CS 341: Algorithms

Lecture 4: Divide and conquer, continued

Slides due to Éric Schost and based on lecture notes by many other CS341 instructors

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Beyond the master theorem

Beyond the master theorem

Some recursive algorithms

- do recursion in uneven sizes (e.g. both $n/3$ and $n/4$), ...
- or in size $\sqrt{n}$, or even $\log(n)$, ...
- or have extra work more complicated than n^y

No master theorem covers everything. But **recursion trees** still work:

- find out how many nodes, of what size, there are at each level
- gives the total work per level
- **warning**, in case of uneven size recursion, some levels may not be complete
  $\rightarrow$ not always clear that you get a Θ (example: linear time median, later)

Remark:

- to be 100% clean, we should justify that we can work with sloppy recurrences.
- don't worry about it; it can be done (but gets more difficult than for merge sort)

Example: "generalized" master theorem

Suppose that $\boldsymbol{a \geq 1}$ and $\boldsymbol{b > 1}$. Consider the recurrence

$$T(n) = a\,T\left(\frac{n}{b}\right) + \boldsymbol{f(n)}, \qquad T(n) = d \quad (n \leq 1), \qquad \inf_n f(n) > 0$$

Let $\boldsymbol{x = \log_b a}$ (so $a = b^x$).

$$\text{Then } T(n) \in \begin{cases} \Theta(\boldsymbol{f(n)}) & \text{if } f(n) \in \Omega(n^{x+\varepsilon}), \text{ for some } \varepsilon > 0 \\ \Theta(\boldsymbol{n^x \log n}) & \text{if } f(n) \in \Theta(n^x) \\ \Theta(\boldsymbol{n^x}) & \text{if } f(n) \in O(n^{x-\varepsilon}), \text{ for some } \varepsilon > 0 \end{cases}$$

Proof: by recursion tree, no details.

Remark: we can either say "for n a power of b" or extend the definition of T to all $n \in \mathbb{R}$

Examples and non-examples

$T(n) = 3T(n/4) + n\log(n)$.

- $x = \log_4(3) < 1$
- $f(n) \in \Omega(n^1) = \Omega(n^{x+\varepsilon})$ where $\varepsilon = 1 - x > 0$
- so $T(n) \in \Theta(n\log(n))$

$T(n) = 2T(n/2) + n\log(n)$.

- $x = \log_2(2) = 1$
- $f(n) \in \Omega(n)$ implies not case three
- $f(n) \notin \Theta(n)$ implies not case two
- $f(n)/n^{x+\varepsilon} = \log(n)/n^{\varepsilon} \to 0$ implies $f(n) \in o(n^{x+\varepsilon})$ for all $\varepsilon > 0$
- therefore $f(n) \notin \Omega(n^{x+\varepsilon})$ and not case one
- No case of the master theorem applies

Linear time median

Median of medians

Selection: given $A[0..n-1]$ and k in $\{0, \ldots, n-1\}$, find the entry that would be at index k **if A was sorted**

Minimum: select from $A[0..n-1]$ at index $k = 0$
Maximum: select from $A[0..n-1]$ at index $k = n-1$

Median: select from $A[0..n-1]$ at index $\lfloor n/2 \rfloor$

Model: unit cost

Known results: sorting A in $\Theta(n \log(n))$, or a simple randomized algorithm in expected time $\Theta(n)$

The selection algorithm

quick-select(A, k)
A: array of size n, k: integer s.t. $0 \leq k < n$
1. $p \leftarrow$ **choose-pivot**(A)
2. $i, j \leftarrow$ **partition**(A, p)
3. **if** $i \leq k \leq j$ **then**
4. **return** p
5. **else if** $k < i$ **then**
6. **return quick-select**$(A[0, 1, \ldots, i-1], k)$
7. **else if** $j < k$ **then**
8. **return quick-select**$(A[j+1, j+2, \ldots, n-1], k-j-1)$

partition(A, p)**:**

- reorders A so that $[< p,\ \ A[i] = p, \ldots, A[j] = p,\ \ > p]$ **in linear time**

Goal: find a pivot such that both i and $n - j - 1$ are not too large

Median of medians

Sketch of the algorithm:

- divide A into $n/5$ groups $G_1, \ldots, G_{n/5}$ of size 5
- find the medians $m_1, \ldots, m_{n/5}$ of each group $\Theta(n)$
- pivot p is the median of $[m_1, \ldots, m_{n/5}]$ $T(n/5)$

