CS 341: ALGORITHMS

Lecture 14: graph algorithms V – single source shortest path

Readings: see website

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DIJKSTRA’S ALGORITHM

Single-source shortest path in a graph with non-negative edge weights
**PROBLEM: SINGLE SOURCE SHORTEST PATHS (SSSP)**

- **Input:** graph $G = (V, E)$ and a non-negative weight function $w(e)$ defined for every edge $e$.

- **Problem:** for every node $v \neq s$, output a path $s \xrightarrow{} v$ with the **smallest total weight** (among all paths $s \xrightarrow{} v$).

- I.e., each path $P$ should minimize $w(P) = \sum_{e \in P} w(e)$.

Suppose this is $s$

Shortest path to $d$

Shortest path to $c$

Shortest path to $i$

And so on... one path for each node.
APPLICATION: DRIVING DISTANCE TO MANY POSSIBLE DESTINATIONS

- Single source: from where you are
- Shortest paths: to all destinations
  - Display a subset of destinations
  - Include the optimal distances computed using SSSP algorithm
- Other heuristics… traffic? Lights?
  - Weights can combine many factors
<table>
<thead>
<tr>
<th>Game AI: path finding with waypoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Divide game world into <strong>linear paths</strong>, then send game characters in <strong>straight lines</strong> between waypoints</td>
</tr>
<tr>
<td>If some linear paths are much faster/slower, use <strong>weighted SSSP</strong></td>
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</tbody>
</table>

Otherwise use BFS to find shortest sequence of waypoints (with **fewest** waypoints)
**DIJKSTRA’S ALGORITHM**

**ILLUSTRATIVE EXAMPLE**

**Start node** $s$ is here

**Showing** $dist$-values

This $dist$ is optimal!

Can we use this optimal $dist$ to improve the $dist$ of neighbours

We call this relaxing the neighbours

Key insight: after relaxing all, the smallest $dist$ (that we didn’t already know was optimal) is now optimal
```python
Dijkstra(adj[1..n], s)

pred[1..n] = [null, null, ..., null]
dist[1..n] = [infty, infty, ..., infty]
pq = new priority queue

dist[s] = 0
for u = 1..n
    pq.enqueue(u, dist[u])

while pq is not empty
    u = pq.dequeueMin()
    for v in adj[u]
        if dist[u] + w(u,v) < dist[v]
            dist[v] = dist[u] + w(u,v)
            pred[v] = u
            pq.changePriority(v, dist[v])

return pred, dist
```

- Maintain nodes in priority order, ordered by smallest distance
- Enqueue all nodes with distance $\infty$ except for $s$ with distance 0
- Eventually dequeue all nodes (no more enqueues)
- Each dequeued node $u$ has optimal $dist$
- Relax neighbour $v$
Dijkstra’s algorithm iteratively constructs a set $OPT$ of nodes for which we know the shortest path from $s$ (initially $OPT = \{s\}$).

After each relaxation step, we grow $OPT$ by adding the node in $V \setminus OPT$ with the smallest $dist$.
PROOF

- **Theorem:** At the end of the algorithm, for all \( u \), \( \text{dist}[u] \) is exactly the total weight of the shortest \( s \rightarrow u \) path

- We prove this in two parts
  - \( \text{dist}[u] \leq \) the total weight of the shortest \( s \rightarrow u \) path (case \( \leq \))
  - \( \text{dist}[u] \geq \) the total weight of the shortest \( s \rightarrow u \) path (case \( \geq \))
Let $P$ be any arbitrary $s \rightarrow u$ path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_\ell$ where $v_0 = s$ and $v_\ell = u$.

For any index $j$ let $L_j$ denote $w(v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_j)$.

We prove by induction: $\text{dist}[v_j] \leq L_j$ for all $j$.

$w($orange edges$) = L_j$
Prove by induction: $\forall j: dist[v_j] \leq L_j$

Base case: $dist[v_0] = dist[s] = 0 = L_0$

Ind. step: **suppose** $\forall j > 0: dist[v_{j-1}] \leq L_{j-1}$

- When `dequeueMin()` returns $v_{j-1}$:
  - we **check** if $dist[v_{j-1}] + w(v_{j-1}, v_j) < dist[v_j]$
  - If so, we **set** $dist[v_j] = dist[v_{j-1}] + w(v_{j-1}, v_j)$
  - If not, $dist[v_j] \leq dist[v_{j-1}] + w(v_{j-1}, v_j)$

In **both cases**, $dist[v_j] \leq dist[v_{j-1}] + w(v_{j-1}, v_j)$
- By I.H. $dist[v_{j-1}] \leq L_{j-1}$ so $dist[v_j] \leq L_{j-1} + w(v_{j-1}, v_j)$
- And $L_{j-1} + w(v_{j-1}, v_j) = L_j$ by definition
- So $dist[v_j] \leq L_j$

$w(\text{orange edges}) = L_j$

This proves $dist[u] \leq L_u$, the weight of an arbitrary $s \xrightarrow{} u$ path.

