = N[i-1][t] \)
   - \( J[i][t] = 0 \)
4. Try \( j > 0 \) coins of type \( d[i] \)
   - \( i = 2 \ldots \) floor \( (t / d[i]) \)
   - If \( N[i-1][t-jd[i]] < N[i][t] \)
     - \( N[i][t] = j + N[i-1][t-jd[i]] \)
     - \( J[i][t] = j \) // best is currently \( j \) of \( d[i] \)

### Return

- Return \( N[n][T] \) // can also return \( N, J \)
### Outputting Optimal Set of Coins

Recall \( J[i, t] \) = \# of coins of type \( d_i \) used in \( N[i, t] \)

We start at \( J[n][T] \) = \# of coins of type \( d_n \) used in the optimal solution

Exercise for later:
compute the correct output
without using \( J[i, t] \)
(i.e., using only \( N, d, T \))

```python
def CoinChangingDP_coins(d[1..n], J[1..n][0..T]):
    counts = new array[1..n]
    t = T
    for i = n..1
        counts[i] = J[i][t]
        t = t - counts[i]*d[i]
    return counts
```
CoinChangingDP(d[1..n], T)
N = new table[1..n][0..T]
J = new table[1..n][0..T]

for t = 0..T     // base cases where i=1
    N[1][t] = t
    J[1][t] = t

for i = 2..n     // general cases
    for t = 0..T
        // initially best solution is 0 of d[i]
        N[i][t] = N[i-1][t]
        J[i][t] = 0

        // try j>0 coins of type d[i]
        for j = 1..floor(t / d[i])
            if N[i-1][t-j*d[i]] < N[i][t]
                N[i][t] = j + N[i-1][t-j*d[i]]
                J[i][t] = j  // best is currently j

return N[n][T]     // can also return N, J

Time complexity?

Unit cost computational model is reasonable here

Consider instance \( I = (d, T) \)

Runtime \( R(I) \in O\left(\sum_{i=2}^{n} \sum_{t=0}^{T} \frac{t}{d_i}\right) \)

\[ R(I) \in O\left(\sum_{i=2}^{n} \frac{1}{d_i} \sum_{t=0}^{T} t \right) \]

\[ R(I) \in O\left(\sum_{i=2}^{n} \frac{1}{d_i} \frac{T(T+1)}{2} \right) \]

\[ R(I) \in O(\sigma_{D^T}) \]

where \( D = \sum_{i=2}^{n} \frac{1}{d_i} < n. \)

Is this polynomial time? Not for every input!
POLYNOMIAL TIME

• An algorithm runs in (worst case) polynomial time IFF its runtime $R(I)$ on every input is upper bounded by a polynomial in the input size $S$

• i.e., $R(I) \in O(c_0 + c_1 S + c_2 S^2 + c_3 S^3 + \cdots + c_k S^k)$ for constants $k$ and $c_0, \ldots, c_k$

• … so is $O(nT^2)$ polynomial in our input size $S$?
INPUT SIZE

• $S = \text{bits}(T) + \text{bits}(d_1) + \ldots + \text{bits}(d_n)$

• It takes $\lceil \log_2 T \rceil$ bits to store $T$

• It takes $\lceil \log_2 d_i \rceil$ bits to store each $d_i$

• **Assume** $d_i \leq T$ (otherwise $d_i$ cannot be used at all, and should be omitted from the input)
  
  • Then we have $\lceil \log_2 d_i \rceil \in O(\log T)$
  
  • So, $S \in O(n \log T)$
COMPARING $T(I)$ TO $S$

- Recall $R(I) \in O(nT^2)$ and $S \in O(n \log T)$
- As an example, if $n$ is fixed at 10 and $T$ is allowed to vary, then $S \in O(\log T)$ and $R(I) \in O(T^2)$
  - In this case, $R(I)$ is exponential in $S$
- However, if $T$ is fixed at 10 and $n$ is allowed to vary, then $S \in O(n)$ and $R(I) \in O(n)$
  - In this case, $R(I)$ is linear in $S$
- So, large $n$ and small $T$ is where this DP solution shines!
A BIT MORE ANALYSIS

• Recall $R(I) \in O(nT^2)$ and $S \in O(n \log T)$

• If $T \in O(n)$, then $S \in O(n \log n)$ and $R(I) \in O(n^3)$
  • Note $O(n^3)$ is a smaller runtime than $O(S^3) = O(n^3 \log n)$
  • And $S^3$ is polynomial in $S$, so $O(n^3)$ is a polynomial runtime

• So, for some inputs with relatively small $T$, we can get polynomial runtimes!
  • In particular, for $T \in O(n^k)$ where $k$ is constant, $R(I) \in O\left(n(n^k)^2\right) = O(n^{2k+1})$ and $S \in O(n \log n^k) = O(n \log n)$
  • And $R(I) \in O(n^{2k+1}) \subseteq O\left((n \log n)^{2k+1}\right) = O(S^{2k+1})$