Claim

With this choice of p, the indices i and $n - j - 1$ are **at most** $7n/10$

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Proof

- **half** of the m_i's are **greater than or equal to** p $n/10$
- for each m_i, there are **3** elements in G_i **greater than or equal to** m_i
- so **at least** $3n/10$ elements **greater than or equal to** p
- so **at most** $7n/10$ elements **less than** p
- so i is **at most** $7n/10$. Same thing for $n - j - 1$

Median of medians

Sketch of the algorithm:

- divide A into $n/5$ groups $G_1, \ldots, G_{n/5}$ of size 5
- find the medians $m_1, \ldots, m_{n/5}$ of each group — $\Theta(n)$
- pivot p is the median of $[m_1, \ldots, m_{n/5}]$ — $T(n/5)$

Claim

With this choice of p, the indices i and $n - j - 1$ are **at most** $7n/10$

Consequence: (sloppy) recurrence

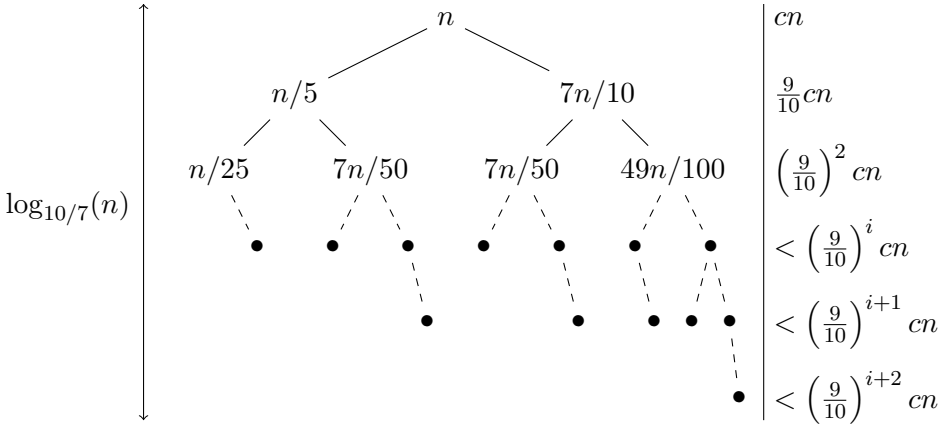
$$T(n) = T(n/5) + T(7n/10) + cn$$

Claim

This gives worst-case $T(n) \in \Theta(n)$ ($\Omega(n)$ clear)

The recursion tree for $T(n)$

Can prove: we can set $T(n) = 0$ for $n \leq 1$.



Key point: $\boldsymbol{T(n) < cn}$ $\times$ (geometric sum of ratio $\mathbf{9/10 < 1}$), so $\boldsymbol{T(n) \in O(n)}$.

Why not median of three?

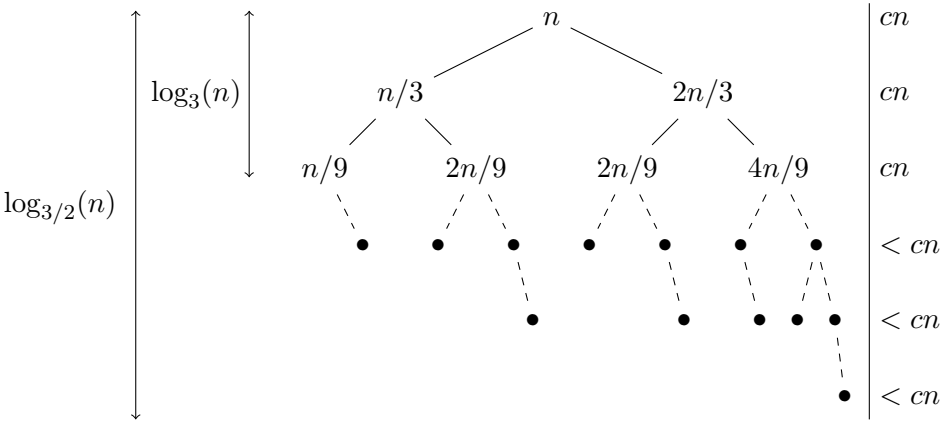
- we do $n/3$ **groups of 3** and find their medians $m_1, \ldots, m_{n/3}$ — $\Theta(n)$
- p is the median of $[m_1, \ldots, m_{n/3}]$ — $T(n/3)$
- half of the m_i's are greater than or equal to p — $n/6$
- in each group, **2** elements greater than or equal to m_i
- so overall at least $n/3$ elements greater than or equal to p
- so at most $2n/3$ elements less than p
- so $i \leq 2n/3$, and $n - j - 1 \leq 2n/3$

Recurrence: $T(n) = T(n/3) + T(2n/3) + cn$

Exercise

This gives $T(n) \in \Theta(n \log(n))$

The recursion tree for $T(n)$



Key point: $\boldsymbol{cn \log_3 \leq T(n) \leq cn \log_{3/2}(n)}$ so $\boldsymbol{T(n) \in \Theta(n \log n)}$.

