THIS TIME

• Intractability (hardness of problems)
  • Decision problems
  • Complexity class P
  • Polynomial-time Turing reductions
  • Introductory reductions
    • Three flavours of the traveling salesman problem
INTRACTABILITY

Studying the hardness of problems
Decision Problems

**Decision Problem:** Given a problem instance $I$, answer a certain question “yes” or “no”.

**Problem Instance:** Input for the specified problem.

**Problem Solution:** Correct answer (“yes” or “no”) for the specified problem instance. $I$ is a **yes-instance** if the correct answer for the instance $I$ is “yes”. $I$ is a **no-instance** if the correct answer for the instance $I$ is “no”.

**Size of a problem instance:** $Size(I)$ is the number of bits required to specify (or encode) the instance $I$. 
The Complexity Class $\mathbb{P}$

**Algorithm Solving a Decision Problem:** An algorithm $A$ is said to **solve** a decision problem $\Pi$ provided that $A$ finds the correct answer ("yes" or "no") for every instance $I$ of $\Pi$ in finite time.

**Polynomial-time Algorithm:** An algorithm $A$ for a decision problem $\Pi$ is said to be a **polynomial-time algorithm** provided that the complexity of $A$ is $O(n^k)$, where $k$ is a positive integer and $n = \text{Size}(I)$.

**The Complexity Class $\mathbb{P}$** denotes the set of all decision problems that have polynomial-time algorithms solving them. We write $\Pi \in \mathbb{P}$ if the decision problem $\Pi$ is in the complexity class $\mathbb{P}$.
Knapsack Problems

Problem 7.3

0-1 Knapsack-Dec

Instance: a list of profits, \( P = [p_1, \ldots, p_n] \); a list of weights, \( W = [w_1, \ldots, w_n] \); a capacity, \( M \); and a target profit, \( T \).

Question: Is there an \( n \)-tuple \( [x_1, x_2, \ldots, x_n] \in \{0, 1\}^n \) such that \( \sum w_i x_i \leq M \) and \( \sum p_i x_i \geq T \)?
Cycles in Graphs

Problem 7.1

Cycle
Instance: An undirected graph $G = (V, E)$.
Question: Does $G$ contain a cycle?

Problem 7.2

Hamiltonian Cycle
Instance: An undirected graph $G = (V, E)$.
Question: Does $G$ contain a hamiltonian cycle?

A hamiltonian cycle is a cycle that passes through every vertex in $V$ exactly once.
Polynomial-time Turing Reductions

Suppose $\Pi_1$ and $\Pi_2$ are problems (not necessarily decision problems). A (hypothetical) algorithm $B$ to solve $\Pi_2$ is called an oracle for $\Pi_2$.

Suppose that $A$ is an algorithm that solves $\Pi_1$, assuming the existence of an oracle $B$ for $\Pi_2$. ($B$ is used as a subroutine within the algorithm $A$.) Then we say that $A$ is a Turing reduction from $\Pi_1$ to $\Pi_2$, denoted $\Pi_1 \leq^T \Pi_2$.

A Turing reduction $A$ is a polynomial-time Turing reduction if the running time of $A$ is polynomial, under the assumption that the oracle $B$ has unit cost running time.

If there is a polynomial-time Turing reduction from $\Pi_1$ to $\Pi_2$, we write $\Pi_1 \leq^T_P \Pi_2$.

Informally: Existence of a polynomial-time Turing reduction means that if we can solve $\Pi_2$ in polynomial time, then we can solve $\Pi_1$ in polynomial time.
Travelling Salesperson Problems

Problem 7.5
TSP-Optimization
Instance: A graph $G$ and edge weights $w : E \to \mathbb{Z}^+$. 
Find: A Hamiltonian cycle $H$ in $G$ such that $w(H) = \sum_{e \in H} w(e)$ is minimized.

Problem 7.6
TSP-Optimal Value
Instance: A graph $G$ and edge weights $w : E \to \mathbb{Z}^+$. 
Find: The minimum $T$ such that there exists a Hamiltonian cycle $H$ in $G$ with $w(H) = T$.

