CS 341: Algorithms

Lecture 2: Solving recurrences

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based on lecture notes by many other CS341 instructors

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From exact to sloppy recurrences

Overview

Consider a recursive algorithm Algo.

Assumption: for an input size $n > 1$, Algo does

- $\boldsymbol{a}$ recursive calls, in size either $\lfloor \boldsymbol{n/b} \rfloor$ or $\lceil \boldsymbol{n/b} \rceil$ ($\boldsymbol{a > 0}$ and $\boldsymbol{b > 1}$, constants)
- between $\boldsymbol{c'n^y}$ and $\boldsymbol{cn^y}$ extra operations. ($\boldsymbol{c}$ and $\boldsymbol{c'}$ nonzero constants, $\boldsymbol{y}$ constant)

Claim

Solving the sloppy recurrence $\boldsymbol{T(n) = aT(n/b) + cn^y}$ for powers of b gives a valid Θ-bound for best and worst-case runtimes.

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Remark 1: if we only know that we do **at most** $\boldsymbol{cn^y}$ extra operations, we only get a big-O.

Remark 2: to be concrete, we'll do the proof for mergesort.

- one recursive call with $\lfloor n/2 \rfloor$, the other with $\lceil n/2 \rceil$, and roughly n extra operations.
- so $a = b = 2$ and $y = 1$

Best and worst-case recurrence relations

Let $T^w(n), T^b(n)$ be the **worst case**, resp. **best case** in size n.

Worst-case recurrence: $T^w(1) = d$ and

$$T^w(n) \leq T^w\left(\left\lceil \frac{n}{2} \right\rceil\right) + T^w\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + cn \quad \text{if } n > 1$$

Best-case recurrence: $T^b(1) = d'$ and

$$T^b(n) \geq T^b\left(\left\lceil \frac{n}{2} \right\rceil\right) + T^b\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c'n \quad \text{if } n > 1$$

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Remark: could be possible to write $=$ instead or $\leq$ or $\geq$, but harder to prove

Worst-case analysis

Use an equal sign: define T by

$$T(1) = d, \qquad T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + cn \quad \text{if } n > 1$$

Exercise

$T^w(n) \leq T(n)$ and $T(n)$ increasing (easy induction)

Remark: same thing can be done for $T^b(n)$.

Worst-case analysis (cont.)

Sloppy recurrence:

$$t(1) = d, \qquad t(n) = 2t\left(\frac{n}{2}\right) + cn \quad \text{if } n > 1$$

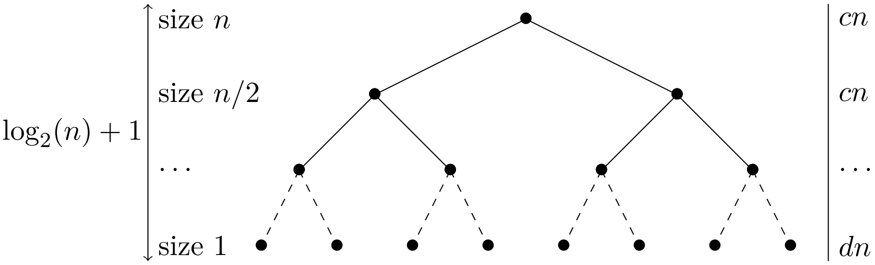
Observations

- this only defines $t(n)$ for powers of 2.
- $T(2^k) = t(2^k)$ for any k
- T is increasing so $T(n) \leq T(\text{next power of 2}) = t(\text{next power of 2})$

Conclusion:

- enough to analyze $t(n)$, n a power of 2
- we'll do it using the recursion tree

The mergesort recursion tree



Total: $t(n) = cn\log_2(n) + dn$ for n a power of 2.