So $dist[u] \leq$ the weight of EVERY $s \xrightarrow{} u$ path.

Including the shortest $s \xrightarrow{} u$ path!
CASE ≥

- Let $P'$ be the path $s \to \cdots \to \text{pred[pred[u]]} \to \text{pred[u]} \to u$
  - I.e., the reverse of following pred pointers from $u$ back to $s$
- We show $\text{dist}[u]$ is as long as this path (and hence as long as the shortest path)
- Denote the nodes in $P'$ by $v_0, v_1, \ldots, v_{\ell}$ where $v_0 = s$ and $v_{\ell} = u$
- Let $L_j = w(v_0 \to v_1 \to \cdots \to v_j)$
- **Prove by induction:** $\forall j > 0 : \text{dist}[v_j] = L_j$
- Base case: $\text{dist}[v_0] = \text{dist}[s] = 0 = L_0$
CASE $\geq$

- $P' = v_0 \rightarrow \cdots \rightarrow v_\ell = s \rightarrow \cdots \rightarrow \text{pred}[\text{pred}[u]] \rightarrow \text{pred}[u] \rightarrow u$
- $L_j = w(v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_j)$
- **Inductive step**: suppose $\forall j > 0 : \text{dist}[v_{j-1}] = L_{j-1}$
- When we set $\text{pred}[v_j] = v_{j-1}$, we set $\text{dist}[v_j] = \text{dist}[v_{j-1}] + w(v_{j-1}, v_j)$

Recall:

- By I.H., $\text{dist}[v_j] = L_{j-1} + w(v_{j-1}, v_j)$
- By definition $L_j = L_{j-1} + w(v_{j-1}, v_j)$
- So $\text{dist}[v_j] = L_j$

So $\text{dist}[u] = \text{length of a particular path } P' \text{ in the graph}$

And length of $P'$ is $\geq$ length of shortest path

So $\text{dist}[u] \geq \text{length of shortest path } s \rightarrow u$

So $\text{dist}[u]$ is both $\leq$ and $\geq$ to the length of the shortest path $s \rightarrow u$ path!

That means it’s **equal** to the length of the shortest path!
Dijkstra(adj[1..n], s)
pred[1..n] = [null, null, ..., null]
dist[1..n] = [infty, infty, ..., infty]
pq = new priority queue

\[ O(n) \]

\[ O(\log n) \]
for \( u = 1..n \)
\[ \text{pq.enqueue}(u, \text{dist}[u]) \]
\[ O(n \log n) \]

while pq is not empty
\[ u = \text{pq.dequeueMin}() \]
\[ O(\log n) \]
for \( v \) in adj[u]
\[ \text{if dist}[u] + w(u,v) < \text{dist}[v] \]
\[ \text{dist}[v] = \text{dist}[u] + w(u,v) \]
\[ \text{pred}[v] = u \]
\[ \text{pq.changePriority}(v, \text{dist}[v]) \]
\[ O(\log n) \]

return pred, dist

\[ O(n \log n) \]

\[ \times \]

\[ O(\log n) \]

\[ O(n \log n) \]

\[ O(n \log n) \]

\[ O(m \log n) \]

w/adjacency lists

\[ \text{Total time } O((n + m) \log n) \]

Space complexity?
OUTPUTTING ACTUAL SHORTEST PATH(S)?

- To compute the actual shortest path $s \rightarrow t$
- Inspect $pred[t]$
  - If it is NULL, there is no such path
  - Otherwise, follow $pred$ pointers back to $s$, and return the reverse of that path
AN ALTERNATIVE IMPLEMENTATION

- Instead of using a priority queue
- Find the minimum dist[] node to add to OPT via linear search
- Runtime?
  - $O(n^2)$
- Better or worse than $O((n + m) \log n)$?
WEBSITE DEMONSTRATING DIJKSTRA’S ALG

BELLMAN-FORD

Single-source shortest path in a graph with possibly negative edge weights but no negative cycles
Shortest Paths and Negative Weight Cycles

Subsequent algorithms we will be studying will solve shortest path problems as long as there are no cycles having negative weight.