Problem 7.7
TSP-Decision
Instance: A graph $G$, edge weights $w : E \to \mathbb{Z}^+$, and a target $T$. 
Question: Does there exist a Hamiltonian cycle $H$ in $G$ with $w(H) \leq T$?

Return type “a path/cycle $H$”

Return type “a positive integer $T$”

Is TSP-Dec $\leq_T$ TSP-Optimal Value?

Is TSP-Dec $\leq_T$ TSP-Optimization?

Return type “yes/no”
We will use polynomial-time Turing reductions to show that different versions of the **TSP** are polynomially equivalent: if one of them can be solved in polynomial time, then all of them can be solved in polynomial time. (However, it is believed that none of them can be solved in polynomial time.)

- We already know
  - TSP-Dec $\leq_T^p$ TSP-Optimal Value
  - TSP-Dec $\leq_T^p$ TSP-Optimization
- We show
  - TSP-Optimal Value $\leq_T^p$ TSP-Dec
  - TSP-Optimization $\leq_T^p$ TSP-Dec
Problem 7.6
TSP-Optimal Value
Instance: A graph $G$ and edge weights $w : E \rightarrow \mathbb{Z}^+$. 
Find: The minimum $T$ such that there exists a hamiltonian cycle $H$ in $G$ with $w(H) = T$. 

Problem 7.7
TSP-Decision
Instance: A graph $G$, edge weights $w : E \rightarrow \mathbb{Z}^+$, and a target $T$. 
Question: Does there exist a hamiltonian cycle $H$ in $G$ with $w(H) \leq T$?

How can we learn the exact optimal value by making such calls?
Is this a "poly-time reduction?" I.e., if we assume TSP-Dec-Solver runs in $O(1)$ time, is the runtime a polynomial in the input size?

Questions: (1) What’s the input size? (2) What’s the runtime?

Use binary search! How to define the starting range $(lo, hi)$ to search?

Largest possible cycle could include every edge

0 is smallest possible weight for any cycle

Maybe there is no Hamiltonian cycle, at all

This is a standard binary search technique.
What's the size of the input $I = (G, w)$?

$\text{Size}(I) = \text{Size}(G) + \text{Size}(w)$

But wait... $G$ and $w$ could be represented in many different ways. Could the choice of representation affect our complexity result?

Only for very inefficient representations (that are **exponentially larger than optimal**).

For example if we store weights in **unary**

We **rule out such inefficient representations** for the purpose of proving polynomial runtime.

Polynomial differences in size do not matter. Exercise: if $T \in \text{poly}(|\text{Size}(I)|^{40})$ then $T \in \text{poly}(\text{Size}(I))$
What’s the size of the input $I = (G, w)$?

Size($I$) = Size($G$) + Size($w$)

So, suppose $G$ is represented as an **array of adjacency lists** (one list for each vertex), with each list containing **edges** to neighbouring vertices, and an edge is represented by a **weight** and the **name** of the target vertex.

**Array** of empty lists for all vertices $v$

**Bits** to store **weight** of the edge (storing $w(e)$ takes $\log w(e) + 1$ bits)

**Bits** to store the **name** of the target vertex (in 1..|V|)

For all edges:

Size($I$) = $|V|$ + $\sum_{e \in E} (\log w(e) + 1 + \log|V| + 1)$

Let’s relate this to runtime... what’s the runtime?
TSP-Optimal Value $\leq T_P$ TSP-Dec

Let's assume $O(1)$ time for operations on weights. Later we'll see this isn't needed to show polytime.

**Algorithm:** $TSP$-Optimal$\text{Value-Solver}(G, w)$

external $TSP$-Dec-Solver

$h_i \leftarrow \sum_{e \in E} w(e)$  \hspace{1cm} $O(|E|)$

$lo \leftarrow 0$  \hspace{1cm} $O(1)$

if not $TSP$-Dec-Solver$(G, w, h_i)$ then return $(\infty)$

while $h_i > lo$

# iterations: $O(\log(h_i - lo))$

return $(h_i)$

Runtime $T(I) \in O(|E| + \log \sum_{e \in E} w(e))$
COMPARING $T(I)$ AND $\text{Size}(I)$

