Lecture 2: Divide and Conquer & Recurrences

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Overview

- Divide-and-Conquer Paradigm
- Solving Recurrences
- Optional: Maximum Subarray Sum
- Acknowledgements
Divide-and-Conquer

Many problems can be efficiently solved by *dividing* them into *smaller subproblems*, and then *combining* the subproblems to give solution to original problem.
Divide-and-Conquer

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**Examples:**

1. Sorting: merge sort
2. Searching: binary search
3. Matrix Multiplication
4. Polynomial Multiplication

many more, (see [CLRS 2009])
Divide-and-Conquer

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- Structure of divide-and-conquer:
  - **Divide:** given instance $I$, construct smaller instances $I_1, \ldots, I_a$ (*subproblems*)
    - Ideally want $|I_j|$ small compared to $|I|$ (say constant fraction)
    - Recursion for running time:
      $$ T(I) = T(I_1) + \cdots + T(I_a) + \text{time to combine} $$
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Example: Merge Sort

- **Input:** array $A$ with $n$ elements
- **Output:** sorted array $A$

Divide and Conquer algorithm:

$$\text{sort}(A[\alpha, \beta]) :$$

1. If $\beta - \alpha < 10$, then trivially sort array and return.
2. $B = \text{sort}(A[\alpha, \lfloor (\alpha + \beta)/2 \rfloor])$, $C = \text{sort}(A[\lfloor (\alpha + \beta)/2 \rfloor + 1, \beta])$
3. return merge($B$, $C$)

Merging algorithm: (input arrays sorted in increasing order)

$$\text{merge}(B, C) :$$

1. Let $D = []$ be an empty array, and let $i, j$ be two pointers, indexing position on arrays $B, C$, initialized at 1.
2. Until we are done scanning both $B, C$:
   - If $B[i] \leq C[j]$, then $D$.append($B[i]$) and $i \leftarrow i + 1$
   - Else, $D$.append($C[j]$) and $j \leftarrow j + 1$
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Analysis: Merge Sort

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- Recursion tree (see board): \( T(n) = \Theta(n \cdot \log n) \)
- Can also “guess and check” the answer
  \[
  T(n) = cn \log n \\
  T(n) = 2 \cdot \left( c \cdot \frac{n}{2} \log(n/2) \right) + cn \\
  = cn(\log n - 1) + cn = cn \log n
  \]
• Divide-and-Conquer Paradigm

• Solving Recurrences

• Optional: Maximum Subarray Sum

• Acknowledgements
Recurrences

- Divide-and-conquer leads naturally to the problem of solving recurrences (to get runtime bounds)
- Mergesort recurrence was easy to analyze.
  
  What about in general?
  How can we deal with more general recurrences?

Theorem (Master Theorem (simple))

Given recurrence

\[ T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^c) \]

with \( T(1), a \geq 1, b > 1, c \geq 0 \) (constants), then

\[ T(n) = \begin{cases} 
\Theta(n^c), & \text{if } c > \log_b a \\
\Theta(n^{c \log_b a}), & \text{if } c = \log_b a \\
\Theta(n^c \log n), & \text{if } c < \log_b a
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Proof of Master Theorem

- *Remark:* it is more important to remember the method than the result  
  - Prof. Lau

Proof method works more generally.
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- Recursion tree: (see board)
  - If $c > \log_b a$, then top level dominates
    - *decreasing* geometric sequence, ratio $a/b^c < 1$
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Recursion tree: (see board)

1. If $c > \log_b a$, then top level dominates
   * * * * * 
   \textit{decreasing} geometric sequence, ratio $a/b^c < 1$
2. If $c = \log_b a$, then every layer same, and $\Theta(\log n)$ layers

$\frac{21}{34}$
Proof of Master Theorem

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Proof method works more generally.

- **Recursion tree**: (see board)
  1. If $c > \log_b a$, then top level dominates
     - *decreasing* geometric sequence, ratio $a/b^c < 1$
  2. If $c = \log_b a$, then every layer same, and $\theta(\log n)$ layers
  3. If $c < \log_b a$, then bottom level dominates
     - *increasing* geometric sequence, ratio $a/b^c > 1$
General Master Theorem

Theorem (Master Theorem)

Given recurrence

\[ T(n) = aT(n/b) + f(n) \]

with \( T(1), f(1), a \geq 1, b > 1 \) (constants), then

\[
T(n) = \begin{cases} 
\Theta(n^{\log_b a}), & \text{if } f(n) = O(n^{\log_b a - \varepsilon}), \text{ for some } \varepsilon > 0 \\
\Theta(n^{\log_a n \log n}), & \text{if } f(n) = \Theta(n^{\log_b a}) \\
\Theta(f(n)), & \text{if } f(n) = \Omega(n^{\log_b a + \varepsilon}), \text{ for some } \varepsilon > 0 \\
\text{and if } af(n/b) \leq cf(n) \text{ for some } 0 < c < 1 
\end{cases}
\]

Same proof
More recurrences

- Imbalanced trees:
  - $T(n) = T(n/3) + T(2n/3) + c \cdot n$
  - $T(n) = \Theta(n \log n)$
More recurrences

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  - $T(n) = T(n/3) + T(2n/3) + c \cdot n$
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More recurrences

- Imbalanced trees:
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More recurrences

- **Imbalanced trees:**
  - \( T(n) = T(n/3) + T(2n/3) + c \cdot n \)
  - \( T(n) = T(n/2) + 1 \)
  - \( T(n) = T(n/2) + n \)
  - \( T(n) = T(\sqrt{n}) + 1 \)

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at level \( i \), subproblem of size \( n^{2^{-i}} \)
More recurrences

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  - $T(n) = T(n/2) + n$ \quad $T(n) = O(n)$
  - $T(n) = T(\sqrt{n}) + 1$ \quad $T(n) = O(\log \log n)$

  at level $i$, subproblem of size $n^{2^{-i}}$

- **Exponential time recurrences:**
  - $T(n) = n \cdot T(n - 1) + 1$ \quad $T(n) = O(n!)$
  - Fibonacci: $T(n) = T(n - 1) + T(n - 2)$ \quad $T(n) = O(\phi^n)$

  $\phi = \frac{1 + \sqrt{5}}{2}$
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Maximum Subarray Sum

- **Input:** array $A = (a_1, \ldots, a_n)$ where each $a_i$ is an integer
- **Output:** indices $1 \leq i \leq j \leq n$ and $s$ such that

\[
s = a_i + \cdots + a_j, \quad \text{and} \quad s = \max_{\alpha \leq \beta} \sum_{k=\alpha}^{\beta} a_k
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- Divide and conquer approach:
  1. divide array in the middle
  2. largest sum either on left subarray, right subarray, or crossing the middle
  3. recurse on left subarray and on the right subarray
  4. compute max sum that crosses the middle
  5. output the max of items 3 and 4
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- for more details, see [CLRS 2009, Chapter 4.1]
Acknowledgement

- Based on Prof Lau’s lecture
  https://cs.uwaterloo.ca/~lapchi/cs341/notes/L02.pdf
Cormen, Thomas and Leiserson, Charles and Rivest, Ronald and Stein, Clifford. (2009)
MIT Press