Closest pairs
(time permitting)

Closest pairs

Goal: given n points (x_i, y_i) in the plane, find a pair (i, j) that minimizes the distance

$$d_{i,j} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

Equivalent to minimize

$$d_{i,j}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$$

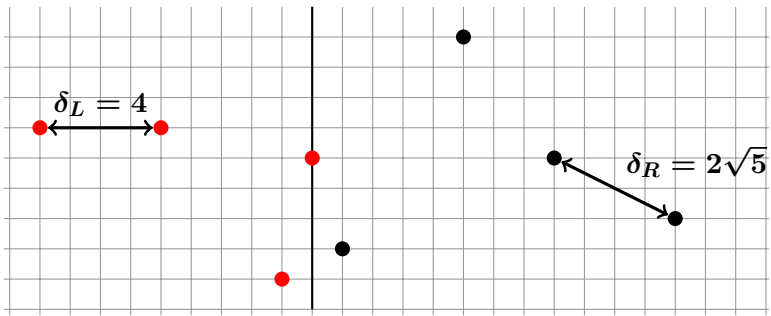
Assumption

all x_i's distinct

Divide-and-conquer

Idea: separate the points into two halves L, R at the median x-value

- L = all $n/2$ points with $x \leq x_{\text{median}}$ (no other point on the median line)
- R = all $n/2$ points with $x > x_{\text{median}}$
- find the closest pairs in both L and R recursively
- the closest pair is either **between points in L** (done), or **between points in R** (done), or **transverse** (one in L, one in R)



Divide-and-conquer

Familiar pattern ...

ClosestPair(P)
P: array of points of size n
1. $L, R \leftarrow$ **PartitionByMedianX**(P)
2. $\delta_L \leftarrow$ **ClosestPair**(L)
3. $\delta_R \leftarrow$ **ClosestPair**(R)
4. $\delta_T \leftarrow$ **TransverseClosestPair**(L, R) ???
5. **return** $\min(\delta_L, \delta_R, \delta_T)$

Divide-and-conquer

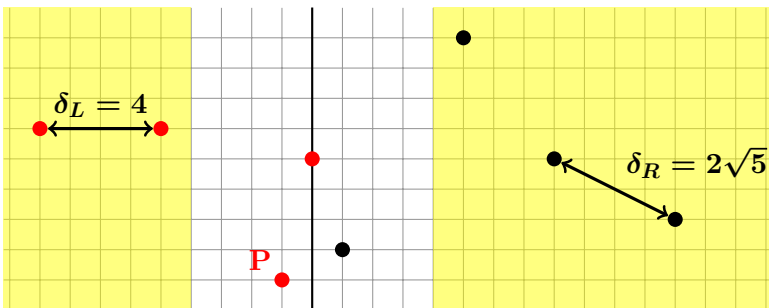
Idea: We only have to find transverse pairs closer than δ

ClosestPair(P)
P: array of points of size n
1. $L, R \leftarrow$ **PartitionByMedianX**(P)
2. $\delta_L \leftarrow$ **ClosestPair**(L)
3. $\delta_R \leftarrow$ **ClosestPair**(R)
4. $\delta \leftarrow \min(\delta_L, \delta_R)$
5. **return** **TransverseCase**(L, R, δ) ???