- $T(I) \in O(|E| + \log \sum_{e \in E} w(e))$
- $\text{Size}(I) = |V| + \sum_{e \in E} (\log w(e) + 1) + \log |V| + 1$
  \[= |V| + \sum_{e \in E} (\log w(e) + 1) + \sum_{e \in E} (\log |V| + 1)\]
  \[= |V| + \sum_{e \in E} (\log w(e) + 1) + \sum_{e \in E} (\log |V|) + |E|\]

- Want to show $T(I) \in O(\text{Size}(I)^c)$ for some constant $c$ (we show $c=1$)
  \[O(|E| + \log \sum_{e \in E} w(e)) \subseteq? O(|V| + \sum_{e \in E} (\log w(e) + 1) + \sum_{e \in E} \log |V| + |E|)\]
  \[\iff O(\log \sum_{e \in E} w(e)) \subseteq? O(|V| + \sum_{e \in E} (\log w(e) + 1) + \sum_{e \in E} \log |V|)\]

**How to compare** $\log \sum_{e \in E} w(e)$ and $\sum_{e \in E} (\log w(e) + 1)$?
COMPARING $T(I)$ AND $\text{Size}(I)$

- How to compare $\log \sum_{e \in E} w(e)$ and $\sum_{e \in E} (\log w(e) + 1)$?
- $\sum_{e \in E} (\log w(e) + 1) = (\log w(e_1) + 1) + (\log w(e_2) + 1) + \cdots + (\log (w(e_{|E|})) + 1)$
- Can we combine these terms into one log using $\log x + \log y = \log xy$?
- $\sum_{e \in E} (\log w(e) + 1) = (\log w(e_1) + \log 2) + + \cdots + (\log (w(e_{|E|})) + \log 2)$
- $\sum_{e \in E} (\log w(e) + 1) = \log 2w(e_1) 2w(e_2) \cdots 2w(e_{|E|}) = \log \prod_{e \in E} 2w(e)$
- So how to compare $\log \prod_{e \in E} 2w(e)$ and $\log \sum_{e \in E} w(e)$?
  - All $w(e)$ are positive integers, so $\prod_{e \in E} 2w(e) \geq \sum_{e \in E} w(e)$
  - Since log is increasing on $\mathbb{Z}^+$, $\log \prod_{e \in E} 2w(e) \geq \log \sum_{e \in E} w(e)$
COMPARING $T(I)$ AND $\text{Size}(I)$

• We in fact show $T(I) \in O(\text{Size}(I))$

$$O(\log \sum_{e \in E} w(e)) \leq O(|V| + \sum_{e \in E} (\log w(e) + 1) + \sum_{e \in E} \log |V|)$$

How to compare $\log \sum_{e \in E} w(e)$ and $\sum_{e \in E} (\log w(e) + 1)$?

We just saw $\sum_{e \in E} (\log w(e) + 1) = \log \prod_{e \in E} 2w(e) \geq \log \sum_{e \in E} w(e)$

So $T(I) \in O(\text{Size}(I)^c)$ where $c = 1$

So this reduction has runtime that is polynomial in the input size!
Exercise: show the variant of this reduction where linear search is used instead of binary search is not $\text{poly}(\text{Size}(I))$. 

```plaintext
Algorithm: TSP-OptimalValue-Solver(G, w)

external TSP-Dec-Solver

hi ← $\sum_{e \in E} w(e)$
lo ← 0

if not TSP-Dec-Solver(G, w, hi) then return (∞)

while hi > lo
    mid ← $\left\lfloor \frac{hi + lo}{2} \right\rfloor$
    do
        if TSP-Dec-Solver(G, w, mid)
            then hi ← mid
        else lo ← mid + 1
    return (hi)
```

TSP-Optimal Value $\leq T_P$ TSP-Dec
REACHED THIS POINT

(but will recap the comparison of $T(I)$ and $\text{Size}(I)$ next time)
So TSP-OptimalValue-Solver is polytime... But is it a correct reduction from TSP-Optimal Value to TSP-Dec?

Need to prove:
TSP-OptimalValue-Solver(G,w) returns the weight $W$ of the shortest Hamiltonian Cycle (HC) in $G$

**Sketch:** We return $\infty$ iff there is no HC. Loop invariant: $W \in [lo, hi]$. So, at termination when $hi = lo$, we return exactly $hi = W$. 