If there is a negative weight cycle, then there is no shortest path (why?).

There is still a shortest simple path, but there are apparently no known efficient algorithms to find the shortest simple paths in graphs containing negative weight cycles.

If there are no negative weight cycles, we can assume WLOG that shortest paths are simple paths (any path can be replaced by a simple path having the same weight).

Negative weight edges in an undirected graph are not allowed, as they would give rise to a negative weight cycle (consisting of two edges) in the associated directed graph.
The Bellman-Ford algorithm solves the single source shortest path problem in any directed graph without negative weight cycles.

The algorithm is very simple to describe:

Repeat $n - 1$ times: relax every edge in the graph (where relax is the updating step in Dijkstra's algorithm).

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<table>
<thead>
<tr>
<th>Line</th>
<th>Code</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>BellmanFord(n, E[1..m], s)</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>2</td>
<td>pred[1..n] = new array filled with null</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>3</td>
<td>D[1..n] = new array filled with infinity</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>4</td>
<td>D[s] = 0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>for $i = 1..n$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>for $(u,v,w)$ in E</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>if $D[u] + w &lt; D[v]$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$D[v] = D[u] + w$</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>pred[v] = u</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>return ($D$, pred)</td>
<td></td>
</tr>
</tbody>
</table>

**Complexity Analysis:**
- **$O(n)$ outer iterations**
- **$O(n)$ work per outer iteration**
- **$O(m)$ inner iterations per outer iteration**
- **$O(m)$ total work**

Total complexity is $O(nm)$. Could be $O(n^3)$.
Edges happen to be processed left to right by the inner loop.

It technically suffices to do one iteration of the outer loop.
WORST CASE EXECUTION

Edges happen to be processed right to left by the inner loop.

Dijkstra’s is similar, but consistently achieves good ordering using its priority queue.

Need \( n \) iterations of outer loop.

Since the longest possible path without a cycle can be \( n - 1 \) edges, the edges must be scanned \( n - 1 \) times to ensure the shortest path has been found for all nodes.
**WHY BELLMAN-FORD WORKS**

- **Not** going to prove this (by induction), but the crucial **lemma** is:
  - After $i$ iterations of the outer for-loop,
    - if $D[u] \neq \infty$, it is equal to the weight of some path $s \rightarrow u$; and
    - if there is a path $P = (s \rightarrow u)$ with **at most $i$ edges**, then $D[u] \leq w(P)$
  - So, after $n - 1$ iterations, if $\exists$ path $P$ with at most $n - 1$ edges, then $D[u] \leq w(P)$. (Note: any more edges would create a cycle.)
  - So, if $u$ is reachable from $s$, then $D[u]$ is the length of the shortest simple path (no cycles) from $s$ to $u$

Of course every simple path has at most $n - 1$ edges

So what if we do **another iteration**, and some $D[u]$ improves?

There is a negative cycle!
A MORE DETAILED IMPLEMENTATION

- With early stopping
- and checking for negative cycles

```python
BellmanFordCheck(n, E[1..m], s)
    pred[1..n] = new array filled with null
    D[1..n] = new array filled with infinity
    D[s] = 0
    for i = 1..n
        changed = false
        for (u,v,w) in E
            if D[u] + w < D[v]
                D[v] = D[u] + w
                pred[v] = u
                changed = true
        if not changed
            exit loop
    if i == n // assert: changed == true
        return NEGATIVE_CYCLE
    return (D, pred)
```
BONUS SLIDE

- Why can’t you just modify a graph with negative weights by: finding the minimum edge weight $W_{min}$, and adding that to each edge, so you no longer have negative edges and can run Dijkstra’s algorithm?

- **Exercise:** can you find a graph for which this will cause Dijkstra’s algorithm to return the wrong answer?

- **Solution:**
  - Consider a graph with 5 nodes: $s$, $a$, $b$, $c$, $t$
  - And edges $s$->a with weight -10, $b$->t with weight 10
  - $s$->b weight -1, $b$->c weight -1, $c$->t weight -1
  - What happens if you modify this graph as proposed, then run Dijkstra’s to find the shortest path from $s$ to $t$?