Finding the shortest transverse distance

Set $\delta = \min(\delta_L, \delta_R)$

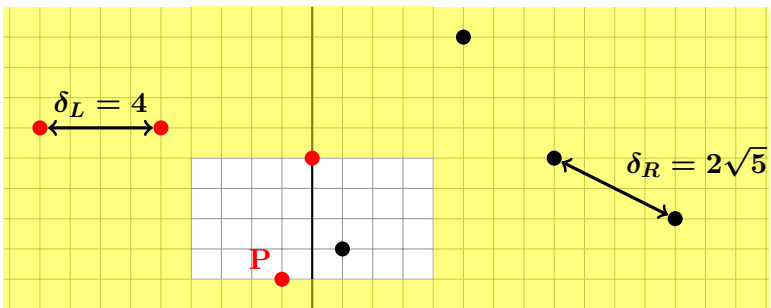
- only need to consider transverse pairs (P, Q) with $\mathbf{dist}(\boldsymbol{P}, \boldsymbol{R}) \leq \boldsymbol{\delta}$ and $\mathbf{dist}(\boldsymbol{Q}, \boldsymbol{L}) \leq \boldsymbol{\delta}$.



Finding the shortest transverse distance

Set $\delta = \min(\delta_L, \delta_R)$

- for any $P = (x_P, y_P)$, enough to look at points with $y_P \leq y \leq y_P + \delta$

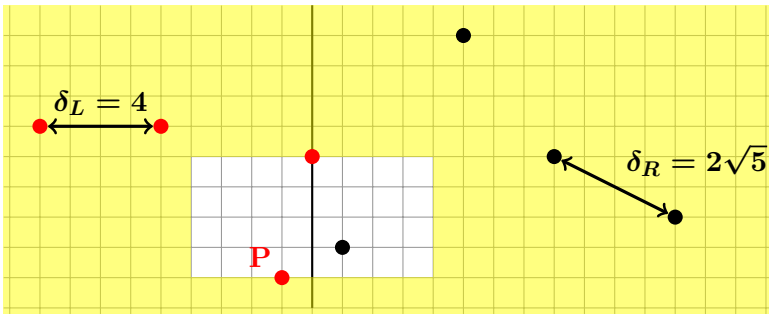


Finding the shortest transverse distance

Set $\delta = \min(\delta_L, \delta_R)$

- for any $P = (x_P, y_P)$, enough to look at points with $y_P \leq y \leq y_P + \delta$

So it is enough to check distances $d(P, Q)$ for Q in the rectangle.



How many points in the rectangle?

Claim

There are at most **8** points from our initial set (including P) in the rectangle.

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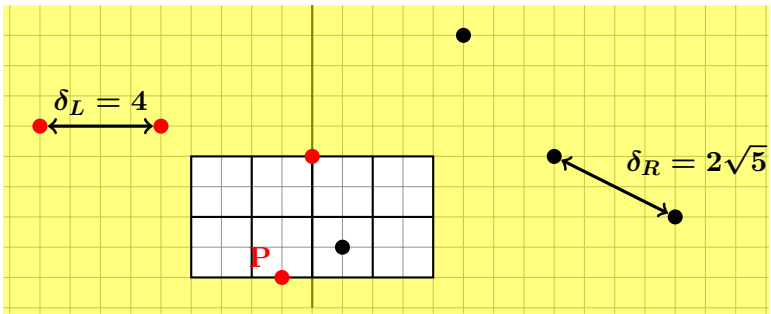
Proof. Cover the rectangle with **8** squares of side length $\delta/2$

- they overlap along lines, but it's OK



How many points in the rectangle?

Claim

There are at most **8** points from our initial set (including P) in the rectangle.

Proof. Cover the rectangle with **8** squares of side length $\delta/2$

- they overlap along lines, but it's OK
- a square on the left contains **at most one point from L**
- a square on the right contains **at most one point from R**

Consequence: at most 8 points in the range $y_P \leq y \leq y_P + \delta$

Data structures and runtime

Initialization: sort the points **twice**, with respect to x and to y.
One-time cost $\boldsymbol{O(n \log(n))}$, before recursive calls cf kd-trees

Main algo: given two arrays representing the **same points** (A_x sorted in x, A_y sorted in y)

- find the x-median using A_x $\boldsymbol{\Theta(1)}$
- split both A_x and A_y for recursions $\boldsymbol{\Theta(n)}$
- recurse on L and R $\boldsymbol{2T(n/2)}$
- remove the points at distance $\geq \delta$ from the x-median line from A_y $\boldsymbol{\Theta(n)}$
- inspect all remaining points P in A_y (=in increasing y-order)
  for each P, compute the distance to points Q in A_y with $y_P \leq y_Q \leq y_P + \delta$ min.
  at most 8 points Q need to be checked per P $\boldsymbol{\Theta(n)}$

Runtime: $\boldsymbol{T(n) = 2T(n/2) + cn}$ so $\boldsymbol{T(n) \in \Theta(n \log(n))}$