TSP-Optimal Value $\leq_T^P$ TSP-Dec

**Algorithm:** TSP-OptimalValue-Solver($G, w$)

external TSP-Dec-Solver

$hi \leftarrow \sum_{e \in E} w(e)$

$lo \leftarrow 0$

if not TSP-Dec-Solver($G, w, hi$) then return ($\infty$)

while $hi > lo$

mid $\leftarrow \left\lceil \frac{hi + lo}{2} \right\rceil$

if TSP-Dec-Solver($G, w, mid$)

then $hi \leftarrow mid$

else $lo \leftarrow mid + 1$

return ($hi$)
We have therefore shown:

TSP-Optimal Value is polytime reducible to TSP-Dec

So, if an $O(1)$ implementation of TSP-Dec-Solver exists, then we have a polytime implementation of TSP-Optimal-Value-Solver!

In fact, TSP-OptimalValue-Solver remains polytime even if the implementation of the oracle runs in polytime instead of $O(1)$!
The key idea is: Consider polynomials $P_R(s)$ and $P_O(s)$ representing the runtime of a reduction and its oracle, respectively, on an input of size $s$.

Worst possible runtime happens if every step in the reduction is a call to the oracle.

This is $P_R(s)P_O(s)$ --- multiplication of polynomials.

But multiplying polynomials of degrees $d_1, d_2$ results in a polynomial of degree $\leq d_1 + d_2$. Example:

$P_1(x) = 5x^2 + 10x + 100$

$P_2(x) = 20x^3 + 20$

$P_1(x)P_2(x) = (5x^2 + 10x + 100)(20x^3 + 20)$

$= 100x^5 + 200x^4 + 2000x^3 + 100x^2 + 200x + 2000$
In more complex reductions where we transform the input before calling the oracle, we will need a more complex proof:

(A) If there is a(n optimal) solution in the input, our transformation will preserve that solution so the oracle can find it, and

(B) Our transformation doesn’t introduce new solutions that are not present in the original input

(i.e., if we find a solution in the transformed input, there was a corresponding solution in the original input)
## Input Size Cheat Sheet

**Perfectly fine choices of Size(I)**

<table>
<thead>
<tr>
<th>Input I</th>
<th>int x</th>
<th>Graph (V, E) with weights W:</th>
</tr>
</thead>
<tbody>
<tr>
<td>int x</td>
<td>1 or $\lfloor \log(x) \rfloor + 1$ (can simplify to $\log(x) + 1$ or $\log x$)</td>
<td>$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Input I</th>
<th>Examples of BAD choices of Size(I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>int x</td>
<td>$x$</td>
</tr>
<tr>
<td>Graph (V, E)</td>
<td>$2^{</td>
</tr>
<tr>
<td>$A[1..n]$ of int</td>
<td>$2^n$ or $\sum_i A[i]$</td>
</tr>
</tbody>
</table>

**Exponentially larger than optimal representation!**

**Pseudo-polynomial** ~ no exponentiation of non-constant terms

**Technically any pseudo-polynomial combination** of these terms is fine. For example, the following is fine: $(|E|^{100} + |V|^2) \cdot \sum_{e \in E} (\log(w(e)) + 1)$
BONUS SLIDES

efficient vs inefficient input representations
What's the size of the input $I$?

$$Size(I) = Size(G) + Size(w)$$

**But wait...** G and $w$ could be represented in many different ways. Could the choice of representation affect our complexity result?

**Representation 1:** What if the entire graph is simply represented as a **weight matrix** $W$ which contains a weight $w_{uv}$ for each $u, v \in V$ ($\infty$ if an edge does not exist).

Consider weight $w_{uv}$. It takes $\Theta(\log w_{uv})$ bits ($\log(w_{uv}) + 1$) to store this weight.

We would then have:

$$Size(R_1) = \sum_{u \in V} \sum_{v \in V} \log(w_{uv}) + 1$$

What would it mean to have a runtime $T$ that is **polynomial in $Size(R_1)$**?

We say $T$ is polynomial in $Size(R_1)$ (denoted $T \in \text{poly}(Size(R_1))$) iff:

$$\exists \text{ constant } c \text{ s.t. for all } I, \text{ we have } T \in O(Size(R_1)^c)$$
Representation 2: What if the graph were represented as an array of adjacency lists (one list for each vertex), with each list containing edges to neighbouring vertices, where an edge is represented by a weight and the name of the target vertex?