Consequences

- $\boldsymbol{T(n) \in O(n\log(n))}$
- $\boldsymbol{T^w(n) \in O(n\log(n))}$

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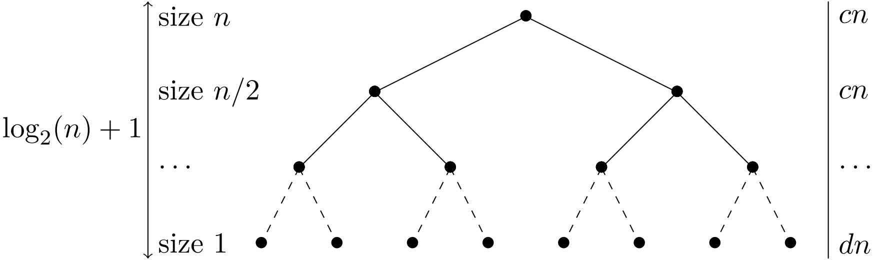

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- $\boldsymbol{T^w(n) \in O(n\log(n))}$

Remark: same approach proves $\boldsymbol{T^b(n) \in \Omega(n\log(n))}$, and so

$$\boldsymbol{T^b(n), T^w(n) \in \Theta(n\log(n))}$$

The master theorem

The master theorem

Solves many recurrence relations coming from divide-and-conquer algorithms.

Suppose that $\boldsymbol{a \geq 1}$ and $\boldsymbol{b > 1}$. Consider the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + cn^y \quad n > 1$$

Let

$$\boldsymbol{x = \log_b a} \quad (\text{so } a = b^x).$$

Then

$$T(n) \in \begin{cases} \Theta(\boldsymbol{n^y}) & \text{if } \boldsymbol{y > x} \quad (\text{root heavy}) \\ \Theta(\boldsymbol{n^y \log n}) & \text{if } \boldsymbol{y = x} \quad (\text{balanced}) \\ \Theta(\boldsymbol{n^x}) & \text{if } \boldsymbol{y < x} \quad (\text{leaf heavy}) \end{cases}$$

We do the proof for n a power of b; result true for $n \in \mathbb{R}_{\geq 0}$.

Recursion tree

Suppose that $n = b^j$, $a \geq 1$, $b \geq 2$ are integers and

$$T(n) = a\,T\left(\frac{n}{b}\right) + c\,n^y, \qquad T(1) = d.$$

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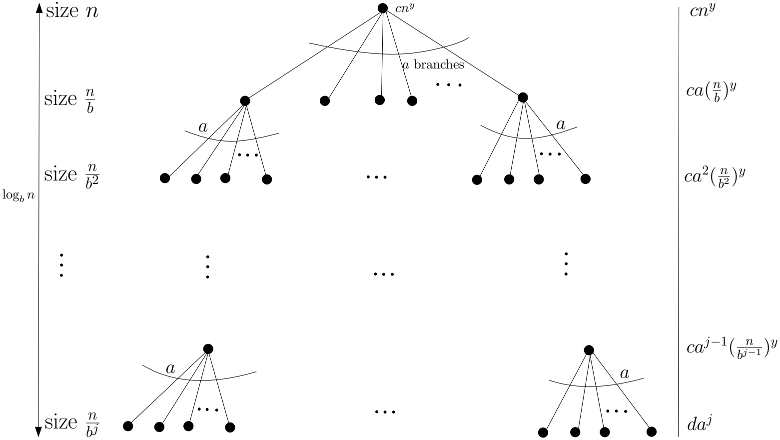

Breakdown of the cost

Suppose that $a \geq 1$ and $b \geq 2$ are integers and

$$T(n) = a\,T\left(\frac{n}{b}\right) + c\,n^y, \qquad T(1) = d.$$

Let $n = b^j$.

size of subproblem	# nodes	cost/node	total cost
$n = b^j$	1	$c\,n^y$	$c\,n^y$
$n/b = b^{j-1}$	a	$c\,(n/b)^y$	$c\,a\,(n/b)^y$
$n/b^2 = b^{j-2}$	a^2	$c\,(n/b^2)^y$	$c\,a^2\,(n/b^2)^y$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n/b^{j-1} = b$	a^{j-1}	$c\,(n/b^{j-1})^y$	$c\,a^{j-1}\,(n/b^{j-1})^y$
$n/b^j = 1$	a^j	d	$d\,a^j$

Computing $T(n)$

Total:

$$T(n) = d\,a^j + c\,n^y \sum_{i=0}^{j-1} \left(\frac{a}{b^y}\right)^i = dn^x + cn^y \sum_{i=0}^{j-1} \left(\frac{a}{b^y}\right)^i .$$

Proof: $a = b^x$ and $n = b^j$, so $a^j = (b^x)^j = (b^j)^x = n^x$.