We would then have:

\[
\text{Size}(R_2) = |V| + \sum_{(u,v) \in E} \left( \log(w_{uv}) + 1 + \log |V| + 1 \right)
\]

- **Array** with one list per vertex \(v\)
- **Weight** of the edge
- **Name** of the target vertex

Compare with representation 1:

\[
\text{Size}(R_1) = \sum_{u \in V} \sum_{v \in V} \log (w_{uv}) + 1
\]
**Representation 3:** What if we were to represent the graph as a weight matrix \( W \) but write all weights in *unary*, instead of binary (so it takes \( w_{uv} \) bits to store weight \( w_{uv} \)).

For this (very stupid) representation, we would then have:

\[
\text{Size}(R_3) = \sum_{u \in V} \sum_{v \in V} (w_{uv})
\]

This can be exponentially larger than \( \text{Size}(R_1) \)!

Compare with representation 1:

\[
\text{Size}(R_1) = \sum_{u \in V} \sum_{v \in V} (\log w_{uv}) + 1
\]

For example, in a graph where there are \( O(1) \) nodes and all edges have weight \( w \):

\[
\text{Size}(R_1) = \Theta(\log_2 w) \quad \text{and} \quad \text{Size}(R_3) = \Theta(w).
\]

In this case, \( \text{Size}(R_3) \in \Theta(2^{\text{Size}(R_1)}) \)

So, some algorithms could be polynomial in \( \text{Size}(R_3) \) but exponential in \( \text{Size}(R_1) \).

We should **rule out** this highly inefficient representation for the purpose of proving polynomial runtime.

**Idea:** determine whether runtime is polynomial in the size of the **optimal representation of the input**

Problem: it’s not clear what the **optimal** representation is…

What if we can argue the runtime is polynomial in some **lower bound** on the size of the input?
LOWER BOUNDING $\text{Size}(I)$

- To prove that a reduction's runtime $T(I)$ on input $I$ is polynomial in the size of $I$:
  - Define a lower bound $L(I)$ on the size of $I$
  - For every possible representation $I_R$ of $I$, $L(I) \leq \text{Size}(I_R)$ should hold
    - Can be proved with information theory, or ad-hoc; outside the scope of the course
  - In this course, we can be a bit sloppy, and just use the table of valid choices here to obtain a term for each variable in $I$
- Then, if we can show $T(I) \leq \text{poly}(L(I))$, we have actually shown $T(I) \leq \text{poly}(\text{size}(I))$

The following are valid choices of $L(I)$ for various input types:

<table>
<thead>
<tr>
<th>Input $I$</th>
<th>$L(I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>int $x$</td>
<td>1 or $\log(x) + 1$</td>
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<tr>
<td>Graph $(V, E)$ possibly with weights $W$</td>
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<td>$n$ or $\sum_i (\log(A[i]) + 1)$</td>
</tr>
<tr>
<td>$n \times n$ matrix $m$</td>
<td>$n^2$ or $\sum_{i,j} (\log(m_{ij}) + 1)$</td>
</tr>
</tbody>
</table>

Justifying sloppy analysis:

**Polynomial differences** in choices of $L(I)$, such as $|V|$ vs $|V|^2$ vs $(|E| + |V|)^{40}$ don't matter.

Such differences cannot change whether a runtime $T(I)$ is in $\text{poly}(L(I))$ or not.

Exercise: $T(I) \in \text{poly}(L(I)^{40})$ iff $T(I) \in \text{poly}(L(I))$
So what’s a valid $L(I)$ for an input $I$ to TSP-OptimalValue-Solver?

Input is a graph $G$ with weight matrix $w$. From the table of valid $L(I)$ choices, we let $L(I) = |E| + \sum_{e \in E} (\log(w(e)) + 1)$.

What’s the relationship between the reduction’s runtime $T(I)$ and $L(I)$?

As we argued earlier, $T(I) \in poly(L(I))$

And thus $T(I) \in poly(\text{Size}(I))$

So this reduction has runtime that is polynomial in the input size!