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Proof: $a = b^x$ and $n = b^j$, so $a^j = (b^x)^j = (b^j)^x = n^x$.

Observation: geometric sum with ratio $\boldsymbol{r = \frac{a}{b^y} = b^{x-y}}$:

- if $\boldsymbol{r < 1} \iff \boldsymbol{x < y}$: $\sum r^i \in \Theta(1)$, so $T(n) \in \Theta(n^y)$
- if $\boldsymbol{r = 1} \iff \boldsymbol{x = y}$: $\sum r^i \in \Theta(\log n)$, so $T(n) \in \Theta(n^y \log n)$
- if $\boldsymbol{r > 1} \iff \boldsymbol{x > y}$: $\sum r^i \in \Theta(r^j)$, so $T(n) \in \Theta(n^x)$

Proof (last item):

$$r^j = \frac{a^j}{b^{yj}} = \frac{n^x}{n^y}$$

Examples

$T(n) = 4T(n/2) + n$ multiplying polynomials

- $a = 4$, $b = 2$, $y = 1$ so $x = \log_b a = 2$ and $\boldsymbol{T(n) = \Theta(n^2)}$

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$\boldsymbol{T(n) = 2T(n/2) + n^2}$

- $a = 2$, $b = 2$, $y = 2$ so $x = \log_b a = 1$ and $\boldsymbol{T(n) = \Theta(n^2)}$

Examples

$T(n) = 4T(n/2) + n$ multiplying polynomials

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$T(n) = 2T(n/2) + n^2$

- $a = 2$, $b = 2$, $y = 2$ so $x = \log_b a = 1$ and $\boldsymbol{T(n) = \Theta(n^2)}$

$T(n) = 2T(n/4) + 1$ kd-trees

- $a = 2$, $b = 4$, $y = 0$ so $x = \log_b a = 1/2$ and $\boldsymbol{T(n) = \Theta(\sqrt{n})}$

Examples

$\boldsymbol{T(n) = T(n/2) + 1}$ binary search

- $a = 1$, $b = 2$, $y = 0$ so $x = \log_b a = 0$ and $\boldsymbol{T(n) = \Theta(\log n)}$

Examples

$\boldsymbol{T(n) = T(n/2) + 1}$ binary search

- $a = 1$, $b = 2$, $y = 0$ so $x = \log_b a = 0$ and $\boldsymbol{T(n) = \Theta(\log n)}$

$\boldsymbol{T(n) = T(n/2) + n}$ amortized analysis of dynamic arrays

- $a = 1$, $b = 2$, $y = 1$ so $x = \log_b a = 0$ and $\boldsymbol{T(n) = \Theta(n)}$

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- does not fit in our framework, but obvious

Examples

$T(n) = T(n/2) + 1$ binary search

- $a = 1$, $b = 2$, $y = 0$ so $x = \log_b a = 0$ and $\boldsymbol{T(n) = \Theta(\log n)}$

$T(n) = T(n/2) + n$ amortized analysis of dynamic arrays

- $a = 1$, $b = 2$, $y = 1$ so $x = \log_b a = 0$ and $\boldsymbol{T(n) = \Theta(n)}$

$T(n) = T(n/2)$

- does not fit in our framework, but obvious

$T(n) = 2T(n/2) + n \log(n)$

- does not fit in our framework, have to redo the recursion tree analysis

Alternative: guess and prove

Consider $T(n) = 2T(n/2) + n$, $T(1) = 0$, n power of 2.

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Guess: $T(n) \leq n$. Proof by induction? Assume $T(n/2) \leq n/2$.

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Guess: $T(n) \le kn$, k TBD? Assume $T(n/2) \le kn/2$.

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Guess: $T(n) \leq kn \log_2 n$, k TBD? Assume $T(n/2) \leq kn/2 \log_2(n/2)$.

$$T(n) = 2T(n/2) + n \leq 2(kn/2 \log_2(n/2)) + n = \boldsymbol{kn \log_2 n - kn + n}$$

proof by induction OK if $k \geq 1$.

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Remark: usually harder to prove $\boldsymbol{T(n) = \cdots}$